Transformations for Bivariate Random Variables

- **Two-to-One**, e.g., $Z = X + Y, Z = X^2/Y$, etc.
  - CDF approach
  - Convolution Formula for some special cases, e.g., $Z = X + Y$.
  - From joint to marginal: the convolution formulae are special cases of this approach.
  - MGF approach

- **Two-to-Two**, e.g., $Y_1 = u_1(X_1, X_2), Y_2 = u_2(X_1, X_2)$.
  - For one-to-one differentiable transformations, i.e., we can solve

$$x_1 = w_1(y_1, y_2), \quad x_2 = w_2(y_1, y_2).$$

Then

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(w_1(y_1, y_2), w_2(y_1, y_2))\,|J|, \quad (y_1, y_2) \in \mathcal{S}_Y \tag{1}$$

where J is the **Jacobian** of the transformation and $\mathcal{S}_Y$ is the two-dimensional support for the pdf of (Y_1, Y_2), which can be derived from the support of (X_1, X_2).

Steps for the Two-to-Two transformation

1. Get some initial guess of the support of (Y_1, Y_2)

2. Solve (x_1, x_2) as functions of (y_1, y_2)

$$\begin{cases} y_1 = u_1(x_1, x_2) \\ y_2 = u_2(x_1, x_2) \end{cases} \Longrightarrow \begin{cases} x_1 = w_1(y_1, y_2) \\ x_2 = w_2(y_1, y_2) \end{cases}$$

3. Compute the Jacobian $J = \begin{vmatrix} \frac{\partial w_1}{\partial y_1} & \frac{\partial w_1}{\partial y_2} \\ \frac{\partial w_2}{\partial y_1} & \frac{\partial w_2}{\partial y_2} \end{vmatrix}$.

4. Plug everything back to the formula (1)

5. Refine the support of (Y_1, Y_2): plug $w_1(y_1, y_2)$ and $w_2(y_1, y_2)$ into any inequalities involve (x_1, x_2) and check whether you need add any more constraint to your initial guess of the support of (Y_1, Y_2).