Tukey's Method for All Contrasts

Consider a balanced one-way ANOVA model with n_0 number of observations in each group. Let $\sum_{i=1}^{g} c_i\alpha_i$ be a contrast and we have learned that the corresponding LS estimate is given by $\Big(\sum c_i\bar{y}_{i\cdot}\Big)$.

The goal is to find $C_{\alpha,\boldsymbol{c}}$ such that

$$\mathbb{P}\Big[\sum c_i\alpha_i \in \Big(\sum c_i\bar{y}_{i\cdot} \pm \frac{\hat{\sigma}}{\sqrt{n_0}} C_{\alpha,\boldsymbol{c}}\Big), \text{ for any } \boldsymbol{c}\Big] \geq 1-\alpha.$$

For any contrast vector $\boldsymbol{c}$,

$$\begin{aligned}\sum c_i\alpha_i \in \Big(\sum c_i\bar{y}_{i\cdot} \pm \frac{\hat{\sigma}}{\sqrt{n_0}} C_{\alpha,\boldsymbol{c}}\Big) \quad &\Longleftrightarrow \quad \frac{|\sum_{i=1}^{g} c_i\bar{y}_{i\cdot} - \sum_{i=1}^{g} c_i\alpha_i|}{\hat{\sigma}/\sqrt{n_0}} \leq C_{\alpha,\boldsymbol{c}} \\ &\Longleftrightarrow \quad \frac{|\sum_{i=1}^{g} c_i(\bar{y}_{i\cdot} - \alpha_i - \mu)|}{\hat{\sigma}/\sqrt{n_0}} \leq C_{\alpha,\boldsymbol{c}} \\ &\Longleftrightarrow \quad \frac{|\sum_{i=1}^{g} c_i Z_i|}{\hat{\sigma}/\sigma} \leq C_{\alpha,\boldsymbol{c}}, \quad (*)\end{aligned}$$

where Z_i's are independent $N(0,1)$ random variables.

Using the result that

$$\Big|\sum_i c_i z_i\Big| \leq \Big(\frac{1}{2}\sum_i |c_i|\Big)\Big(\max_i z_i - \min_i z_i\Big),$$

we have (*) hold true if

$$\Big(\frac{1}{2}\sum_i |c_i|\Big)\frac{\max_i Z_i - \min_i Z_i}{\hat{\sigma}/\sigma} \leq C_{\alpha,\boldsymbol{c}}.$$

Note that the 2nd term on the left side of the inequality above has the studentized range distribution $T_{g,n-g}$. So to make the above inequality to hold true with probability at least $(1-\alpha)$, it suffices to set

$$C_{\alpha,\boldsymbol{c}} = \Big(\frac{1}{2}\sum_i |c_i|\Big) T_{g,n-g,\alpha},$$

that is, the simultaneous $(1-\alpha)$ CI for any contrast $\sum c_i\alpha_i$ is

$$\sum c_i\bar{y}_{i\cdot} \pm T_{g,n-g,\alpha}\big(\frac{1}{2}\sum_i |c_i|\big)\frac{\hat{\sigma}}{\sqrt{n_0}}.$$