Apparent thermal conductivity of periodic two-dimensional composites

M. Jiang a, I. Jasiuk b,*, M. Ostoja-Starzewski c

Received 1 July 2001; received in revised form 10 February 2002; accepted 1 March 2002

Abstract

Effects of scale and boundary conditions, as well as mismatch between component phases, on apparent thermal conductivity of periodic, square-array, two-dimensional composites are investigated in this paper. The apparent conductivities are defined as those under either essential, natural, mixed or periodic boundary conditions applied to finite size material domains. It is shown that apparent conductivities obtained under mixed and periodic conditions are the same within numerical accuracy, and they are bounded by those obtained under essential and natural boundary conditions. Bounds are very sensitive to the mismatch of phase conductivities: the higher the mismatch, the wider the bounds. Bounds tighten as the window size increases. It is also observed that the window’s location strongly affects the bounds’ sharpness.

1. Introduction

Scale and boundary conditions effects have been investigated during the last decade for random inhomogeneous materials—for both linear [1–3], and nonlinear materials [4,5]. The basic idea is to use variational principles to obtain bounds, and then their order structures and hierarchy structures. The details of these structures, critically depending on the mismatch of phase properties, volume fractions, phase microgeometries (distribution information) and scale sizes can only be determined by experiment or computational micromechanics.

Few investigations on these issues have been done for periodic composites, even though they are easier to carry out compared with those of random composites. Hollister and Kikuchi [6], Pecullan et al. [7] and Jiang et al. [8] investigated scale and boundary conditions effects in elastic or elasto-plastic behavior of periodic two-dimensional (2D) composites. The motivations for such investigations include:

(a) The understanding of periodic composites can give better insights into random composites.
The properties of periodic structures consisting of a finite number of periodic cells may be very different from the effective ones. In this paper, we investigate thermal conductivity of 2D composites with locally isotropic phases; this is equivalent to anti-plane elasticity problem, electrical conductivity problem, etc., (see the Appendix A for a list of equivalent problems). It is governed by the 2D Laplace's equation, which is easier to solve than the governing equation of 2D (in-plane) and three-dimensional (3D) elasticity problems. In addition, because only one material constant, i.e., thermal conductivity, is needed to describe each phase, the parametric investigation of such a problem is much simpler.

The influence of several factors on apparent conductivity is investigated, such as the window (a piece of composite with the same volume fraction as the original composite, in ensemble average sense for the random composite) size effect, the mismatch in material properties effect, and the boundary conditions effect. The boundary conditions include essential, natural, mixed and periodic. The property obtained under a certain boundary condition applied to a window is referred to as an apparent property, following Huet [2]. The apparent property of a periodic composite under periodic boundary conditions (PBCs) is just an effective property.

We show that the apparent properties under mixed and periodic conditions are the same within numerical accuracy, and are bounded by those obtained under essential and natural boundary conditions (NBCs). Bounds are very sensitive to mismatch of phase conductivities: the higher the mismatch, the wider the bounds. We also investigate the effect of a window's location on bounds' sharpness.

2. Composite structure, unit cells and boundary conditions

We study the composite with a square packing of inclusions as shown in Fig. 1. The two choices of unit cells are sketched in Fig. 2. We choose these unit cells on the basis of two considerations. First, we should be able to construct the whole composite by repeating of unit cell in $x_1$ and $x_2$ directions. Second, the effective conductivity obtained from the unit cell should be isotropic. If we define the scale of a square unit cell of Fig. 2 to be $\delta = \delta_0$ ($\delta_0 = L_0/d$, where $L_0$ is the edge length of a unit cell and $d$ is the diameter of inclusion), by applying the second requirement on any scale of window of edge length $L$, we choose the windows of sizes $\delta = 2\delta_0$, $\delta = 4\delta_0$ and $\delta = 8\delta_0$, respectively. $n\delta_0$ means that the window's edge is $n$ times larger than the unit cell in both $x_1$ and $x_2$ directions.

For each phase the system is described by

$$C\left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2}\right) = 0$$

(1)

with $C(\vec{x}) = \chi_1(\vec{x})C_1 + \chi_2(\vec{x})C_2$. $\vec{x}$ is a position vector, $T$ is temperature, $C_i = k_i I$ ($i = 1$ for matrix,
or 2 for inclusions) is the conductivity tensor for isotropic phase with a constant conductivity \( k \), \( I \) is the identity tensor of second order, and \( \chi_i(\vec{x}) \) are the indicator functions of the regions occupied by phase \( i \). Locally, we also have

\[
\vec{q}(\vec{x}) = C(\vec{x}) \cdot \nabla T(\vec{x})
\]  

(2)

where \( \vec{q}(\vec{x}) \) is the local heat flux vector and \( \nabla T(\vec{x}) \) is the local temperature gradient.

The apparent properties of a given size window are obtained under four kinds of boundary conditions, which are:

**Periodic boundary condition (PBC)**

\[
T(\vec{x} + \vec{L}) = T(\vec{x}) + \nabla T \cdot \vec{L},
\]

\[
\vec{q}(\vec{x} + \vec{L}) = -\vec{q}(\vec{x}), \quad \forall \vec{x} \in \partial B_\delta
\]  

(3)

which yields the effective conductivity tensor \( C^\text{eff} \). \( \partial B_\delta \) is the boundary of window domain \( B_\delta \). \( \nabla T \) is the spatial average temperature gradient, \( \vec{L} = L\vec{e} \) (\( L \) is the length of periodicity or, equivalently, \( \delta \); \( \vec{e} \) is the unit vector) and \( \vec{q} \) is the heat flux.

**Essential boundary condition (EBC)**

\[
T = \nabla T \cdot \vec{n}, \quad \forall \vec{x} \in \partial B
\]  

(4)

which yields a tensor \( C^\text{E} \) (E stands for an essential condition).

**Natural boundary condition (NBC)**

\[
\vec{q} \cdot \vec{n} = \overline{\vec{q}} \cdot \vec{n}
\]  

(5)

which yields a tensor \( C^\text{N} = (S^\text{N})^{-1} \) (N stands for a natural condition), where \( \overline{\vec{q}} \) is the spatial average heat flux, and \( \vec{n} \) is the outer normal to the window’s boundary.

**Mixed boundary condition (MBC):** We apply essential condition on one pair of edges and natural condition on the other pair of edges. This condition yields a tensor \( C^\text{M} \) (M stands for a mixed condition).

Following Huet [2], it can be proved that

\[
C^R \equiv (S^R)^{-1} \leq (S^N)^{-1} \leq (S^N_{\eta})^{-1} \leq (S^N_{\eta'})^{-1} \leq C^\text{eff} \leq C^\text{M} \leq C^\text{E} \leq C^V, \quad \forall 1 < n' < n
\]  

(6)

where \( C^V \) and \( C^R \) denote the Voigt and Reuss bounds of conductivity tensor, corresponding to assumptions of uniform temperature gradient and uniform heat flux, respectively, and \( S \) is the resistivity tensor. Also following [9,1], one can prove that, for an arbitrary window of size \( \delta \)

\[
(S^N_{\delta})^{-1} \leq C^\text{M} \leq C^\text{E}_{\delta}
\]  

(7)

The relationships in (6) and (7) are to be understood as follows: for two second-rank tensors \( A \) and \( B \), the order relation \( B \leq A \) means that \( \vec{v} \cdot B \cdot \vec{v} \leq \vec{v} \cdot A \cdot \vec{v} \) for any vector \( \vec{v} \neq \vec{0} \). For the composite studied in this paper, the microstructure is isotropic in the sense of thermal conductivity, i.e., \( C^\text{eff} = k^\text{eff}I \), where constant \( k^\text{eff} \) depends on phases’ conductivities and volume fraction of inclusions. From Eqs. (6) and (7) we know that apparent properties depend on boundary conditions and window size, and the influence of these factors disappears as the window size goes to infinity.

### 3. Numerical experiments

To find apparent properties as defined above, one must solve the corresponding boundary value problems by a numerical method. In this paper, we employ the finite element software ANSYS 5.4. In particular, the volume fraction of inclusions is chosen to be 0.35. The thermal conductivity of matrix is assumed (without loss of generality) to be 1, and the mismatch between the thermal conductivities of inclusion and matrix varies from 0.001 to 1000 (in this paper, we refer to the mismatch less than unity as poorly conducting inclusion case, while mismatch greater than unity to be a highly conducting inclusion case). After the temperature field is obtained, we can use two methods to find the apparent properties, i.e., by

\[
\vec{q} = C^\text{app} \cdot \nabla T
\]  

(8)

or

\[
\Phi = \frac{1}{2} V_\delta \cdot \nabla T \cdot C^\text{app} \cdot \nabla T
\]  

(9)

where \( \Phi \) is the entropy rate, \( C^\text{app} \) is an apparent conductivity tensor under a certain boundary condition, \( V_\delta \) is the area of a given window.

It has been shown that, under EBCs, apparent properties obtained from Eqs. (8) and (9) are
equivalent. The same holds separately for natural and properly constructed mixed conditions. This results from the Hill condition [1]. In our numerical approach, we use Eq. (9) to obtain the apparent properties.

During our numerical calculation of

$$C_{\text{app}}^{-1} = \begin{bmatrix} C_{11}^{\text{app}} & C_{12}^{\text{app}} \\ C_{21}^{\text{app}} & C_{22}^{\text{app}} \end{bmatrix}$$

a given $\nabla \eta$ is used in PBCs and EBCs, and a given $\bar{\eta}$ is used in NBCs. Specifically, for PBCs and EBCs, we apply $\nabla \eta = (1, 0)^T$ to calculate $C_{11}^{\text{app}} = C_{22}^{\text{app}}$, and we apply $\nabla \eta = (1, 1)^T$ to calculate $C_{12}^{\text{app}} = C_{21}^{\text{app}}$. For NBCs, we apply $\bar{\eta} = (1, 0)^T$ to calculate $S_{11}^{\text{app}} = S_{22}^{\text{app}}$, then we apply $\bar{\eta} = (1, 1)^T$ to calculate $S_{12}^{\text{app}} = S_{21}^{\text{app}}$. When boundary condition is mixed, on the pair of edges in the $x_1$-direction, we fix the temperature of nodes according to their $x_1$ coordinates, and we specify the heat flux to be zero on the pair of edges in $x_2$-direction. First we illustrate some results under EBCs and NBCs. For the unit cell (1) of Fig. 2, for example, when $\delta = \delta_0$ and $k_1 = 1, k_2 = 10$, we have

$$C^E = \begin{bmatrix} 3.1128 & 0.0019 \\ 0.0019 & 3.1128 \end{bmatrix} \quad \text{and} \quad S^N = \begin{bmatrix} 0.5919 & 3.3 \times 10^{-5} \\ 3.3 \times 10^{-5} & 0.5919 \end{bmatrix}$$

and when $k_1 = 1, k_2 = 1000$, we have

$$C^E = \begin{bmatrix} 208.84 & 0.3350 \\ 0.3350 & 208.84 \end{bmatrix} \quad \text{and} \quad S^N = \begin{bmatrix} 0.5251 & 3.9 \times 10^{-5} \\ 3.9 \times 10^{-5} & 0.5251 \end{bmatrix}$$

We can see that $C_{11}^{\text{app}} = C_{22}^{\text{app}}$, $C_{12}^{\text{app}} \ll C_{11}^{\text{app}}$, $S_{11}^{\text{app}} = S_{22}^{\text{app}}$ and $S_{12}^{\text{app}} \ll S_{11}^{\text{app}}$, which is trivial for $C^{\text{eff}}$ and $S^{\text{eff}}$, but not for $C^E$ and $S^N$. When window size is larger, $C^{\text{app}}$ becomes more isotropic. For example, for $\delta = 4\delta_0$, when $k_1 = 1, k_2 = 10$, we have

$$C^E = \begin{bmatrix} 2.1308 & 1.1 \times 10^{-8} \\ 1.1 \times 10^{-8} & 2.1308 \end{bmatrix} \quad \text{and} \quad S^N = \begin{bmatrix} 1.7725 & 2.4 \times 10^{-5} \\ 2.4 \times 10^{-5} & 1.7725 \end{bmatrix}$$

So, in the following, we take the form of $C^{\text{app}}$ to be

$$C^{\text{app}} = \begin{bmatrix} k^{\text{app}} & 0 \\ 0 & k^{\text{app}} \end{bmatrix}$$

and we only report $k^{\text{app}}$ for different window sizes, mismatches and boundary conditions. From (6) and (9), we can obtain the following hierarchy structure

$$k^R \leq k_{\delta_0}^N \leq k_{n \delta_0}^N \leq k_{n \delta_0}^N \leq k^{\text{eff}}_m \leq k^{\text{E}} \leq k^{\text{V}}_m, \quad \forall 1 < n' < n$$

Table 1

<table>
<thead>
<tr>
<th>NBC or MBC</th>
<th>EBC</th>
<th>Window size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2/k_1 = 0.001$</td>
<td>0.480e−2</td>
<td>0.5264</td>
</tr>
<tr>
<td>0.954e−2</td>
<td>0.5052</td>
<td>$\delta = 2\delta_0$</td>
</tr>
<tr>
<td>0.0187</td>
<td>0.4939</td>
<td>$\delta = 4\delta_0$</td>
</tr>
<tr>
<td>0.0732</td>
<td>0.4855</td>
<td>$\delta = 8\delta_0$</td>
</tr>
<tr>
<td>0.0458</td>
<td>0.4890</td>
<td>0.5327</td>
</tr>
<tr>
<td>0.0841</td>
<td>0.5118</td>
<td>$\delta = 2\delta_0$</td>
</tr>
<tr>
<td>0.1435</td>
<td>0.5007</td>
<td>$\delta = 4\delta_0$</td>
</tr>
<tr>
<td>0.3278</td>
<td>0.4980</td>
<td>$\delta = 8\delta_0$</td>
</tr>
<tr>
<td>0.3217</td>
<td>0.5546</td>
<td>0.5927</td>
</tr>
<tr>
<td>0.4078</td>
<td>0.5747</td>
<td>$\delta = 2\delta_0$</td>
</tr>
<tr>
<td>0.4703</td>
<td>0.5651</td>
<td>$\delta = 4\delta_0$</td>
</tr>
<tr>
<td>0.5544</td>
<td>0.5601</td>
<td>$\delta = 8\delta_0$</td>
</tr>
<tr>
<td>10</td>
<td>1.689</td>
<td>3.113</td>
</tr>
<tr>
<td>1.744</td>
<td>2.459</td>
<td>$\delta = 2\delta_0$</td>
</tr>
<tr>
<td>1.772</td>
<td>2.131</td>
<td>$\delta = 4\delta_0$</td>
</tr>
<tr>
<td>1.773</td>
<td>1.872</td>
<td>$\delta = 8\delta_0$</td>
</tr>
<tr>
<td>100</td>
<td>1.879</td>
<td>2.044</td>
</tr>
<tr>
<td>1.960</td>
<td>11.97</td>
<td>$\delta = 2\delta_0$</td>
</tr>
<tr>
<td>2.002</td>
<td>7.009</td>
<td>$\delta = 4\delta_0$</td>
</tr>
<tr>
<td>2.011</td>
<td>3.281</td>
<td>$\delta = 8\delta_0$</td>
</tr>
<tr>
<td>1000</td>
<td>1.902</td>
<td>2.074</td>
</tr>
<tr>
<td>1.986</td>
<td>105.6</td>
<td>$\delta = 2\delta_0$</td>
</tr>
<tr>
<td>2.030</td>
<td>53.85</td>
<td>$\delta = 4\delta_0$</td>
</tr>
<tr>
<td>2.040</td>
<td>15.10</td>
<td>$\delta = 8\delta_0$</td>
</tr>
</tbody>
</table>
Table 2
Apparent property as a function of mismatch, boundary condition and window size; windows are generated using the unit cell (2)

<table>
<thead>
<tr>
<th>$k_2/k_1$</th>
<th>NBC</th>
<th>PBC or MBC</th>
<th>EBC</th>
<th>Window size</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.4703</td>
<td>0.4821</td>
<td>0.4955</td>
<td>$\delta = \delta_0$</td>
</tr>
<tr>
<td></td>
<td>0.4768</td>
<td>0.4894</td>
<td>$\delta = 2\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4796</td>
<td>0.4859</td>
<td>$\delta = 4\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4817</td>
<td>0.4832</td>
<td>$\delta = 8\delta_0$</td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>0.4775</td>
<td>0.4890</td>
<td>0.5021</td>
<td>$\delta = \delta_0$</td>
</tr>
<tr>
<td></td>
<td>0.4838</td>
<td>0.4961</td>
<td>$\delta = 2\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4866</td>
<td>0.4928</td>
<td>$\delta = 4\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.4881</td>
<td>0.4912</td>
<td>$\delta = 8\delta_0$</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.5462</td>
<td>0.5546</td>
<td>0.5648</td>
<td>$\delta = \delta_0$</td>
</tr>
<tr>
<td></td>
<td>0.5511</td>
<td>0.5604</td>
<td>$\delta = 2\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5533</td>
<td>0.5580</td>
<td>$\delta = 4\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5540</td>
<td>0.5553</td>
<td>$\delta = 8\delta_0$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.771</td>
<td>1.803</td>
<td>1.831</td>
<td>$\delta = \delta_0$</td>
</tr>
<tr>
<td></td>
<td>1.786</td>
<td>1.816</td>
<td>$\delta = 2\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.794</td>
<td>1.809</td>
<td>$\delta = 4\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.799</td>
<td>1.804</td>
<td>$\delta = 8\delta_0$</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1.993</td>
<td>2.044</td>
<td>2.095</td>
<td>$\delta = \delta_0$</td>
</tr>
<tr>
<td></td>
<td>2.018</td>
<td>2.070</td>
<td>$\delta = 2\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.032</td>
<td>2.058</td>
<td>$\delta = 4\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.041</td>
<td>2.049</td>
<td>$\delta = 8\delta_0$</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>2.019</td>
<td>2.074</td>
<td>2.127</td>
<td>$\delta = \delta_0$</td>
</tr>
<tr>
<td></td>
<td>2.046</td>
<td>2.100</td>
<td>$\delta = 2\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.061</td>
<td>2.088</td>
<td>$\delta = 4\delta_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.071</td>
<td>2.079</td>
<td>$\delta = 8\delta_0$</td>
<td></td>
</tr>
</tbody>
</table>

From Table 1 it is found that, for the same volume fraction, under PBCs and MBCs, the apparent properties are the same for both unit cells (1) and (2), within numerical accuracy. This tells us that one can use either of the unit cells to estimate the apparent—which happens to be effective—property under either MBC or PBC. It should be pointed out that the MBC is a common setup in experiments. The above observation gives us reasons to use the mixed condition in practice to obtain the effective conductivity for 2D isotropic periodic composite. Because of the equivalence of the apparent properties obtained under MBCs and PBCs, there are no scale effects for those properties, i.e., such apparent properties are the same for other window sizes. The contour plots of temperature are shown in Fig. 4. It should be noted that the contour plots of temperature under MBCs and PBCs are not exactly the same. The effective conductivity and apparent properties under the mixed condition are bounded from above and below by the apparent conductivities obtained under EBCs and NBCs, respectively, as is shown earlier in Eqs. (6) and (7).

For the unit cell (2), shown in Fig. 2, the apparent properties under different boundary conditions are close. The temperature contour plots for $\delta = \delta_0$ are given in Fig 5. We can see that the two plots in Fig. 5 are similar. However, for the
Fig. 4. Temperature contour plots under different boundary conditions, unit cell (1), $\delta = \delta_0, k_2/k_1 = 10$: (a) PBC; (b) MBC; (c) EBC; (d) NBC.

Fig. 5. Temperature contour plots under EBC and NBC, unit cell (2), $\delta = \delta_0, k_2/k_1 = 10$: (a) EBC; (b) NBC.
unit cell (1), the apparent conductivities obtained under EBCs and NBCs are much different from the effective conductivity and they give very wide bounds. Their temperature contour plots given in Fig. 4(c) and (d) are quite different. We further observe that for a poorly conducting inclusion case, $j_k^{E}/C_0^k < j_k^{N}/C_0^k$, while for a highly conducting inclusion case, $j_k^{N}/C_0^k < j_k^{E}/C_0^k$.

From the Tables 1 and 2 and Fig. 3, we see that, as window size increases, the difference between apparent properties under different conditions becomes smaller and bounds become tighter. This convergence is a function of mismatch in conductivity constants: the larger the mismatch, the slower the bounds’ convergence. It is further observed that for a highly conducting inclusion case, the effective conductivity is closer to the lower bound, while for a poorly conducting inclusion case, the effective conductivity is closer to the upper bound. The details of this phenomenon can be seen in Figs. 6–8. Actually, the temperature contour plot patterns under EBC and NBC are not very distinct as shown in Fig. 6. However, from the contour plots of entropy rate, we can see that the patterns are totally different under essential and natural conditions. From Fig. 7, we observe that, for a highly conducting inclusion case, there is $\Phi$ concentration on the pair of edges where linear temperature constraint is applied, while $\Phi$ under natural conditions is pretty uniformly distributed. From Fig. 8, we see that, for a poorly conducting inclusion case, the concentration of $\Phi$ is on the pair of edges where uniform heat flux is applied, while $\Phi$ is quite uniformly distributed under EBCs, which is similar to those under MBCs and PBCs.

We have observed that choice of unit cells has a big influence on bounds. From Fig. 3, we see the big difference between bounds obtained for unit cells (1) and (2). It can be seen that for unit cell (2), when $\delta = \delta_0$, the bounds are already very close. For unit cell (1), on the other hand, even when the window size is pretty big, the bounds are still very wide. The convergence of bounds for this case may require a very large window, which will exceed the capacity of computer simulation, as illustrated in Fig. 3 for $k_2/k_1 = 0.1$ and $k_2/k_1 = 10$.

Concentration of $\Phi$ caused by location of inclusions on boundary, when the conditions are essential and natural, gives us some explanation on the wide bounds obtained for random composite [10]. For random composite, when we cut a window from the whole material, the probability that there is at least one inclusion falling on the window edges is almost 1. Thus, the entropy (energy in elasticity problem) concentration on edges, which is the reason for wide bounds, is inevitable. This also indicates that for random composite, the only...
A way to obtain close bounds is to choose a large window, whose size depends on mismatches and volume fraction of inclusions. Recently, a comprehensive study on the apparent out-of-plane elasticity of random composite under different boundary conditions was done in [11] by using a spring network method.

Fig. 7. Contour plot of $\Phi$ under MBC, EBC and NBC, unit cell (1), $\delta = 4\delta_0$, $k_2/k_1 = 10$. (The plots under MBC and PBC are almost identical, so only that of PBC is shown.) (a) PBC; (b) EBC; (c) NBC.

Fig. 8. Contour plot of $\Phi$ under MBC, EBC and NBC, unit cell (1), $\delta = 4\delta_0$, $k_2/k_1 = 0.1$. (The plots under MBC and PBC are almost identical, so only that of PBC is shown.) (a) PBC; (b) EBC; (c) NBC.
4. Conclusions

(1) Windows of square packing composite yield isotropic apparent conductivity tensor in the form of \( k_{\text{app}} I \), approximately, where \( I \) is a 2 \( \times \) 2 identity matrix.

(2) When mixed and periodic boundary conditions are applied, unit cells (1) and (2) yield the same apparent conductivity, within numerical accuracy, which, in this case, is the effective conductivity. This important observation provides us the way in numerical simulation or real experiment to obtain effective conductivity of periodic composite by applying the special mixed boundary conditions introduced in present paper.

(3) Effective conductivity is bounded from above and below by apparent conductivities obtained under essential and natural conditions.

(4) The apparent conductivity under either essential or natural condition is very sensitive to the unit cell choice.

(5) Bounds become tighter as window size increases and mismatch decreases.

(6) For highly conducting inclusion case, lower bound approaches to effective conductivity faster; for poorly conducting inclusion case, the opposite is true.

(7) Inclusions on edges are the major reason for wide bounds, which implies the need for very big windows for the investigation of random composites.

(8) The investigation in the present paper can also be generalized to 3D periodic composites, and we believe similar conclusions would be obtained. However, for the 3D case, the convergence rate of bounds under essential and natural boundary conditions could be different quantitatively from that of 2D case, with the same properties’ mismatch between fibers and matrix. This is an interesting topic for future investigation. For 3D composites, the computer requirements to do the FEA simulation would be much higher.

Acknowledgements

Support by the NSF under grants CMS-9713764 and CMS-9753075, and by the Canada Research Chairs program, is gratefully acknowledged.

Appendix A

The thermal conductivity problem is equivalent to a series of problems governed by the Laplace equation. A list of analogous quantities is given in Table 3 (e.g., [12]).

References


