Convergence in Probability

Go through the definitions and examples in `Week9_ConvergeProbans.pdf`.

How to show an estimator in consistent? Recall the four types of estimators described in the estimation notes.

1. $\hat{\theta}$ = the sample mean $\bar{X}$ or linear functions of the sample mean, e.g., $2\bar{X}$.
   - $\hat{\theta} = a\bar{X}$ and $\theta = a\mathbb{E}X_1 = a\mu$, that is, $\hat{\theta}$ is unbiased.
   - By WLLN, $\bar{X} \xrightarrow{P} \mu$.
   - Use the result

     $$X_n \xrightarrow{P} X, \quad a \text{ is a constant} \quad \Longrightarrow \quad aX_n \xrightarrow{P} aX,$$

     to show that $\hat{\theta} = a\bar{X} \xrightarrow{P} a\mu = \theta$, i.e., consistent.

2. $\hat{\theta}$ = the average of a transformation of X_i's, e.g., $\frac{1}{n}\sum_{i=1} \log X_i$ or $\frac{1}{n}\sum_{i=1} X_i^2$.
   - Define $Y_i = \log X_i$ then $\hat{\theta} = \frac{1}{n}\sum_{i=1} \log X_i = \bar{Y}$.
   - Find the distribution of Y_i and then you are back to the previous case.

3. $\hat{\theta} = g(\bar{X})$, a function of the sample mean, e.g., $1/\bar{X}$.
   - Suppose $\theta = g(\mu)$ where $\mu = \mathbb{E}X_1$.
   - By WLLN, $\bar{X} \xrightarrow{P} \mu$.
   - Use the result

     $$X_n \xrightarrow{P} a, \quad g \text{ is continuous at } a \quad \Longrightarrow \quad g(X_n) \xrightarrow{P} g(a),$$

     to show that $\hat{\theta} = g(\bar{X}) \xrightarrow{P} g(\mu) = \theta$, i.e., consistent.

4. $\hat{\theta}$ = order statistics, e.g., $Y_1 = \min_i X_i$ or $Y_n = \max_i X_i$.
   - Find the distribution of Y_1 or Y_n.
   - Calculate the the following probability

     $$\mathbb{P}\big(|\hat{\theta} - \theta| \geq \epsilon\big)$$

     and check whether it goes to zero when $n \to \infty$ for any small $\epsilon > 0$: if yes, $\hat{\theta}$ is consistent.