ROBUST CONTROL FOR SYSTEMS WITH SECTOR-BOUNDED, SLOPE-RESTRICTED, AND ODD MONOTONIC NONLINEARITIES USING LINEAR MATRIX INEQUALITIES

BY

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Abstract

Nonlinear systems are everywhere in engineering and science contexts including aerospace, chemical, mechanical, and electrical engineering, and biology, chemistry, mathematics, and physics. Since analysis and synthesis problems of nonlinear systems are usually impossible to solve analytically, much research has focused on developing for computational tools for solving these problems for various classes of systems. Linear matrix inequalities (LMIs) have been developed as a useful approach to developing such tools.

Many dynamical system analysis and controller synthesis problems based on optimization have been formulated as linear matrix inequalities. The first LMIs in systems theory appeared in the 1890s with Lyapunov's stability theory. More recently, with the development of high-speed computers, computationally tractable and efficient interior-point methods [79, 17] for solving convex optimization problems have been developed and allowed stable and efficient computation of solutions of LMIs that arise in control systems theory [100].

This thesis focuses on analysis and control based on Lyapunov functions which enable the analysis of global asymptotic stability, decay rate, dissipativity, L_2 (or l_2), and RMS gains, to name a few. LMI-based problem formulations are applied to a specific class of nonlinear systems called Lur'e systems, which consist of a nominal LTI plant and feedback connected nonlinear functions whose properties are characterized by the set-valued nonlinear functions.

This thesis consists of two main contributions. First, a less conservative tool is derived for stability and performance analysis for Lur'e systems by exploiting the static (or memoryless), sector-bounded (or bounded-gain), and slope-restricted (or Lipschitz-bounded) nature of the feedback-connected nonlinear operators. Second, numerically tractable feasibility and optimization problems are constructed to solve LMIs for the design of stabilizing controllers for a certain class of Lur'e systems. Chapter 1 provides an introduction for nonlinear systems analysis and control, especially for absolute stability with a review of LMIs in systems and control theory. The connections between classical nonlinear systems analysis based on frequency-domain inequalities (FDIs) and LMIs of state-space realization are also described. Chapter **??** introduces some useful results from convex analysis and mathematical preliminaries with proofs, which are frequently used in this

thesis. In Chapter 2, the nonlinear stability analysis tools are formulated for certain classes of discrete-time Lur'e systems as feasibility problems over LMIs. A superset of the Lur'e-Postnikov function is used and specified properties of the nonlinear operators are exploited to reduce conservatism. Chapter 3 provides LMI-based methods to design a stabilizing controller, where a simple class of Lur'e systems is considered. Chapter 4 extends the results of Chapter 3 to robust control, where the nominal LTI plant is replaced by uncertain models that are dependent on unknown parameters and the set-valued uncertain mapping, which defines a set of plants of which the true process is in the set, corresponds to the plant/model mismatch. Finally, Chapter **??** suggests a design tool for the design of a stabilizing controller, where the size of the set of feedback-connected nonlinear operators is reduced by exploiting the slope-restrictedness (or Lipschitz-bound property) as well as the sector-bounded property.

This thesis is dedicated to my parents, Jin-Bong Kim and Jung-Ja Kwon.

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Table of Contents

List of Tables . . . . **ix**

List of Figures . . . . **x**

Chapter 1 Introduction **1**
1.1 Nonlinear Systems Analysis 1
1.1.1 Lyapunov Stability 2
1.1.2 Passivity and Small-Gain Theorems 7
1.1.3 Absolute Stability–Lur'e Problems 16
1.1.4 Loop Transformations 18
1.2 Literature Review for Absolute Stability–FDI and LMI Approaches 22
1.2.1 Circle Criterion 22
1.2.2 Popov Criterion 27
1.2.3 Tsypkin's Criterion 32
1.3 Linear and Bilinear Matrix Inequalities 35
1.3.1 The Linear Matrix Inequality and Optimization Problems 36
1.3.2 The Bilinear Matrix Inequality and Optimization Problems 39
1.3.3 The S-Procedure 41

Chapter 2 Stability and Performance Analysis for Lur'e Systems **46**
2.1 Introduction 46
2.2 Sufficient Conditions for Asymptotic Stability–Lyapunov Stability Analysis 47
2.2.1 Modified Lur'e-Postnikov Stability of a Discrete Time Lur'e System 47
2.2.2 Lagrange Relaxations 48
2.2.3 Discrete-Time Lur'e Systems with Slope-restricted Nonlinearities 49
2.2.4 Discrete-Time Lur'e Systems with Slope-restricted and Odd Monotonic Nonlinearities 52
2.3 Numerical Examples for Stability Analysis 54
2.4 Performance Analysis for Discrete-Time Lur'e Systems 58
2.5 Summary 59

Chapter 3 Controller Synthesis Problems for Lur'e Systems **60**
3.1 Introduction 61
3.2 State-Feedback Control via LMI Optimization 67
3.2.1 Continuous-Time Lur'e Systems 67
3.2.2 Discrete-Time Lur'e Systems 68
3.2.3 Illustrative Examples 70
3.3 Observer-Based State-Feedback Control via LMI Optimization 72
3.3.1 Discrete-Time Lur'e Systems 72
3.3.2 Illustrative Examples 78
3.4 Output-Feedback Control via LMI Optimization 81
3.4.1 Static Output-Feedback Control via LMI Optimization 81
3.4.2 Extension for Dynamic Output-Feedback Control 88

3.4.3 Static Output-Feedback Control Based on Differentiator-Free Control Model 89
3.4.4 Dynamic Output-Feedback Control via LMI Optimization 94
3.4.5 Illustrative Examples 98
3.5 Computational Issues in SOF Controller Synthesis Problems 107
3.5.1 Introduction 107
3.5.2 Successive Projection Algorithm 109
3.5.3 Computational Issues in Fixed-Order Output-Feedback Controller Design 111

Chapter 4 Robust Controller Synthesis Problems for Lur'e Systems 114
4.1 Introduction 114
4.2 Static Output-Feedback Control via LMI Optimization for a Certain Class of Lur'e Systems under Matching Uncertainties 116
4.2.1 SOF Control via LMI Optimization for a Certain Class of Lur'e Systems under Matching Uncertainties–Case I 117
4.2.2 SOF Control via LMI Optimization for a Certain Class of Lur'e Systems under Matching Uncertainties–Case II 123
4.3 Static Output-Feedback Control via LMI Optimization for a Certain Class of Lur'e Systems in the Presence of Polytopic Uncertainties 127
4.3.1 Parameter-Dependent Lyapunov Function (PLDF) and Its Applications 127
4.3.2 SOF Control via LMI Optimization for a Certain Class of Lur'e Systems under Polytopic Uncertainties–Case I 132
4.3.3 SOF Control via LMI Optimization for a Certain Class of Lur'e Systems under Polytopic Uncertainties–Case II 141
4.3.4 Illustrative Examples 150
4.4 Parameter Dependent State-Feedback Control via LMI Optimization for Lur'e Systems with Uncertainties 151
4.4.1 Continuous-Time LTI Systems with Time-Invariant Parametric Uncertainty 152
4.4.2 Continuous Lur'e Systems with Time-Invariant Parametric Uncertainty 156

References 161

Author's Biography 168

List of Tables

2.1 The maximal upper bound on the sector bound 55

3.1 The maximal upper sector-bound achieved by the output-feedback control schemes 104

4.1 The maximal upper sector-bound achieved by the output-feedback control schemes 150

List of Figures

1.1 Standard Nonlinear Feedback System . . . 4
1.2 Reduced Standard Nonlinear Feedback System . . . 4
1.3 Passivity Theorem . . . 8
1.4 Passive Multiplier . . . 8
1.5 Loop Transformations for Sector Conditions . . . 18
1.6 Geometric Circle Criterion in Nyquist Diagram: $q = j\omega$ for CT cases or $e^{j\omega}$ for DT cases . . . 24
1.7 Loop Transformations for Popov Criterion . . . 28
1.8 Geometrical Popov Criterion in Nyquist Diagram . . . 30
1.9 Geometrical Tsypkin's Criterion in Nyquist Diagram . . . 34

3.1 Trajectory of solution for the closed-loop system . . . 71
3.2 State-feedback control law $u(k) = Kx(k)$ . . . 71
3.3 Trajectory of solution for the closed-loop system . . . 72
3.4 State-feedback control law $u(k) = Kx(k)$ . . . 72
3.5 Observer-Based State-Feedback Control . . . 73
3.6 Error Dynamics . . . 73
3.7 Closed-Loop System with State Observer . . . 77
3.8 Trajectory of solution for the closed-loop system and the error dynamics . . . 80
3.9 State-feedback control law $u(k) = K_s\hat{x}(k)$ . . . 80
3.10 Trajectory of solution for the closed-loop system and the error dynamics . . . 80
3.11 State-feedback control law $u(k) = K_s\hat{x}(k)$ . . . 80
3.12 LFT representation for DOF Control . . . 88
3.13 LFT representation for Equivalent SOF Control . . . 88
3.14 Example 5: Theorem 27 . . . 104
3.15 Example 5: Theorem 28 . . . 104
3.16 Example 5: Theorem 29 . . . 105
3.17 Example 5: Theorem 30 . . . 105
3.18 Example 8: Section 3.4.2 . . . 105
3.19 Example 8: Section 3.4.4 . . . 105
3.20 Example 6: Theorem 27 . . . 105
3.21 Example 6: Theorem 28 . . . 105
3.22 Example 6: Theorem 29 . . . 106
3.23 Example 6: Theorem 30 . . . 106
3.24 Example 9: Section 3.4.2 . . . 106
3.25 Example 9: Section 3.4.4 . . . 106
3.26 Example 7: Theorem 27 . . . 106
3.27 Example 7: Theorem 28 . . . 106
3.28 Example 7: Theorem 29 . . . 107
3.29 Example 7: Theorem 30 . . . 107
3.30 Example 10: Section 3.4.2 . . . 107
3.31 Example 10: Section 3.4.4 . . . 107

3.32 Successive Projections: rel int $\mathcal{C}_1 \cap$ rel int $\mathcal{C}_2 = \emptyset$ 108
3.33 Successive Projections: rel int $\mathcal{C}_1 \cap$ rel int $\mathcal{C}_2 \neq \emptyset$ 108

4.1 LFT and SNOF for Robust Control for Uncertain Lur'e Systems with External Input-Output 115
4.2 LFT and SNOF for Robust Control for Uncertain Lur'e Systems 115
4.3 Theorem 37 151
4.4 Theorem 38 151
4.5 Theorem 39 151
4.6 Theorem 40 151
4.7 Parameter-Dependent State-Feedback Control for LTI systems 152
4.8 Parameter-Dependent State-Feedback Control for Lur'e systems 152
4.9 Trajectory of solution for the closed-loop system 156
4.10 State-feedback control law $u(k) = K(\hat{\theta})x(k)$ and the adaptatation parameters 156
4.11 Trajectory of solution for the closed-loop system 160
4.12 State-feedback control law $u(k) = K(\hat{\theta})x(k)$ and the adaptation parameters 160

Chapter 1

Introduction

This thesis considers the global stability of the state equation, that is, the internal stability of the system. Although some theorems for the global stability of continuous-time nonlinear systems are described, a main contribution in this thesis is to develop new nonconservative conditions for the stability analysis of discrete-time systems, with the advances being LMI-based conditions that exploit the static (or memoryless), sector-bounded (or bounded-gain), and slope-restricted (or Lipschitz-bounded) nature of the feedback-connected nonlinear operators. The results are extended to address performance for discrete-time systems. Another contribution in this thesis is to design controllers for a certain class of nonlinear systems such that closed-loop stability and robustness is guaranteed. The discrete-time domain was the main focus, because discrete-time systems are more naturally constructed from process input-output data. The global stability tools described and developed here are based on Lyapunov stability analysis. Lyapunov methods provide a simple but powerful way to analyze nonlinear systems and to synthesize stabilizing controllers with optimal performance.

1.1 Nonlinear Systems Analysis

In this section, consider a system of ordinary differential (or difference) equations of the general form

$$\frac{d}{dt}x(t) = f(t, x(t)) \quad (\text{or} \;\; x(k+1) = f(k, x(k))), \tag{1.1}$$

where f is assumed to be a continuous vector field so that, for each initial condition x_0, there exists a solution $x(r)$ for $r \in \{t, k\}$ such that $x(0) = x_0$. This does not necessarily mean that the input to the system is zero in which case (1.1) is called an unforced nonlinear equation. For example, when the input signal $u(t)$ to the system is a known function of (t, x), the overall closed-loop system can be interpreted as (1.1). A special case of (1.1) is the so-called *autonomous* or *time-invariant* system given by

$$\frac{d}{dt}x(t) = f(x(t)) \quad (\text{or} \;\; x(k+1) = f(x(k))). \tag{1.2}$$

If the system (1.1) is not autonomous then it is called *non-autonomous*. For notational simplicity, we will only consider the discrete-time case in the remainder of this section; the results obtained can be trivially extended for the continuous-time case. A central concept in nonlinear systems analysis is the existence of an equilibrium point.

Definition 1. *(Equilibrium points) The system state $x = x_e$ is called an equilibrium for (1.2), if $x(k_1) = x_e$ for some $k_1 \in \mathbb{Z}$ implies that $x(k) = x_e$ for all future time $k \geq k_1$ on the maximum interval of existence of the solution for (1.2). Furthermore, x_e is an equilibrium point if and only if $f(x_e) = x_e$ for the discrete-time case and $f(x_e) = 0$ for the continuous-time case.*

A primary concern is to determine whether an equilibrium point is stable or unstable.

1.1.1 Lyapunov Stability

This section is devoted to the internal stability of nonlinear autonomous systems. The internal stability of an equilibrium point is investigated in several different behaviors of the trajectories of the solution for the system (1.2). Let define $\mathcal{X}_{x_0}$ for the set of all the solutions such that $x(0) = x_0$. In addition, assume that the origin $x \equiv 0$ is an equilibrium point without loss of generality. Note that any arbitrary equilibrium can be translated to the origin via a change of coordinates (see page 147 of [56] for details).

Definition 2. *(Definitions of stability for the equilibrium point at the origin) The equilibrium point $x \equiv 0$ for the system (1.2) is*

(i) (Lyapunov) stable if for each $\epsilon > 0$ there exists $\delta(\epsilon, t_0)$ such that

$$\|x(t)\| \leq \epsilon, \quad \forall t \geq t_0 \quad \textit{whenever} \quad \|x(t_0)\| \leq \delta(\epsilon, t_0).$$

(ii) uniformly (or Lagrange) stable if it is stable with $\delta = \delta(\epsilon)$ that is independent of the initial time t_0.

(iii) asymptotically stable (AS) if it is stable and there exists a constant $\delta(t_0)$ such that

$$\lim_{t \to \infty} x(t) = 0 \quad \textit{whenever} \quad \|x(t_0)\| < \delta(t_0).$$

In other words, for every $\epsilon > 0$ there exists $T(\epsilon) > 0$ and $\delta(t_0) > 0$ such that

$$\|x(t)\| \leq \epsilon, \quad \forall t \geq t_0 + T(\epsilon) \quad \textit{whenever} \quad \|x(t_0)\| < \delta(t_0).$$

(iv) uniformly asymptotically stable (UAS) if it is uniformly stable and asymptotically stable with δ such that δ is independent of t_0 and the convergence of the solution $x(t)$ to the origin is uniform in t_0. In other words, for every $\epsilon > 0$ there exists $T(\epsilon) > 0$ and $\delta > 0$ such that

$$\|x(t)\| \leq \epsilon, \quad \forall t \geq t_0 + T(\epsilon) \quad \textit{whenever} \quad \|x(t_0)\| < \delta.$$

(v) exponentially stable (ES) if there exist some positive numbers β and λ such that for sufficiently small $x_0 = x(t_0)$

$$\|x(t)\| \leq \beta \|x_0\| e^{-\lambda(t-t_0)} \quad \forall t \geq t_0.$$

(vi) globally uniformly asymptotically stable (GUAS) if it is uniformly asymptotically stable and ϵ and γ can be arbitrary large values.

(vii) globally exponentially stable (GES) it is exponentially stable for arbitrary initial condition x_0.

It is easy to see that there exists no nonuniform version of the definitions for exponential stability. Noting that for stability analysis of autonomous systems the Lyapunov stability theorem [115, 13, 56] is the most popular and powerful approach, we will give a brief introduction for the so-called Lyapunov direct method, which is also called the second method of Lyapunov.

The Second Method of Lyapunov

Let us consider the linear time invariant (LTI) system with a memoryless, nonlinear uncertainty

$$\begin{aligned} x(k+1) &= Ax(k) + B_p p(k) + B_w w(k) + B_u u(k) \\ q(k) &= C_q x(k) + D_{qp} p(k) + D_{qw} w(k) + D_{qu} u(k) \\ z(k) &= C_z x(k) + D_{zp} p(k) + D_{zw} w(k) + D_{zu} u(k) \\ p(k) &= \phi(q(k)) \end{aligned} \tag{1.3}$$

where p, w, and u indicate the feedback input from the (unknown) nonlinear mapping $\phi(\cdot)$, the disturbances, and the control input to the LTI system, respectively. To investigate the internal stability of the system in Figure 1.1, let us assume that $u = w = z = 0$ such that

$$G(s) = C_q(sI - A)^{-1} B_p + D_{qp} =: \left[\begin{array}{c|c} A & B_p \\ \hline C_q & D_{qp} \end{array} \right]$$

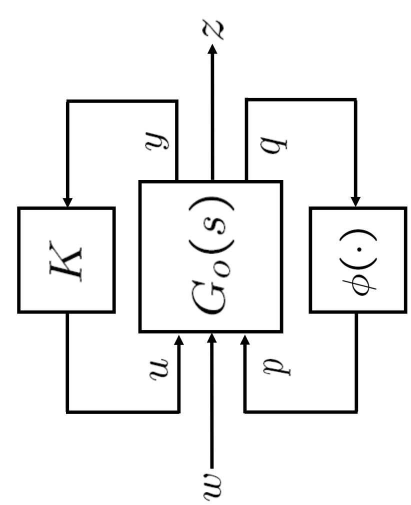


Figure 1.1: Standard Nonlinear Feedback System

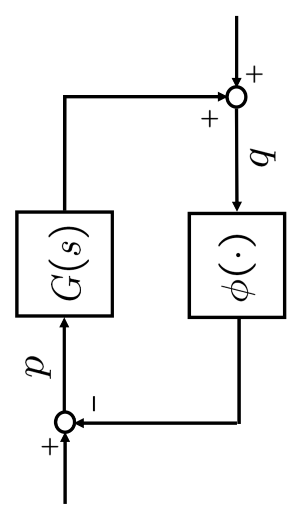


Figure 1.2: Reduced Standard Nonlinear Feedback System

is a stable and proper rational transfer function, i.e., A is Hurwitz and D_{qp} is a constant matrix. The reduced block diagram is in Figure 1.2. The nonlinear function described in (1.3) is assumed to belong to the family of nonlinear functions denoted by Φ. The classifications for some families of nonlinear function will be given in Section 1.1.3. The development of a mathematical framework to analyze and synthesize generic nonlinear systems should fulfill the following conditions:

i Large applicability.

ii Simplicity.

iii Computational feasibility.

iv Unification of theoretical results.

The so-called *second method of Lyapunov* leads to conclusions regarding the stability of well-defined families of systems. This method made it natural to introduce the notion of *asymptotic stability.* Its origin can be traced to Lyapunov's book that was first published in 1892, but received little attention outside Russia until the late 1950s [65]. Lyapunov's method is a general approach used to verify the stability of nonlinear systems without solving any differential equations. In this method, a "bowl-shaped" energy function $V(x)$ is defined, where $V : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives such that $V(x) > 0$ and $V(x_{eq}) = 0$ for $x_{eq} = 0$, and x_{eq} is the equilibrium point. Such functions are known as *Lyapunov functions* and in some sense they can be considered to define the total "energy" of the system. If the system is perturbed to a new energy level (or initial state) then, the trajectories of a solution of the system show some distinct nature of the Lyapunov function and its time-difference (or time-derivative for continuous-time systems). Here, we give a description only for Lyapunov function properties of a discrete-time system to determine whether the origin is an asymptotically stable equilibrium point and recommend the reader to see the nonlinear system books [115, 13, 56] for more details. Theorem 1 provides conditions for the Lyapunov stability of the discrete-time system. The disadvantage of the second method of Lyapunov is that it provides only sufficient conditions (except in specialized cases such as the linear time invariant systems, where its conditions are necessary and sufficient [20]). If the method fails to prove stability no definite conclusion may be reached, which is the weakest aspect of the Lyapunov method for nonlinear system stability analysis. A very thorough discussion of Lyapunov's direct method can be found in [61]. The following theorem formalizes the discussion on the second method of Lyapunov for the discrete-time case.

Theorem 1 (Discrete Time Lyapunov GAS [20])**.** *A system of the form*

$$x(k+1) = f(x(k))$$

has a globally asymptotically stable equilibrium point at the origin if:

1. $V(x_{eq} = 0) = 0$ *and* $V(x(k)) > 0$ *for all* $x(k) \neq 0$,

2. $\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0$ *for all* $x(k) \neq 0$, *and*

3. $V(x(k)) \to \infty$ *as* $\|x(k)\| \to \infty$, *i.e.,* $V(\cdot)$ *is radially unbounded.*

The most commonly used Lyapunov function is quadratic:

$$V(x) = x^T P x \tag{1.4}$$

where P is a positive-definite matrix. Investigations into global asymptotic stability using quadratic Lyapunov functions can provide conservative results, which in some cases may be unsatisfactory. Because of this reason, many researchers have looked for alternative forms of the Lyapunov function [69, 77, 63, 86, 121]. Construction of suitable Lyapunov functions is something of an art, however a number of standard functions have been studied extensively, for example, quadratic forms, quadratic plus integral of the feedback-connected nonlinear functions, and parameter-dependent Lyapunov functions (PDLFs). Researchers have focussed on developing Lyapunov functions that are both flexible in addressing general nonlinear systems and restrictive in excluding nonlinearities that are not meaningful. The key idea used in many researches is to introduce new variables and enlarge the dimension of the linear matrix inequalities (LMIs) to obtain sufficient conditions for the existence of a PDLF.

Comment 1. *(Existence and necessity of a PDLF in the stability analysis in robust and nonlinear system theory) When a nominal LTI plant has a feedback connection with uncertain and/or nonlinear mappings whose input-output relation is the only information available, the quadratic form for the Lyapunov functions can be hugely conservative in the analysis and synthesis of the closed-loop system. It is natural to have an intuition that some known or estimated properties, if any, of the uncertain and/or nonlinear parts should be considered in the analysis and synthesis problems. One of the primary purposes of this thesis is to reduce potential conservatism by using knowledge about the system as much as possible.*

For a single-input and single-output (SISO) system given by

$$x(k+1) = Ax(k) - b\phi(q(k)), \qquad q(k) = c^T x(k), \tag{1.5}$$

which is a special case of (1.3), one such way is to use a term involving an integral of the nonlinearity [33, 68, 64], which is well-known and for which there is a large literature:

$$V(x(k)) = x^T(k)Px(k) + \lambda \int_0^q \phi(\sigma)d\sigma \tag{1.6}$$

where P is a positive-definite symmetric matrix and $\lambda \geq 0$. Equation (1.6) is known as Lur'e-Postnikov equation [97, 118, 69, 86] and the existence of such a Lypunov function is a key question in absolute stability theory described in Section 1.1.3. It is a well-known fact that Lur'e-Postnikov equations provide less conservative results than quadratic Lyapunov functions for a certain class of nonlinear functions Φ, whose output enters an LTI plant. By restricting the input-output properties of the feedback-connected nonlinear functions to satisfy the local slope restriction or the Lipschitz bound condition given by

$$0 < \frac{\phi(q_{k+1}) - \phi(q_k)}{q_{k+1} - q_k} < \mu, \quad \forall k \in \mathbb{Z}_+ \tag{1.7}$$

where $\mu \in \mathbb{R}$ represents the maximum local slope for the nonlinearity, less conservative analysis and synthesis results can be obtained. A continuous-time version of a slope-restricted Lyapunov function and a description of the kind of Lyapunov functions to be used for different admissible nonlinearities can be found in [33] and [42].

1.1.2 Passivity and Small-Gain Theorems

A. The Passivity Theorem and The Positive Real Lemma

The definitions for input-output stability and passive operators follow the standard nomenclature [53, 28, 115, 56].

Definition 3. *(Input-output stability) A causal operator $H : L_2 \to L_2$ is said to be L_2 stable if $Hx \in L_2$ for all $x \in L_2$. More specifically, if there exist $\gamma \geq 0$ and γ_0 such that*

$$\|Hx\|_{L_2} \leq \gamma \|x\|_{L_2} + \gamma_0 \quad \forall x \in L_2$$

then the operator H is said to be finite-gain L_2 stable.

Definition 4. *(Passivity) A causal operator $H : L_2 \to L_2$ is said to be passive if there exist $\gamma \geq 0$ and some γ_0 such that*

$$\langle x_T | (Hx)_T \rangle \geq \gamma \|x_T\|^2_{L_2} + \gamma_0 \quad \forall T \in \mathbb{R}_+, \forall x \in L_{2e},$$

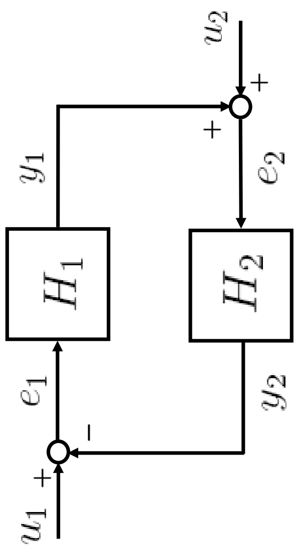


Figure 1.3: Passivity Theorem

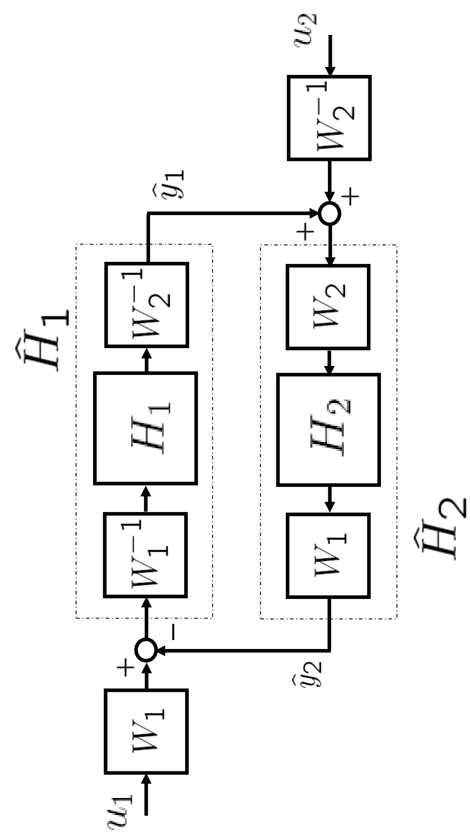


Figure 1.4: Passive Multiplier

where the subscript T and an inner product $\langle \cdot, \cdot \rangle$ are defined as

$$x_T(t) \triangleq \begin{cases} x(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases} \quad \text{and} \quad \langle x|y \rangle \triangleq \int_{-\infty}^{+\infty} x^T(t)y(t)dt.$$

If $\gamma > 0$, then the operator H is said to be strictly passive.

The connection between passivity and stability of a closed-loop system which is composed of successive feedback interconnections of operators is discussed in many classical control and networks theory books and papers [7, 6, 53, 115, 28, 56]. The fundamental approach has been to investigate if a network consisting of successively connected passive subsystems is necessarily stable. The following so-called passivity theorem answers this question.

Theorem 2. *(Passivity theorem) Consider the feedback system shown in Figure 1.3, where the negative feedback-connected operators $H_1, H_2 : L_{2e} \to L_{2e}$ are any general causal operators. Assume that for any $u_1, u_2 \in L_2$, there exist solutions $e_1, e_2, y_1, y_2 \in L_{2e}$. That is, the system in Figure 1.3 is well-posed. Suppose that there exist scalar constants $\gamma, \gamma_1, \gamma_2, \gamma_0, \gamma_{10}, \gamma_{20}$ such that*

$$\begin{aligned} \|(Hx)_T\|_{L_2} &\leq \gamma \|x_T\|_{L_2} + \gamma_0 \\ \langle x_T|(H_1x)_T \rangle &\geq \gamma_1 \|x_T\|^2_{L_2} + \gamma_{10} \qquad (1.8) \\ \langle x_T|(H_2x)_T \rangle &\geq \gamma_2 \|x_T\|^2_{L_2} + \gamma_{20} \end{aligned}$$

$$(1.9)$$

for all $T \in \mathbb{R}_+$ and $x \in L_{2e}$. Then, $u_1, u_2 \in L_2$ implies that $e_1, e_2, y_1, y_2 \in L_2$ provided $\gamma_1 + \gamma_2 > 0$. Furthermore, the mapping from the inputs (u_1, u_2) to the internal states (e_1, e_2, y_1, y_2) is finite-gain L_2 stable whenever there exists the constants $\gamma_0, \gamma_{10}, \gamma_{20}$ whose values are all zeros such that the conditions in (1.8).

Proof: There are many good monographs that review the passivity theorem and related results. For example, see [53, 115, 28, 56]. □

Lemma 1. *(Successive feedback connection of passive systems) The negative feedback connection of two passive systems is passive.*

Proof: The proof of the lemma is nothing but an application of the result in Theorem 2. See [53, 115, 28, 56] for the details. □

Remark 1. *(Passive multiplier [53, 115, 28, 56]) It is well known that the system represented by the interconnection of H_1 and H_2 in Figure 1.3 is stable if and only if the system represented by the interconnection of $\hat{H}_1$ and $\hat{H}_2$ in Figure 1.4 is stable, provided that the multipliers W_1 and W_2 in Figure 1.4 are rational, proper, and stable transfer functions, or constant gains.*

The positive real lemma

The positive real lemma is a fundamental result concerning stability of the Lur'e problem as well as of uncertain systems. It is also known as the *Kalman-Yakubovich lemma* [124, 127, 125, 126, 115, 92] due to the fact that Kalman and Yakubovich independently discovered it in the early 1960s. It provides an algebraic criterion for the transfer function of a continuous-time system to be positive real. This criterion states conditions for a quadratic Lyapunov function to guarantee the stability of the Lur'e problem with a single memoryless nonlinearity. These ideas were extended to the multivariable case by Anderson [6, 7] and were later written in terms of an algebraic Ricatti equation [100, 34]. The Kalman-Yakubovich lemma and the equivalent algebraic Ricatti equation are described Section **??**.

I. The continuous time positive real lemma

Before stating the positive real lemma [33, 6, 120, 67], consider following definitions and results.

Definition 5. *Consider the following definitions:*

1. *A square transfer function $G(s)$ is* positive real *if all poles of $G(s)$ are in the closed left-half plane, and $G(s) + G^*(s)$ is positive semi-definite for all $\mathrm{Re}(s) > 0$, where $G^*(s)$ is the conjugate transfer function of $G(s)$.*

2. *A square transfer function $G(s)$ is* strictly positive real *if all poles of $G(s)$ are in the open left-half plane (asymptotically stable), and $G(j\omega) + G^*(j\omega)$ is positive definite for all $\omega \in \mathbb{R}$.*

3. *A square transfer function $G(s)$ is* strongly positive real *if $G(s)$ is strictly positive real, and $D + D^T > 0$ where $D = G(\infty)$.*

Lemma 2. *([115, 56]) Let $G(s) = C_q(sI - A)^{-1}B_p + D_{qp}$ be a transfer function matrix of compatible dimension and (A, B_p, C_q) is controllable and observable. Then, $G(s)$ is strongly positive real if and only if there exist matrices $P = P^T > 0$, L, and W with full row rank and a positive constant ϵ such that*

$$PA + A^T P = -L^T L - \epsilon P, \quad PB_p = C_q^T - L^T W, \quad \text{and} \quad W^T W = D_{qp} + D_{qp}^T.$$

Consider a proper and strongly positive real transfer function in a negative feedback configuration with a time-varying memoryless nonlinearity where the set of nonlinear functions is given by

$$\bar{\Phi}_{pr} = \bar{\Phi}_{sb,tv}^{[0,\infty]} \triangleq \left\{\phi(\sigma(t),t) : \mathbb{R}^{n_q} \times \mathbb{R}_+ \to \mathbb{R}^{n_p} : \phi^T(\sigma(t),t)\sigma(t) \geq 0, \sigma(t) \in \mathbb{R}^{n_q}, \forall t \geq 0\right\} \tag{1.10}$$

and $\phi(\sigma(t),t)$ is Lebesgue measurable for all $(\sigma(t),t) \in \mathbb{R}^{n_q} \times \mathbb{R}_+$. A sufficient condition for the stability of a system (1.3) where $\phi \in \bar{\Phi}_{pr}$ is given by the following theorem.

Theorem 3. *Consider the transfer function $G(s)$ that corresponds to the minimal realization of a nonlinear system of the form described in (1.3). Then the following statements are equivalent:*

1. *A is Hurwitz and $G(s)$ is strongly positive real;*

2. *$D_{qp} + D_{qp}^T > 0$ and there exist positive-definite matrix P such that:*

$$\begin{bmatrix} A^TP + PA & PB_p - C_q^T \\ B_p^TP - C_q & -D_{qp} - D_{qp}^T \end{bmatrix} < 0 \tag{1.11}$$

Furthermore, the negative feedback interconnection of $G(s)$ is asymptotically stable for all $\phi \in \bar{\Phi}_{pr}$.

Proof: The proof using the KYP lemma described in the later part of this section has been described many times in the literature [33, 118, 37, 56]. To prove global asymptotic stability, consider the system realization (A, B_p, C_q, D_{qp}) and consider the passivity of the nonlinear function ϕ. Define the Lyapunov function:

$$V(x) = x^TPx + 2\int_0^t \phi^T(q(\tau),\tau)q(\tau)d\tau \geq 0, \quad \forall t \in \mathbb{R}_+. \tag{1.12}$$

Note that $V(0) = 0$, since $\phi(0) = 0$ which implies that the nonlinear function is zero-observable [56], and that $V(x) > 0, \forall t$ since $\phi \in \bar{\Phi}_{pr}$. Thus,

$$\dot{V} = \begin{bmatrix} x \\ \phi \end{bmatrix}^T \begin{bmatrix} A^TP + PA & -PB_p + C_q^T \\ -B_p^TP + C_q & -D_{qp} - D_{qp}^T \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} \tag{1.13}$$

Historically, in 1971, Willems related the LMI in the positive real lemma for a strictly proper transfer function to an algebraic Riccati equation. Later in 1993, Haddad and Bernstein extended this result for proper transfer functions [33, 100, 123]. The positive real lemma is considered a generating theorem since it can be used to derive other important stability criterions.

II. The Discrete-Time Positive Real Lemma

An analogous result to the continuous-time positive real lemma was derived by Szego and Kalman [109] for SISO discrete-time systems, and was also extended to the multivariable case by Hitz and Anderson [39] who defined the notion of discrete positive real and obtained an algebraic stability criterion. The discrete-time positive real lemma has also been written as algebraic Ricatti equations [36, 118]. Before stating the discrete-time positive real lemma, the following definition describing positive real transfer functions [36, 39, 118] is relevant.

Definition 6. *Consider the following definitions:*

1. *A square transfer function $G(z)$ is* discrete positive real *if all poles of $G(z)$ are in the closed unit disk, and $G(z) + G^*(z)$ is nonnegative definite for $|z| < 1$, where $G^*(z)$ is the conjugate transfer function of $G(z)$.*

2. *A square transfer function $G(z)$ is* strictly discrete positive real *if all poles of $G(z)$ are asymptotically stable, and $G(e^{j\omega}) + G^*(e^{j\omega})$ is positive definite for all $\omega \in [0, 2\pi]$.*

3. *A square transfer function $G(s)$ is* strongly discrete positive real *if $G(z)$ is strictly discrete positive real, and $D + D^T > 0$ where $D = G(\infty)$.*

Lemma 3. *([35, 36]) Let $G(z) = C_q(zI - A)^{-1}B_p + D_{qp}$ be a a transfer function matrix of compatible dimension and (A, B_p, C_q) is controllable and observable. Then, $G(s)$ is strongly positive real if and only if there exist matrices $P = P^T > 0$, L, and W which has full row rank, and a positive constant ϵ such that*

$$A^T PA - P = -L^T L - \epsilon P, \quad A^T PB_p = C_q^T - L^T W, \quad \text{and} \quad B_p^T PB_p + W^T W = D_{qp} + D_{qp}^T;$$

Consider a proper strongly discrete positive real transfer function in a negative feedback configuration with a time-varying memoryless nonlinearity where $\phi \in \bar{\Phi}_{pr}$ which is the discrete time version of (1.10) A sufficient condition for a system of the form shown in (1.3) in terms of an LMI is given by the following theorem.

Theorem 4. *Consider the transfer function $G(z)$ which corresponds to the minimal realization of a nonlinear system of the form described in (1.3). Then the following statements are equivalent:*

1. *A is asymptotically stable and $G(z)$ is strongly positive-real.*

2. *There exists a positive-definite matrix M such that*

$$\begin{bmatrix} A^T PA - P & A^T PB_p - C^T \\ B_p^T PA - C & B_p^T PB_p - D_{qp} - D_{qp}^T \end{bmatrix} < 0. \tag{1.14}$$

Proof Same as [118] and applying Schur complement. □

The discrete-time version of the Kalman-Yakubovich Lemma and how it is written in terms of an algebraic Ricatti equation are also described in Section **??**. The LTI system whose transfer function is (strictly) positive real has the passivity property.

Remark 2. *(Passivity and positive realness [53, 115, 28, 56, 29]) The transfer function $G(s) = C(sI - A)^{-1}B + D$ is (strictly) passive if $G(s)$ is (strictly) positive real.*

B. The Small-Gain Theorem and The Bounded Real Lemma

The bounded real lemma is basically a result of the small-gain theorem. The lemma below reviews the statements of the small-gain theorem and shows that the bounded real lemma is a special case.

Lemma 4. *(The small-gain theorem and its stronger form [8]) Let us consider the system in Figure 1.3 and suppose that the causal operators $H_1, H_2 : L_{2e} \to L_{2e}$ satisfy the following conditions:*

i. H_1 is bounded and H_2 is Lipschitz continuous.

ii. $\|H_1 H_2\|_{L_{2e}} < 1$.

Then, the closed-loop system is stable. Moreover, noting the condition on H_2 is stronger than that on H_1 and the condition $\|H_1\|_{L_{2e}}\|H_2\|_{L_{2e}} < 1$ is stronger than $\|H_1 H_2\|_{L_{2e}} < 1$, we can conclude that the condition $\|H_1\|_{L_{2e}}\|H_2\|_{L_{2e}} < 1$ can replace the conditions i. and ii. as a sufficient condition for the stability of the closed-loop system.

I. The Continuous-Time Bounded Real Lemma

The bounded real lemma is also a version of the KYP lemma and gives a tool to transform H_∞ norm condition into an LMI–this change of the condition to achieve a performance is very helpful for attaining the optimal controller design.

Theorem 5. *Suppose that the system transfer function $G(s)$ has the state space realization (A, B_p, C_q, D_{qp}). Then the following statements are equivalent:*

1. *All poles of $G(s)$ are in the open left-half plane, and $\gamma^2 I - G^*(s)G(s)$ is positive semi-definite for $\mathrm{Re}(s) > 0$, where $G^*(s)$ is the conjugate transfer function of $G(s)$.*

2. *The matrix A is Hurwitz and*

$$\|G(s)\|_\infty < \gamma.$$

3. *There exists a positive definite matrix $P = P^T > 0$ such that*

$$\frac{1}{\gamma^2}\begin{bmatrix} C_q^T \\ D_{qp}^T \end{bmatrix}\begin{bmatrix} C_q & D_{qp} \end{bmatrix} + \begin{bmatrix} A^T P + PA & PB_p \\ B_p^T P & -I \end{bmatrix} < 0.$$

II. The Discrete-Time Bounded Real Lemma

Theorem 6. *Suppose that the system transfer function $G(z)$ has the state space realization (A, B_p, C_q, D_{qp}). Then the following statements are equivalent:*

1. *All poles of $G(z)$ are in the open unit disk, and $\gamma^2 I - G(z)G^*(z)$ is positive semi-definite for $|z| < 1$, where $G^*(z)$ is the conjugate transfer function of $G(z)$.*

2. *The matrix A is Schur and*

$$\|G(z)\|_\infty < \gamma.$$

3. *There exists a positive definite matrix $P = P^T > 0$ such that*

$$\frac{1}{\gamma^2}\begin{bmatrix} C_q^T \\ D_{qp}^T \end{bmatrix}\begin{bmatrix} C_q & D_{qp} \end{bmatrix} + \begin{bmatrix} A^T PA - P & A^T PB_p \\ B_p^T PA & B_p^T PB_p - I \end{bmatrix} < 0.$$

The bounded real lemma previously described is nothing but an application of the results in the small gain theorem and the KYP lemma. So the proofs for Theorem 5 and 6 are omitted to avoid redundancy.

C. The Kalman-Yakubovich-Popov (KYP) Lemma

There are many variations of the KYP lemma, which establishes the equivalence between a frequency domain inequality (FDI) and a conditions given in a Riccati equation or an LMI with respect to the state space realization of the system. In this section, the primary results are from a pioneer paper [123] in frequency domain inequalities and linear matrix inequalities and a recently written paper [96] whose results are parallel to [123]. The description about the KYP lemma and its variations presented in [123, 96] can be considered a general description of the connections between FDIs and LMIs. Here we give a brief introduction for the KYP lemma and its variations. For the details and proofs, readers are recommended to see [123, 96].

Theorem 7. *(The KYP lemma for the continuous-time system) Given the system realization* (A, B_p) *and a matrix* M_q *which characterize the properties of the system, of the compatible dimensions, if* $det(j\omega I - A) \neq 0$ *for all* $\omega \in \mathbb{R}$ *and* (A, B_p) *is controllable then the following statements are equivalent:*

1.
$$\begin{bmatrix} (j\omega I - A)^{-1}B_p \\ I \end{bmatrix}^* M_q \begin{bmatrix} (j\omega I - A)^{-1}B_p \\ I \end{bmatrix} \leq 0 \quad \forall \omega \in \mathbb{R}; \tag{1.15}$$

2. *there exists a matrix* $P = P^T$ *of the compatible dimension such that*
$$M_q + \begin{bmatrix} A^T P + PA & PB_p \\ B_p^T P & 0 \end{bmatrix} \leq 0. \tag{1.16}$$

The equivalence of the corresponding to strict inequalities holds without assuming that (A, B_p) *is controllable.*

Theorem 8. *(The KYP lemma for the discrete-time system) Given the system realization* (A, B_p) *and a matrix* M_q *which characterize the properties of the system, of the compatible dimensions, if* $det(e^{j\omega} I - A) \neq 0$ *for all* $\omega \in \mathbb{R}$ *and* (A, B_p) *is controllable then the following statements are equivalent:*

1.
$$\begin{bmatrix} (e^{j\omega} I - A)^{-1}B_p \\ I \end{bmatrix}^* M_q \begin{bmatrix} (e^{j\omega} I - A)^{-1}B_p \\ I \end{bmatrix} \leq 0 \quad \forall \omega \in \mathbb{R}; \tag{1.17}$$

2. *there exists a matrix* $P = P^T$ *of the compatible dimension such that*
$$M_q + \begin{bmatrix} A^T PA - P & A^T PB_p \\ B_p^T PA & B_p^T PB_p \end{bmatrix} \leq 0. \tag{1.18}$$

The equivalence of the corresponding to strict inequalities holds without assuming that (A, B_p) *is controllable.*

Now, the following statements show that the bounded real lemma and the positive real lemma are special cases of descriptions for the KYP lemma.

Remark 3. *(The bounded real lemma and the positive real lemma) The quadratic inequality (1) and its equivalent LMI (2) recover the bounded and positive real lemma:*

1. *The values of*

$$M_q = \begin{bmatrix} C_q^T \\ D_{qp}^T \end{bmatrix} \begin{bmatrix} C_q & D_{qp} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$$

gives the bounded real lemma for a FDI and an LMI;

2. *The values of*

$$M_q = \begin{bmatrix} C_q^T \\ D_{qp}^T \end{bmatrix} \begin{bmatrix} 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ -I \end{bmatrix} \begin{bmatrix} C_q & D_{qp} \end{bmatrix}$$

gives the positive real lemma for a FDI and an LMI.

1.1.3 Absolute Stability–Lur'e Problems

A problem on the stability analysis of a class of nonlinear systems was formulated by Lur'e and Postnikov in 1944 [69]. This benchmark problem is known as the Lur'e problem and the primary purpose is to ask if the zero equilibrium point $x \equiv 0$ of the continuous-time system given by

$$\frac{d}{dt}x(t) = Ax(t) + b\phi(q(t)), \quad q(t) = c^T x(t) \tag{1.19}$$

is GUAS for any the nonlinear function ϕ satisfying the so-called sector condition

$$0 \leq \sigma\phi(\sigma) \leq \xi\sigma^2, \tag{1.20}$$

where ξ corresponds to the maximum upper bound on the sector with which the input-output relations of the nonlinear function are characterized. In addition, the nonlinear function ϕ is required to provide the existence and uniqueness of the solution of (1.19), which is guaranteed if the origin $x = 0$ is the unique solution for $\phi(c^T x) = 0$ or $\phi(c^T x) = b^\perp$. After the Lur'e problem of absolute stability had been addressed, two famous conjectures were suggested and were disproved by counterexamples:

i. *Aizerman conjecture (1949)*: When the system given by (1.19) is stable for all linear function in the sector $[0, \xi]$ is stable, does it immediately follow that the system with the same system realization and the nonlinear function which is characterized by (1.20) is also stable?

ii. *Kalman conjecture (1957)*: If the nonlinear function characterized by (1.20) is differentiable almost everywhere and its derivative ϕ' is in the sector $[\epsilon_1, \xi - \epsilon_2]$ for some small positive numbers ϵ_1, ϵ_2, then does it immediately follow that the system with the same system realization and the nonlinear function ϕ?

Those two hypotheses have been disproved by some counter examples of higher-order systems. In the wake of Lur'e problem and successive interesting hypotheses, the well known Circle and Popov criteria was used in the absolute stability theory and their geometrical properties in the diagrams of frequency domain gave straightforward tools for the stability analysis in the Lur'e system with a scalar-valued nonlinear function ϕ. More through review for these frequency domain criteria will be given in Section 1.2. Through this thesis, we will consider some certain classes of nonlinear functions that are characterized by their input-output relations.

Through this thesis, some certain classes of nonlinear functions and their interconnections with a nominal linear time-invariant (LTI) plant will be considered.

Definition 7. *(Definitions for some certain classes of nonlinear functions) Let us consider some classes of nonlinear mappings whose properties are characterized by the so-called sector bound and/or Lipschitz bound:*

$$
\begin{aligned}
\Phi_{sb}^{[0,\xi]} &\triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q} \mid \phi_i(\sigma)\left[\xi_i^{-1}\phi_i(\sigma) - \sigma\right] \le 0,\ \forall \sigma \in \mathbb{R} \text{ and for } i = 1, \ldots, n_q\}; \\
\Phi_{sb}^{|\xi|} &\triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q} \mid \left[\xi_i^{-1}\phi_i(\sigma) + \sigma\right]\left[\xi_i^{-1}\phi_i(\sigma) - \sigma\right] \le 0,\ \forall \sigma \in \mathbb{R} \text{ and for } i = 1, \ldots, n_q\}; \\
\bar{\Phi}_{sb}^{\xi} &\triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p} \mid \phi^T(\sigma)\phi(\sigma) \le \xi^2\sigma^T\sigma,\ \forall \sigma \in \mathbb{R}^{n_q}\}; \\
\bar{\Phi}_{pr} &\triangleq \left\{\phi(\sigma(t), t) : \mathbb{R}^{n_q} \times \mathbb{R}_+ \to \mathbb{R}^{n_p} : \phi^T(\sigma(t), t)\sigma(t) \ge 0, \sigma(t) \in \mathbb{R}^{n_q}, \forall t \ge 0\right\} = \bar{\Phi}_{sb,tv}^{[0,\infty]}; \\
\Phi_{sr}^{[0,\mu]} &\triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q} \mid 0 \le \frac{\phi_i(\sigma) - \phi_i(\hat{\sigma})}{\sigma - \hat{\sigma}} \le \mu_i,\ \forall \sigma \ne \hat{\sigma} \in \mathbb{R} \text{ and for } i = 1, \ldots, n_q\}; \\
\Phi_{sr}^{|\mu|} &\triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_q} \mid -\mu_i \le \frac{\phi_i(\sigma) - \phi_i(\hat{\sigma})}{\sigma - \hat{\sigma}} \le \mu_i,\ \forall \sigma \ne \hat{\sigma} \in \mathbb{R} \text{ and for } i = 1, \ldots, n_q\}; \\
\bar{\Phi}_{sr}^{\mu} &\triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p} \mid \|\phi(\sigma) - \phi(\hat{\sigma})\|_2^2 \le \mu^2\|\sigma - \hat{\sigma}\|_2^2,\ \forall \sigma \ne \hat{\sigma} \in \mathbb{R}_{n_q}\}; \\
\Phi_{odd} &\triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p} \mid \phi_i(\sigma) = -\phi_i(-\sigma),\ \forall \sigma \in \mathbb{R} \text{ and for } i = 1, \ldots, n_q\}.
\end{aligned}
$$

Note that $\Phi_{sb}^{[0,\xi]} \subset \Phi_{sb}^{|\xi|}$ *and* $\Phi_{sr}^{[0,\mu]} \subset \Phi_{sr}^{|\mu|}$. *Further, if* $\xi_i = \xi$ *and* $\mu_i = \mu$ *for all indices* i, *and* $n_q = n_p$, *then* $\Phi_{sb}^{|\xi|} \subset \bar{\Phi}_{sb}^{\xi}$ *and* $\Phi_{sr}^{|\mu|} \subset \bar{\Phi}_{sr}^{\mu}$, *respectively. The converses, however, are not true.*

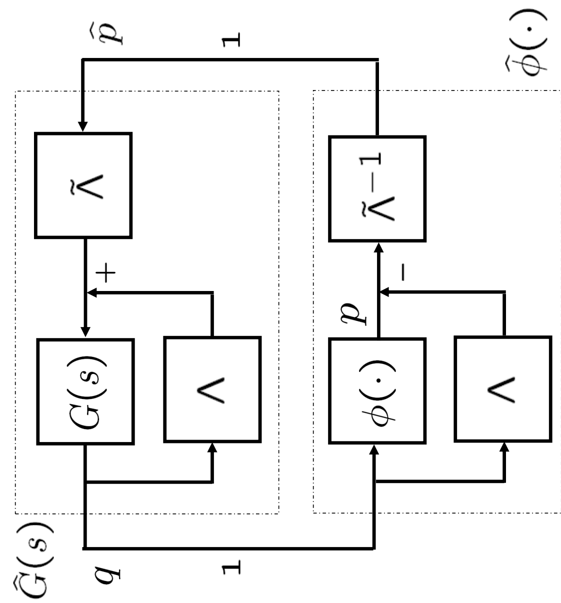


Figure 1.5: Loop Transformations for Sector Conditions

1.1.4 Loop Transformations

Loop Transformations for Sector Conditions

This section shows that any forms of sector-bounded nonlinear functions can be transformed into other forms of sector-bounded nonlinear functions via the so-called loop transformations with constant gains [115, 100, 56]. The purpose is to derive another feedback connected system $\bar{S}$ from the given feedback system S such that the new system realization $\bar{S}$ is stable if and only if the original system representation S is stable.

Example 1: Consider the Lur'e system with general sector conditions:

$$S: \quad \dot{x} = Ax + Bp, \qquad q = Cx + Dp, \tag{1.21}$$

$$p_i(t) = \phi_i(q_i(t)), \qquad \alpha_i\sigma^2 \leq \sigma\phi_i(\sigma) \leq \beta_i\sigma^2. \tag{1.22}$$

Here we have dropped the variables w and z and the subscripts on B, C and D to simplify the presentation. Assume that this system is well-posed, i.e., $\det(I - D\Delta)$ is nonsingular for all diagonal $\Delta = \mathrm{diag}(\Delta_{ii})$ with

$\alpha_i \le \Delta_{ii} \le \beta_i$. Note that Δ might be time-dependent. Define

$$\bar{p}_i \triangleq \frac{2}{\beta_i - \alpha_i}(p_i - \alpha_i q_i) - q_i. \tag{1.23}$$

Then it is readily shown that $|\bar{p}_i(t)| \le |q_i(t)|$ for all t. Define the diagonal matrices:

$$\Lambda \triangleq \text{diag}(\frac{\alpha_i + \beta_i}{2}), \qquad \tilde{\Lambda} \triangleq \text{diag}(\frac{\beta_i - \alpha_i}{2}),$$

so that $p = \tilde{\Lambda}\bar{p} + \Lambda q$. Using our well-posed assumption and change of variables in which $\bar{p}$ is substituted into p, this results in

$$\dot{x} = [A + B\Lambda(I - D\Lambda)^{-1}C]x + [B\Lambda(I - D\Lambda)^{-1}D\tilde{\Lambda} + B\tilde{\Lambda}]\bar{p}, \tag{1.24}$$

$$q = (I - D\Lambda)^{-1}Cx + (I - D\Lambda)^{-1}D\tilde{\Lambda}\bar{p}. \tag{1.25}$$

Hence the Lur'e system can be expressed with general sector conditions as a system with diagonal norm-bounded uncertainties:

$$\bar{S}: \quad \dot{x} = \bar{A}x + \bar{B}\bar{p}, \qquad q = \bar{C}x + \bar{D}\bar{p}, \tag{1.26}$$

$$\bar{p}_i(t) = \bar{\phi}_i(q_i(t)), \qquad |\bar{\phi}_i(\sigma)| \le |\sigma|. \tag{1.27}$$

where

$$\bar{A} = A + B\Lambda(I - D\Lambda)^{-1}C, \qquad \bar{B} = B\Lambda(I - D\Lambda)^{-1}D\tilde{\Lambda} + B\tilde{\Lambda}, \tag{1.28}$$

$$\bar{C} = (I - D\Lambda)^{-1}C, \qquad \bar{D} = (I - D\Lambda)^{-1}D\tilde{\Lambda}. \tag{1.29}$$

Example 2: Consider the discrete-time Lur'e system with general sector conditions:

$$S: \quad x(k+1) = Ax(k) + Bp(k), \qquad q(k) = Cx(k), \tag{1.30}$$

$$p_i(k) = -\phi_i(q_i(k)), \qquad 0 \le \sigma\phi_i(\sigma) \le \mu_i\sigma^2. \tag{1.31}$$

Here the variables w and z and the subscripts on B and C are dropped to simplify the presentation and it is assumed that the system is strictly proper, i.e., $D \equiv 0$. Define a new variable:

$$\bar{\phi}_i \triangleq \alpha_i \left(\frac{2}{\mu_i} \phi_i - q_i \right). \tag{1.32}$$

Then it is readily shown that $|\bar{\phi}_i(k)| \leq \alpha_i |q_i(t)|$ for all k. Define the diagonal matrices:

$$\Lambda \triangleq \text{diag}(-\frac{\mu_i}{2}), \qquad \tilde{\Lambda} \triangleq \text{diag}(\frac{\mu_i}{2\alpha_i}),$$

where $\alpha_i \neq 0$ for all indices i such that $p = \tilde{\Lambda}\bar{p} + \Lambda q$. Hence the Lur'e system with general sector conditions can be expressed as a system with diagonal norm-bounded uncertainties:

$$\bar{S}: \quad x(k+1) = \bar{A}x(k) + \bar{B}\bar{p}(k), \qquad q(k) = Cx(k), \tag{1.33}$$

$$\bar{p}_i(k) = -\bar{\phi}_i(q_i(k)), \qquad |\bar{\phi}_i(\sigma)| \leq \alpha_i |\sigma|, \quad i = 1, \cdots, n_p. \tag{1.34}$$

with

$$\bar{A} = A - B\tilde{\Lambda}C, \qquad \bar{B} = B\tilde{\Lambda}.$$

For the case when $\alpha_i = \alpha$ for all indices i, the transformed norm-bounded condition can be written as

$$||p(k)||_2^2 = \bar{\phi}(q(k))^T \bar{\phi}(q(k)) \leq \alpha^2 ||q(k)||_2^2 = \alpha^2 x^T(k) C^T C x(k) \quad \text{for all } k \in \mathbb{Z}_+.$$

Loop Transformation for a Class of Lipschitz-bound or Slope-restricted Nonlinear Functions

The process for loop transformations with constant gains is illustrated for a class of Lipschitz-bound or slope-restricted nonlinear functions. The purpose is to derive another feedback connected system $\bar{S}$ from the given feedback system S such that the new system realization $\bar{S}$ is stable if and only if the original system representation S is stable. Consider the discrete-time Lur'e system with the slope-restricted nonlinear function:

$$S: \; x(k+1) = Ax(k) + Bp(k), \qquad q(k) = Cx(k), \tag{1.35}$$

$$p_i(k) = -\phi_i(q_i(k)), \qquad \alpha_i \leq \frac{\phi_i(\sigma) - \phi_i(\hat{\sigma})}{\sigma - \hat{\sigma}} \leq \beta_i. \tag{1.36}$$

That is, $\phi \in \Phi_{sr}^{[\alpha,\beta]}$. Further, it is readily seen that the nonlinear function satisfies also a sector-bound condition $\phi \in \Phi_{sb}^{[\alpha,\beta]}$. Here the variables w and z and the subscripts on B and C are dropped to simplify the presentation. Assume that this system is well-posed, i.e., $\det(I - D\Delta)$ is nonsingular for a diagonal nonlinear perturbation $\Delta = \mathrm{diag}(\Delta_i)$. Note that Δ might be time-dependent. Define

$$\bar{\phi}_i(\sigma) \triangleq \frac{2\mu_i}{\beta_i - \alpha_i}(\phi_i(\sigma) - \alpha_i\sigma) - \mu_i\sigma. \tag{1.37}$$

Then it is readily shown that $-\mu_i \leq \frac{\bar{\phi}_i(\sigma)-\bar{\phi}_i(\hat{\sigma})}{\sigma-\hat{\sigma}} \leq \mu_i$ for all $\sigma \in \mathbb{R}$ such that $\bar{\phi} \in \Phi_{sr}^{|\mu|}$. Now, define the diagonal matrices:

$$\Lambda \triangleq \mathrm{diag}(\frac{\alpha_i + \beta_i}{2}), \qquad \tilde{\Lambda} \triangleq \mathrm{diag}(\frac{\beta_i - \alpha_i}{2\mu_i}),$$

so that $p = \tilde{\Lambda}\bar{p} + \Lambda q$ with $p_i(k) := \phi_i(q_i(k))$. The well-posed assumption and change of variables in which $\bar{p}$ is substituted into p implies that

$$\dot{x} = [A + B\Lambda(I - D\Lambda)^{-1}C]x + [B\Lambda(I - D\Lambda)^{-1}D\tilde{\Lambda} + B\tilde{\Lambda}]\bar{p}, \tag{1.38}$$

$$q = (I - D\Lambda)^{-1}Cx + (I - D\Lambda)^{-1}D\tilde{\Lambda}\bar{p}. \tag{1.39}$$

Therefore the Lur'e system with general slope-restricted nonlinear conditions can be expressed as a system with diagonal Lipschitz-bounded nonlinearities:

$$\bar{S}: \quad x(k+1) = \bar{A}x(k) + \bar{B}\bar{p}(k), \qquad q(k) = \bar{C}x(k) + \bar{D}\bar{p}(k), \tag{1.40}$$

$$\bar{p}_i(k) = \bar{\phi}_i(q_i(k)), \qquad -\mu_i \leq \frac{\bar{\phi}_i(\sigma) - \bar{\phi}_i(\hat{\sigma})}{\sigma - \hat{\sigma}} \leq \mu_i. \tag{1.41}$$

where

$$\bar{A} = A + B\Lambda(I - D\Lambda)^{-1}C, \qquad \bar{B} = B\Lambda(I - D\Lambda)^{-1}D\tilde{\Lambda} + B\tilde{\Lambda}, \tag{1.42}$$

$$\bar{C} = (I - D\Lambda)^{-1}C, \qquad \bar{D} = (I - D\Lambda)^{-1}D\tilde{\Lambda}. \tag{1.43}$$

For the case when $\mu_i = \mu$ for all indices i, the transformed diagonal slope-restricted nonlinear function satisfies the following Lipschitz condition:

$$
\begin{aligned}
&\|\bar{p}(k+1) - \bar{p}(k)\|_2^2 \\
&\leq \mu^2 \|q(k+1) - q(k)\|_2^2 \\
&= \mu^2 \begin{bmatrix} x(k) \\ \bar{p}(k) \\ \bar{p}(k+1) \end{bmatrix}^T \begin{bmatrix} (\bar{A}-I)^T \bar{C}^T \\ (\bar{C}\bar{B} - \bar{D})^T \\ \bar{D}^T \end{bmatrix} \begin{bmatrix} \bar{C}(\bar{A}-I) & (\bar{C}\bar{B} - \bar{D}) & \bar{D} \end{bmatrix} \begin{bmatrix} x(k) \\ \bar{p}(k) \\ \bar{p}(k+1) \end{bmatrix} \\
&= \mu^2 \begin{bmatrix} x(k) \\ \bar{p}(k) \\ \bar{p}(k+1) \end{bmatrix}^T \begin{bmatrix} (\bar{A}-I)^T \bar{C}^T \bar{C}(\bar{A}-I) & (\bar{A}-I)^T \bar{C}^T (\bar{C}\bar{B} - \bar{D}) & (\bar{A}-I)^T \bar{C}^T)\bar{D} \\ (\bar{C}\bar{B} - \bar{D})^T \bar{C}(\bar{A}-I) & (\bar{C}\bar{B} - \bar{D})^T (\bar{C}\bar{B} - \bar{D}) & (\bar{C}\bar{B} - \bar{D})^T \bar{D} \\ \bar{D}^T \bar{C}(\bar{A}-I) & \bar{D}^T (\bar{C}\bar{B} - \bar{D}) & \bar{D}^T \bar{D} \end{bmatrix} \begin{bmatrix} x(k) \\ \bar{p}(k) \\ \bar{p}(k+1) \end{bmatrix}
\end{aligned}
$$

for all $k \in \mathbb{Z}_+$.

Limitations of Loop Transformation for a Class of Nonlinear Functions

Let us consider the discrete-time Lur'e system with sector-bounded and slope-restricted nonlinear functions:

$$S: \quad x(k+1) = Ax(k) + Bp(k), \qquad q(k) = Cx(k), \tag{1.44}$$

$$p_i(k) = -\phi_i(q_i(k)), \tag{1.45}$$

where $\phi \in \Phi_{sb}^{[\grave{\alpha},\acute{\alpha}]} \cap \Phi_{sr}^{[\grave{\beta},\acute{\beta}]}$. It can be easily seen that there is no loop transformation such that $\bar{\phi} \in \Phi_{sb}^{|\alpha|} \cap \Phi_{sr}^{|\beta|}$ unless $\grave{\alpha} \equiv \grave{\beta}$ and $\acute{\alpha} \equiv \acute{\beta}$. In other words, if $\phi \in \Phi_{sb}^{[\grave{\alpha},\acute{\alpha}]} \cap \Phi_{sr}^{[\grave{\beta},\acute{\beta}]}$ and either of $\grave{\alpha} \neq \grave{\beta}$ or $\acute{\alpha} \neq \acute{\beta}$ holds, then such a nonlinear function ϕ cannot be transformed into the standard bound representation $\bar{\phi} \in \Phi_{sb}^{|\alpha|} \cap \Phi_{sr}^{|\beta|}$.

1.2 Literature Review for Absolute Stability–FDI and LMI Approaches

1.2.1 Circle Criterion

The circle criterion is a generalization of the sufficient part of the conventional Nyquist stability criterion. It was first produced by Sandberg and Zames for a scalar nonlinear feedback system [130, 131, 102]. They showed that the critical point in the classical Nyquist criterion could be expanded to an on-axis disc with radius proportional to the degree of departure from linearity by the nonlinearity. The circle theorem is geometric in nature since the stability criterion is that the frequency response function $G(j\omega)$ should not intersect or encircle the disc shown in Figure 1.6. The circle theorem is inherently conservative, and it is also applicable to time-varying systems whereas the Nyquist theorem is only applicable to linear time-invariant systems. The circle theorem generalizes the Nyquist theorem since it includes it as a special case. The Nyquist theorem [81, 80, 25] is a well known result and it is not covered in this section due to limited space. Instead, in this section, the frequency domain circle theorem for the scalar case will be first introduced in order to describe its geometrical characteristics of the SISO case. Then a mapping transfer function is described that relates the circle criterion to a positive real function that leads to a formulation in the form of an algebraic Riccati equation. Finally, the scalar algebraic version is extended to the multivariable case.

The Continuous-Time Circle Criterion

In a 1966 paper, Zames [131] (Lemma 1 on page 467) showed that a conic sector of the form shown in (1.20) has a counterpart in the frequency plane in the form of a circular disk, given by the condition

$$\left| G(j\omega) + \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \right| \geq \frac{1}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \tag{1.46}$$

such that the Nyquist plot of $G(s)$ does not encircle the point $-\frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta})$. Equation (1.46) corresponds to the complement of equation (1.20). Therefore, the frequency domain statement of the circle criterion is as follows [18, 19].

Theorem 9. *(Circle criterion in geometrical point of view) Let $n(s)$ and $d(s)$ be coprime polynomials, i.e. they do not have common factors, where $G(s) = \frac{n(s)}{d(s)}$ is strictly proper without loss of generality. Then if $d(s)$ has no zeros for all $s \in \mathbb{C}$ such that its real part is positive then the following statements hold:*

1. *The trajectories of solutions of (1.19) are bounded if $\phi \in \Phi_{sb}^{[\alpha,\beta]}$ and the Nyquist locus of $G(s)$ does not encircle or intersect the open disk $D(\alpha, \beta)$ which is centered on the negative real axis of the $G(s)$ plane and has as a diameter the segment of the negative real axis $(-\frac{1}{\alpha}, -\frac{1}{\beta})$ (see Figure 1.6).*

2. *The trajectories of solutions of (1.19) are bounded and exponentially converge to the origin if there is some $\epsilon > 0$ such that $\phi \in \Phi_{sb}^{[\alpha+\epsilon,\beta-\epsilon]}$ and the Nyquist locus of $G(s)$ does not encircle or intersect the open disk $D(\alpha, \beta)$ which is centered on the negative real axis of the $G(s)$ plane and has as a diameter the segment of the negative real axis $(-\frac{1}{\alpha}, -\frac{1}{\beta})$ (see Figure 1.6).*

Under the conditions, the closed loop system is asymptotically stable in that all sets of initial conditions lead to outputs which are bounded as t approaches infinity if the Nyquist locus of $G(s)$ does not intersect the disk $D(\alpha, \beta)$ and encircles it m_ρ times in the counterclockwise direction.

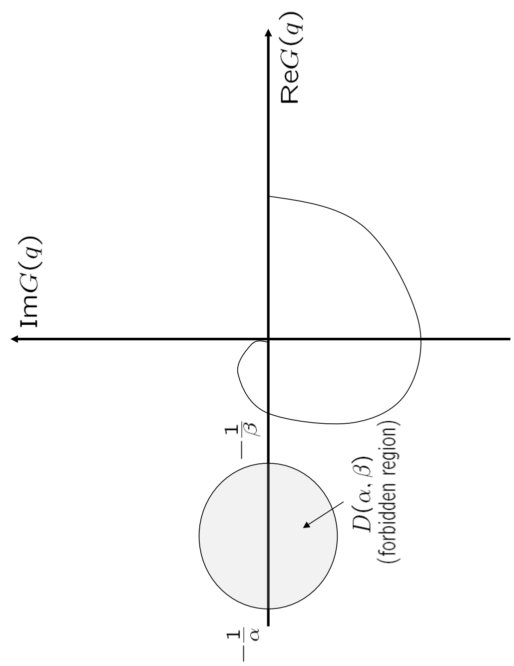


Figure 1.6: Geometric Circle Criterion in Nyquist Diagram: $q = j\omega$ for CT cases or $e^{j\omega}$ for DT cases

Different proofs of the circle criterion based on Lyapunov functions have appeared in the literature [33, 115, 122]. To generate a Lyapunov function for the circle criterion, the positive real lemma [6] is applied to the following mapping of the transfer function for the scalar case [19, 9]:

$$\hat{G}(s) \triangleq \frac{1+\alpha G(s)}{1+\beta G(s)} = \frac{\alpha}{\beta}\frac{\hat{n}(s)}{\hat{d}(s)}, \tag{1.47}$$

where $\hat{n}(s) = n(s) + \frac{1}{\alpha}d(s)$, and $\hat{d}(s) = n(s) + \frac{1}{\beta}d(s)$ Note that the mapping described in (1.47) takes the exterior of the disk $D(\alpha, \beta)$ in the s-domain for the scalar transfer function $G(s)$ into the right-half-plane (RHP) in the s-domain for the transformed transfer function $\hat{G}(s)$, if $G(j\omega)$ is exterior to $D(\alpha, \beta)$ for all $\omega \in \mathbb{R}$ (see Lemma 3 in [19]):

$$0 \leq \mathrm{Re}\left[F(j\omega)\right] = \mathrm{Re}\left[\frac{1+\alpha G(s)}{1+\beta G(s)}\right] \leq \frac{\hat{d}(j\omega)\hat{n}(-j\omega)+\hat{d}(-j\omega)\hat{n}(j\omega)}{2|\hat{d}(j\omega)|^2} \tag{1.48}$$

with $0 \neq \alpha \neq \beta \neq 0$. The above implies that the numerator of equation (1.48) is a positive real function. For a proof of equation (1.48) see [119]. The following statement for asymptotic stability using the circle criterion corresponds to Lemma 4 in [19].

Lemma 5. *Let us consider a strictly proper rational transfer function $G(s)$ and a nonlinear function $\phi \in \Phi_{sb}^{[\alpha,\beta]}$, then if $F(s)$ is positive real the feedback nonlinear system described by equations (1.3) is asymptotically stable.*

Multivariable extensions of the circle criterion have been proposed [133] and a multivariable version by Haddad and Bernstein has been written as an algebraic Ricatti equation [33]. The multivariable version of (1.47) is given by

$$\hat{G}(s) = \left[I + K_1 G(s)\right]\left[I + K_2 G(s)\right]^{-1} \tag{1.49}$$

where K_1, K_2 are the matrices of compatible dimensions and the multivariable conic sector is defined by

$$\bar{\Phi}_{sb}^{[K_1,K_2]} \triangleq \left\{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p} | (\phi(\sigma(t)) - K_1\sigma(t))^T(\phi(\sigma(t)) - K_2\sigma(t)) \leq 0, \forall \sigma(t) \in \mathbb{R}^{n_q}\right\} \tag{1.50}$$

and $\phi(\sigma(t))$ is Lebesgue measurable for all $\sigma(t) \in \mathbb{R}_{n_q}$.

By writing a realization of $\hat{G}(s)$ in (1.49) and using the Kalman-Yakubovich-Popov (KYP) Lemma with the procedure shown in Section **??**, a multivariable algebraic Ricatti equation equivalent to Lemma 5 can be written [34] such that it yields the following theorem in terms of an LMI.

Theorem 10. *Suppose $\hat{G}(s)$ is strongly positive real and $G(s)$ is a minimal realization of the system described* *in (1.3), then there exists a matrix $P > 0$ satisfying:*

$$\begin{bmatrix} \hat{A}^T P + P\hat{A} & P\hat{B} - \hat{C}^T \\ \hat{B}^T P - \hat{C} & -\hat{D} - \hat{D}^T \end{bmatrix} < 0 \tag{1.51}$$

where

$$\begin{aligned} \hat{A} &= A - B(I + K_1 D)^{-1} K_1 C \\ \hat{B} &= B(I + K_1 D)^{-1} \\ \hat{C} &= (K_2 - K_1)(I + DK_1)^{-1} C \\ \hat{D} &= I + (K_2 - K_1)(I + DK_1)^{-1} D \end{aligned} \tag{1.52}$$

Furthermore, the negative feedback interconnection of $G(s)$ and $\phi(q(t))$ is asymptotically stable for all $\phi \in \bar{\Phi}_{sb}^{[K_1,K_2]}$.

Proof: This is nothing but the combination of taking a loop transformation and applying the positive real lemma. The same result has been shown in [34] when applying Schur complement. □

Remark 4. *(Loop transformation (or multiplier theory)) A loop transformation with constant gain matrices does not change the stability of the system. That is, the system which consists of the transfer function $G(s)$ and the negative feedback interconnection with the nonlinear function $\phi \in \bar{\Phi}_{sb}^{[K_1,K_2]}$ is stable if and only if $\hat{G}(s)$ is positive real such that its input-output property is characterized to be passive.*

The circle criterion is an important result because it serves as a generating criterion for various constraining forms of nonlinear functions, Φ, for example, consider the following corollary.

Corollary 1. *The circle criterion reduces to the positive real lemma described in Theorem 3 provided that $K_2 = \infty I$ and $K_1 = 0$.*

The circle criterion is considered to be the least conservative result that can be obtained for time-varying systems for the case of a quadratic Lyapunov function [122].

The Discrete-Time Circle Criterion

All the arguments regarding the formulation of the continuous-time circle theorem carry over to the case of the discrete-time version of the circle criterion [18].

Theorem 11. *(Circle criterion in geometrical point of view) Let $n(z)$ and $d(z)$ be coprime polynomials, i.e. they do not have common factors, where $G(z) = \frac{n(z)}{d(z)}$ is strictly proper without loss of generality. Then if $d(z)$ has no zeros outside the unit circle then the following statements hold:*

1. *The trajectories of solutions of (1.3) are bounded if* $\phi \in \Phi_{sb}^{[\alpha,\beta]}$ *and the Nyquist locus of* $G(z)$ *does not encircle or intersect the open disk which is centered on the negative real axis of the z-domain for* $G(z)$ *and has as a diameter the segment of the negative real axis* $(-\frac{1}{\alpha}, -\frac{1}{\beta})$ *(see Figure 1.6).*

2. *The trajectories of solutions of (1.3) are bounded and exponentially converge to the origin if there is some* $\epsilon > 0$ *such that* $\phi \in \Phi_{sb}^{[\alpha+\epsilon,\beta-\epsilon]}$ *and the Nyquist locus of* $G(z)$ *does not encircle or intersect the open disk which is centered on the negative real axis of the* $G(z)$ *plane and has as a diameter the segment of the negative real axis* $(-\frac{1}{\alpha}, -\frac{1}{\beta})$ *(see Figure 1.6).*

In 1994 Haddad and Bernstein [118] suggested a multivariable version of the discrete-time circle criterion in terms of an algebraic Ricatti equation [36]. Here we give an alternative version of Theorem 12 in terms of LMIs, which is nothing but the combination of taking a loop transformation and applying the positive real lemma.

Theorem 12. *Let us suppose* $\hat{G}(z) \triangleq \frac{1+\alpha G(z)}{1+\beta G(z)}$ *is strongly positive real and* $G(z)$ *is a minimal realization of the system described in (1.3) such that their subscripts are dropped for convenience, then there exists a matrix* $P > 0$ *satisfying:*

$$\begin{bmatrix} \hat{A}^T P \hat{A} - P & \hat{A}^T P \hat{B} - \hat{C}^T \\ \hat{B}^T P \hat{A} - \hat{C} & \hat{B}^T P \hat{B} - \hat{D} - \hat{D}^T \end{bmatrix} < 0 \tag{1.53}$$

where the matrices $\hat{A}, \hat{B}, \hat{C}, \hat{D}$, *are the same as the ones defined in the continuous-time case. Furthermore, similar to the continuous-time case, the negative feedback interconnection of* $G(z)$ *and* $\phi(q(k))$ *is asymptotically stable for all* $\phi \in \bar{\Phi}_{sb}^{[K_1,K_2]}$.

Proof: This is nothing but the combination of taking a loop transformation and applying the positive real lemma. The same result has been shown in [34] when applying Schur complement. □

1.2.2 Popov Criterion

In 1960, V. M. Popov [89, 90, 91] provided a less conservative sufficient criterion for the stability of the system shown in Figure 1.7 in terms of a modified frequency response of the linear part of the feedback system $G(j\omega)$. The idea consists on the removal of a multiplier from the linear element and further conditions on the properties of the negative feedback connected nonlinear functions in the block Φ. Such conditions are that $\phi \in \Phi_{sr}^{[0,\infty]}$ does not depend on time explicitly and is memoryless (static). It was not until Popov's work

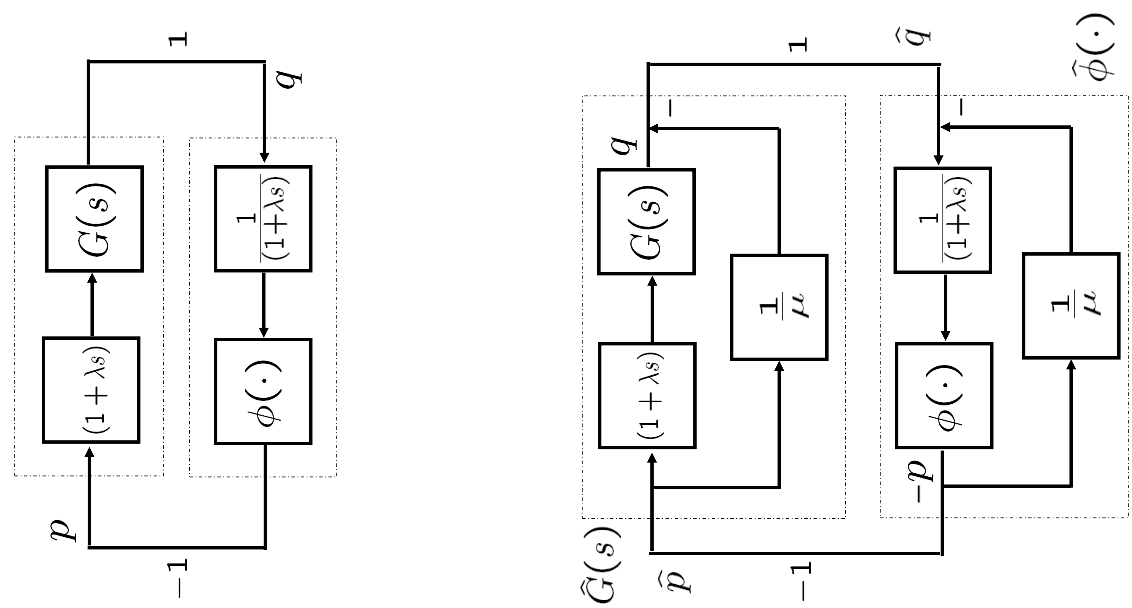


Figure 1.7: Loop Transformations for Popov Criterion

and making use of frequency response methods for nonlinear stability analysis and synthesis that a theory of design for nonlinear systems began to emerge.

Continuous-Time Popov Criterion

In this section, the factorization and loop transformation that lead to the scalar version of the Popov criterion are discussed. The aforementioned transformation leads to a *Modified Phase Amplitude Characteristic* or MPAC; the geometrical interpretation of the Popov criterion in terms of an MPAC plot is described and the previous results will be extended for the case where $\phi \in \Phi_{sr}^{[0,\mu]}$ for the SISO case. Finally, the scalar algebraic version is extended to the multivariable case. Let us consider that the transfer function $G(s)$ can be factored as

$$G(s) = \frac{1}{1+\lambda s}(\hat{G}(s) - \frac{1}{\mu}) \tag{1.54}$$

for some scalar $\lambda \geq 0$, which is shown in Figure 1.7. This factorization leads to the transformed transfer function

$$\hat{G}(s) = \mu^{-1} + (1+\lambda s)G(s) \tag{1.55}$$

The block diagram can be written for (1.55) by means of a loop transformation, as shown in Figure 1.7. The detailed procedure for constructing this loop transformation is shown in the proof of Corollary 2 on page 470 of [131]. Note that the nonlinear block Φ is multiplied by a first-order transfer function, thus the name *Popov multiplier* for such transfer function. Such multiplication results in a modified block $\hat{\Phi}$ whose elements $\hat{\phi}$ belong to the class of nonlinear function defined by $\Phi_{sr}^{[0,\infty]}$ provided the nonlinear functions are memoryless, as shown by Lemma 2 of [131] in page 468. The frequency response of the linear element is modified by the removal, and, in effect, the size of the critical region is reduced. The Popov criterion for continuous-time transfer functions [90, 115] follows.

Theorem 13. *Consider a nonlinear feedback system of the form described in (1.3) where the nonlinear element $\phi \in \Phi_{sr}^{[0,\infty]}$. If there exists a number λ such that*

$$\mathrm{Re}\left[(1+\lambda j\omega)G(j\omega)\right] \geq 0, \forall \omega > 0 \tag{1.56}$$

then the nonlinear feedback system is asymptotically stable.

Proof: The proof consists on showing that $(1+\lambda s)G(s)$ is strictly positive real, thus by means of the KYP lemma there exists a symmetric positive definite matrix $P = P^T > 0$ that is used to establish a Lure-Postnikov Lyapunov function [28, 115, 56] in order to prove global asymptotic stability. The details of the proof are shown in Popov's original paper [90] and on pages 231 to 233 of [115]. □

Note that $(1+\lambda j\omega)G(j\omega) = \frac{\hat{n}(j\omega)}{\hat{d}(j\omega)}$ has the form of a straight line provided $G(j\omega)$ corresponds to the horizontal axis and $\omega G(j\omega)$ corresponds to the vertical axis, as shown in Figure 1.8 which is known as a *Modified Phase Amplitude Characteristic* or MPAC. The inequality expressed by (1.56) means that the MPAC of $G(j\omega)$ corresponds to a half-plane. Thus the Popov criterion reduces to finding the hyperplanes that intersect the negative or positive real axis of a Nyquist plot modified by the multiplier that is to the left of this plot but as close to the origin as possible. Thus, the geometrical interpretation of Theorem 13 is as follows: if there is a straight line situated either in the first and the third quadrants of the $G(j\omega)$ vs $\omega G(j\omega)$ plane or it is the ordinate axis, and in addition it is such that the MPAC is on the right of this straight line, then the trivial solution of the investigated system is absolutely asymptotically stable. Further, the MPAC is on the right of this straight line if any point of the MPAC is either on the straight line or it is in the half-plane bounded by this straight line and containing the point $(0, +\infty)$.

If $\phi \in \Phi_{sr}^{[0,\mu]}$ and $\phi \in \Phi_{sb}^{[0,\xi]}$ such that $\mu \geq \xi$, then a special case of Theorem 13 can be considered [115] by doing the loop transformation shown on Figure 1.7 [28, 115]. In terms of the geometrical interpretation the straight line in Figure 1.8 may be shifted a distance $\frac{1}{\mu}$ to the left as shown in Figure 1.8. Asymptotic

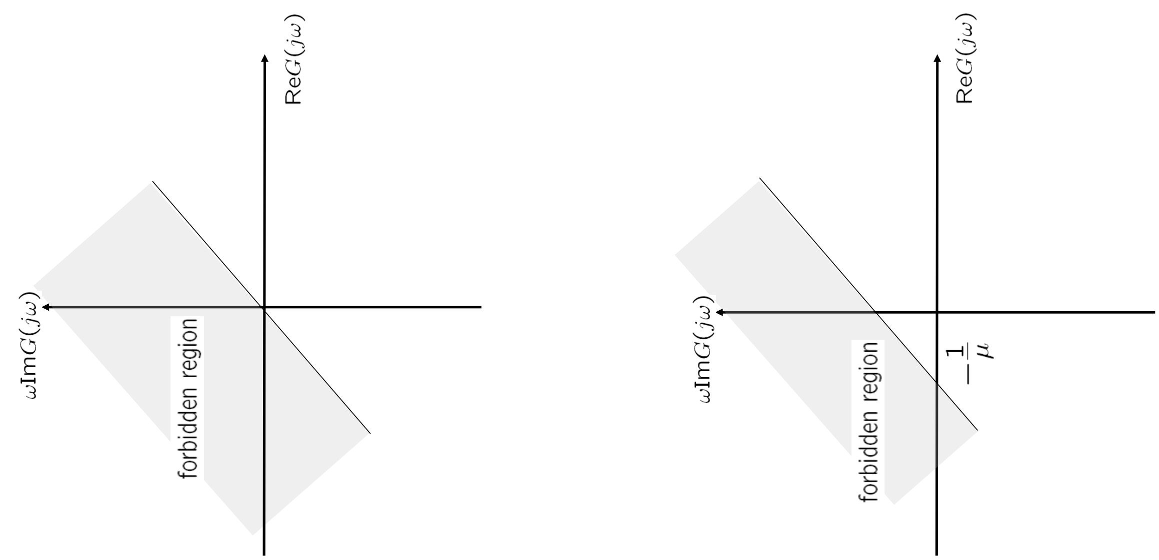


Figure 1.8: Geometrical Popov Criterion in Nyquist Diagram

stability is guaranteed for any ϕ in a specified class of nonlinear functions, which is also called a *relation*.

The Popov criterion was generalized to the multivariable case [12, 7] in the late 1960s; the proof is very similar to that of the scalar case since it is based on applying the KYP lemma to the multivariable version of (1.56) given by

$$W(s) = M^{-1} + (I + \Lambda s)G(s) \tag{1.57}$$

in order to show positive realness of $W(s)$, where $M = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_{n_p}\} \in \mathbb{R}^{n_p \times n_p}$, and Λ is a nonnegative definite diagonal matrix whose diagonal components correspond to the Lagrange multipliers, absolute stability is proved by means of a Lur'e-Postnikov Lyapunov function. Details are shown on pages 490 to 491 of [12]. The set of nonlinear functions is defined in component-wise sense. That is, each component $\phi_i(\sigma_i)$ of ϕ satisfies the restriction $\phi_i \in \Phi_{sr}^{[0,\mu_i]}$ and the components of ϕ are assumed to be decoupled. The multivariable Popov equation described in (1.57) has been written as an algebraic Ricatti equation [12, 33], which leads to the following theorem in terms of an LMI.

Theorem 14. *Suppose $W(s)$ is strongly positive real, where $G(s)$ is a minimal realization of the system described by equation (1.3) where $\Phi_{sr}^{[0,M]}$ is as described previously in Definition 7, then there exist matrices $P > 0$ and $\Lambda \geq 0$ satisfying:*

$$\begin{bmatrix} A^T P + PA & (C + \Lambda CA - B^T P)^T \\ C + \Lambda CA - B^T P & (M^{-1} + \Lambda CB) + (M^{-1} + \Lambda CB)^T \end{bmatrix} < 0 \tag{1.58}$$

Furthermore, the negative feedback interconnection of the LTI system $G(s)$ and the nonlinear mapping ϕ is asymptotically stable for all $\Phi_{sr}^{[0,M]}$.

Proof: The Lur'e-Postnikov-Lyapunov function for this theorem is proved on pages 490 and 491 of [12]. The LMI is obtained by applying the Schur complement to equation (71) on page 328 of [33]. See also pages 278–279 of [56]. □

The Popov criterion is less conservative since it removes the time variation property of the nonlinearities. Further research on Popov criterion involved the consideration of local slope restrictions [101, 38]. In particular, in [106] the Popov multiplier shown in (1.55) was substituted and (1.56) is required to be positive real and μ is a bound on the local slope of the feedback nonlinearity such that

$$0 < \phi'(\sigma) < \mu, \quad \forall \sigma \in \mathbb{R}, \tag{1.59}$$

which is also defined as the set of nonlinear functions $\Phi_{sr}^{[0,\mu]} \subseteq \mathbb{R}$.

Because of the new multiplier, the MPAC plot changes in such a way that the vertical axis corresponds to $\frac{1}{\omega}\mathrm{Im}\left[G(j\omega)\right]$ and the Popov plane is bounded in terms of a straight line with a real axis intercept $-\frac{1}{\mu}$ and slope $-\frac{1}{\lambda}$ for the SISO case.

In 1995, Haddad and Kapila [37] generalized Theorem 13 to the multivariable case containing an arbitrary number of memoryless time-invariant slope-restricted monotonic nonlinearities by considering the multivariable version of (1.56) given in (1.57) with a different Lyapunov function.

Theorem 15. *Let $G(s) = C(sI - A)^{-1}B$ be a stable transfer function, its realization (A, B, C) be minimal, and $CA^{-1}B$ be nonsingular, which implies that the origin is the unique equilibrium point. If there exists a positive semi-definite matrix $\Lambda = diag\{\lambda_1, \cdots, \lambda_{n_p}\}$ so that a new transfer function is defined as $W(s) = M^{-1} + (I + \Lambda s^{-1})G(s)$, then $W(s)$ is strongly positive real if and only if there exists a positive definite matrix $P = P^T > 0$ such that*

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} & (\bar{B}^T P - \Lambda\bar{C}\bar{A}^{-1} - \bar{D})^T \\ \bar{B}^T P - \Lambda\bar{C}\bar{A}^{-1} - \bar{D} & 2M^{-1}I \end{bmatrix} < 0, \tag{1.60}$$

where

$$\bar{A} \triangleq \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \bar{B} \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{C} \triangleq \begin{bmatrix} 0 & I \end{bmatrix}, \quad \bar{D} \triangleq \begin{bmatrix} C & 0 \end{bmatrix} \tag{1.61}$$

Furthermore, the negative feedback interconnection of $G(s)$ and ϕ is asymptotically stable for all $\phi \in \Phi_{sr}^{[0,M]}$.

Proof: Details of the proof are shown in pages 362 and 363 of [37]. Strict positive realness leads to the application of the Kalman-Yakubovich lemma and its equivalent LMI form is written by means of the Schur complement. Absolute stability is proved by means of a modified Lur'e-Postnikov equation of the form:

$$V(\dot{x}(t), q(t)) = \begin{bmatrix} \dot{x}(t) \\ q(t) \end{bmatrix}^T P \begin{bmatrix} \dot{x}(t) \\ q(t) \end{bmatrix} + 2\sum_{i=1}^{m} \lambda_i \int_0^{q_i(t)} \phi'(\sigma)\sigma d\sigma \tag{1.62}$$

Details are shown on page 363 of [37]. □

1.2.3 Tsypkin's Criterion

After Popov published his stability criterion for continuous-time systems, several researchers worked on developing a discrete-time stability test similar to Popov's. The closest analog was developed by Tsypkin [111, 112] who derived the following theorem.

Theorem 16. *The corresponding scalar nonlinear system with the form of equation (1.3), where ϕ belongs to sector condition $[0, \xi]$, i.e., $\phi \in \Phi_{sb}^{[0,\xi]}$, is globally asymptotically stable if the following condition holds:*

$$\mathrm{Re}\left[G^*(j\omega)\right] + \frac{1}{\xi} > 0 \tag{1.63}$$

Proof: See pages 172 and 173 of [112]. □

The geometrical interpretation of Theorem 16 is shown in Figure 1.9. The discrete-time nonlinear system will be absolutely stable if the frequency characteristic of $G(z)$ of the linear part does not intersect the straight line $-1/\xi$. The foundations upon which Popov's criterion is extended to the discrete-time case were established in the work of Hitz and Anderson [39] who described discrete positive-realness in terms of the Kalman-Yakubovich lemma. Another important element was the work of Pearson and Gibson [86] who bounded the integral of the difference equation of a discrete-time version of the Lur'e-Postnikov equation in terms of the maximum slope of the nonlinearity, which makes it possible to further restrict the upper boundary of the sector conditions in comparison with earlier applications of quadratic. In 1964, Jury and Lee [49, 48] proposed a new sufficient condition that considers a local slope restriction that produces a modified response by shifting the frequency response of the discrete-time transfer function $G(z)$ to the right in a manner proportional to the value of the slope μ, that is, the smaller the slope the more the modified frequency is shifted to the right leading to a less conservative solution.

Theorem 17. *The corresponding scalar nonlinear system with the form of equation (1.5), where $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$, is globally asymptotically stable if a non-negative scalar λ exists such that the inequality*

$$\mathrm{Re}\left[G^*(z)\right]\left[\lambda + \lambda(z-1)\right] + \frac{1}{\xi} - \frac{\lambda\mu}{2}\left|(z-1)G^*(z)\right|^2 \geq 0 \tag{1.64}$$

is satisfied on the unit circle $z = \exp(j\omega)$ for $\omega \in [0, 2\pi]$.

Proof: See pages 53 to 56 of [49]. □

The criterion described in (1.64) includes Tsypkin's criterion as a special case when $\lambda = 0$. The multivariable extension of Jury and Lee's criteria was described by Haddad and Bernstein [118, 36] by defining the set Φ_P characterizing a class of sector-bounded time-invariant memoryless nonlinearities. Let $K = \mathrm{diag}\{\xi_1, \cdots, \xi_{n_p}\} \in \Re^{n_p \times n_p}$ be a given positive definite matrix and define:

$$0 \leq \sigma\phi_i(\sigma) \leq \xi_i\sigma^2, \quad \forall\sigma \in \mathbb{R}, \quad \text{for } i = 1, 2, \ldots, n_p \tag{1.65}$$

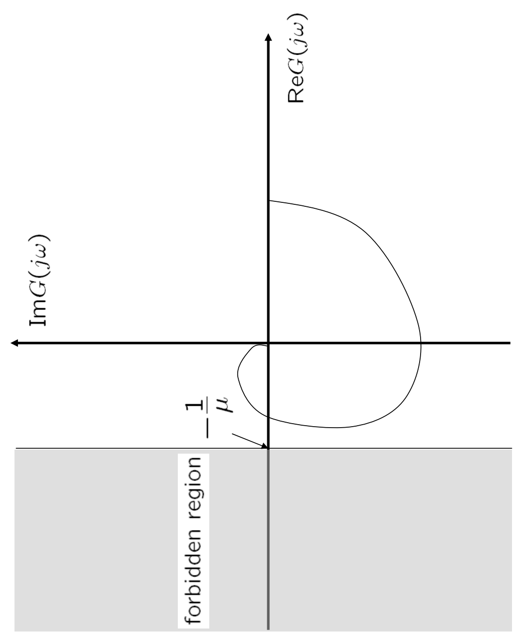


Figure 1.9: Geometrical Tsypkin's Criterion in Nyquist Diagram

and

$$0 < \frac{\phi_i(\sigma) - \phi_i(\hat{\sigma})}{\sigma - \hat{\sigma}} < \mu_i, i = 1, \ldots, n_p. \tag{1.66}$$

Note that components of ϕ are decoupled. The multivariable version of (1.64), as shown in [118] is given by

$$K^{-1} + [I + (z-1)\Lambda]G(z) - \frac{1}{2}|z-1|^2 G^*(z) M \Lambda G(z) \tag{1.67}$$

where Λ is a positive semi-definite matrix. The multivariable version of Theorem 17 is described in terms of an LMI in the following theorem.

Theorem 18. *Consider the nonlinear system described in (1.3). If there exists a nonnegative definite diagonal matrix N such that (1.67) is strongly discrete-time positive real, where $G(z)$ is a minimal realization, then the negative feedback interconnection of $G(z)$ and $\phi(\cdot)$ is asymptotically stable for $\phi \in \Phi_{sb}^{[0,K]} \cap \Phi_{sr}^{[0,M]}$ if the following condition holds:*

$$\begin{bmatrix} \begin{pmatrix} P - A^T P A \\ -(A-I)^T C^T M \Lambda C (A-I) \end{pmatrix} & \begin{pmatrix} -C - B^T C^T M \Lambda C \\ -QCA + B^T C^T M \Lambda C A + \Lambda C + B^T P A \end{pmatrix} \\ \begin{pmatrix} (-C - B^T C^T M \Lambda C \\ -QCA + B^T C^T M \Lambda C A + \Lambda C + B^T P A)^T \end{pmatrix} & \begin{pmatrix} (K^{-1} + \Lambda C B) + (K^{-1} + \Lambda C B)^T \\ -B^T C^T \Lambda C B - B^T P B \end{pmatrix} \end{bmatrix} > 0. \tag{1.68}$$

Proof: See [118] for the Riccati equation version and using the Schur complement lemma one has the equivalent LMI condition above. □

1.3 Linear and Bilinear Matrix Inequalities

In this section, backgrounds on the mathematical theory and control applications of linear matrix inequalities (LMIs) and bilinear matrix inequalities (BMIs) are provided. The research monograph [100] is a useful introduction for locating LMI results scattered throughout the control and system engineering literature. A through review for LMIs and BMIs for process control is given in [113]. More advanced topics of LMIs in control are introduced in the book [58].

1.3.1 The Linear Matrix Inequality and Optimization Problems

Here the definition and some properties of the LMI is given. A Linear Matrix Inequality (LMI) has the form:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{1.69}$$

where $x \in \mathbb{R}^m, F_i \in \mathbb{R}^{n\times n}$. The inequality means that $F(x)$ is a positive definite matrix, that is,

$$z^T F(x) z > 0,\ \forall z \neq 0, z \in \mathbb{R}^n. \tag{1.70}$$

The symmetric matrices $F_i, i = 0, 1, \ldots, m$ are fixed and x is the (decision) variable. Thus, $F(x)$ is an affine function of the elements of x.

Comment 2. *Any feasible nonstrict LMI can be reduced to an equivalent strict LMI that is feasible. To see this, eliminate implicit equality constraints and then reduce the resulting LMI by removing any constant nullspace [100], page 19).*

Proposition 1. *(LMI equivalence to polynomial inequalities [113]) It is informative to represent the LMI in terms of scalar inequalities. More specifically, the LMI (1.69) is equivalent to n polynomial inequalities.*

Moreover, LMIs are not unique to represent a feasibility set.

Proposition 2. *(Non-uniqueness of LMIs) A matrix A is positive (or negative) definite if and only if $T^T AT$ is positive (or negative) definite for any nonsingular matrix T. Thus, the permutation transformation matrix T gives some rearrangements of matrix elements such that definiteness of the transformed matrix does not change.*

Another interesting property of the LMI (1.69) is its convexity. A set C is said to be *convex* if $\lambda x+(1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in (0, 1)$ [2].

Proposition 3. *(Convexity) An important property of LMIs is that the set $\{x|F(x) > 0\}$ is convex, that is, the LMI (1.69) forms a convex constraint on x.*

An important property of LMIs in control and system engineering problems is its ability to consider multiple control requirements or objectives by appending additional LMIs.

Proposition 4. *(Capability to handle multiple constraints) Consider a set defined by q LMIs:*

$$F^1(x) > 0;\ F^2(x) > 0;\ \cdots\ ;\ F^q(x) > 0. \tag{1.71}$$

Then, noting the fact that the eigenvalues of a block diagonal matrix are equal to the union of the eigenvalues of the blocks, or from the definition of positive definiteness, an equivalent single LMI is given by

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i = diag\{F^1(x), F^2(x), \cdots, F^q(x)\} > 0, \tag{1.72}$$

where

$$F_i = diag\{F_i^1, F_i^2, \cdots, F_i^q\}, \ \forall i = 0, \cdots, m \tag{1.73}$$

and $diag\{X_1, X_2, \cdots, X_q\}$ *is a block diagonal matrix with blocks* $X_1, X_2, \ldots, X_q$.

The linear constraints on variables can be written as LMI constraints and the problem to investigate the existence of the solution which satisfies LMIs is called an LMI feasibility problem.

Proposition 5. *(Linear constraints an an LMI) Consider the general linear constraint* $Ax < b$ *written as* n *scalar inequalities:*

$$b_i - \sum_{j=1}^{m} A_{ij} x_j > 0, \quad i = 1, \ldots, n \tag{1.74}$$

where $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times m}$, *and* $x \in \mathbb{R}^m$. *Each of the* n *scalar inequalities is an LMI. Since multiple LMIs can be written as a single LMI, the linear inequalities (1.74) can be expressed as a single LMI.*

In addition to the properties of LMIs, the following lemma is a tool to convert a class of convex nonlinear inequalities to an LMI.

Lemma 6. *(The Schur complement lemma) The convex nonlinear inequalities*

$$R(x) > 0, \qquad Q(x) - S(x)R(x)^{-1}S(x)^T > 0, \tag{1.75}$$

where $Q(x) = Q(x)^T, R(x) = R(x)^T$, *and* $S(x)$ *depend affinely on* x, *are equivalent to the LMI*

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0. \tag{1.76}$$

Proof: A proof of the Schur Complement Lemma using only elementary calculus is given in the appendix. In what follows, the Schur Complement Lemma is applied to several inequalities that appear in control and system engineering problems. □

One of the most useful and promising aspects of LMIs is that fact that many optimization and control problems can be formulated in terms of finding a feasible solution to a set defined by LMIs.

Eigenvalue problems

The following optimization problem is commonly referred to as a semidefinite program (SDP) [5]:

$$\begin{aligned} \inf_{x} \quad & c^T x \\ F(x) > 0 & \end{aligned} \tag{1.77}$$

An SDP that often arises in control applications is the LMI eigenvalue problem (EVP). It is the minimization of the maximum eigenvalue of a matrix that depends affinely on the variable x, subject to an LMI constraint on x. Many performance analysis tests, such as computing the H_∞ norm, can be written in terms of an EVP [114]. Two common forms of the EVP are presented so that readers will recognize them:

$$\begin{aligned} \inf_{x,\lambda} \quad & \lambda \\ \lambda I - A(x) > 0 & \\ B(x) > 0 & \end{aligned} \tag{1.78}$$

and

$$\begin{aligned} \inf_{x,\lambda} \quad & \lambda \\ A(x,\lambda) > 0 & \end{aligned} \tag{1.79}$$

where $A(x, \lambda)$ is affine in x and λ. The equivalence of (1.77), (1.78), and (1.79) can be easily seen [113].

Generalized eigenvalue problems

A large number of the control properties can be computed as a generalized eigenvalue problem (GEVP), including many robustness margins and the minimized condition number discussed [100, 113, 103]. A GEVP is, given square matrices A and B, $B > 0$, find scalars λ and nonzero vectors y such that

$$Ay - \lambda By = 0. \tag{1.80}$$

The following property of GEVPs looks trivial, but its importance cannot be ignored.

Lemma 7. *Let consider a symmetric matrix $F(\delta)$ such that F is convex in $\delta \in \mathbb{R}$. Then the open set $\Delta \triangleq \{\delta \in \mathbb{R} \,|\, F(\delta) < 0\}$ is connected and the inf/sup values of $\delta \in \Delta$ are achieved on the boundary of the* *set. That is,*

$$\delta_{low} \triangleq arg\ inf_{\delta\in\Delta}\, \delta = \inf\{\delta_l, \delta_r\}$$
$$\delta_{upp} \triangleq arg\ sup_{\delta\in\Delta}\, \delta = \sup\{\delta_l, \delta_r\},$$

where $\delta_l, \delta_r \in \bar{\bar{\delta}}$ *such that* $F(\delta_l - \epsilon) \geq 0$ *and* $F(\delta_r + \epsilon) \geq 0$ *for some* ϵ*. Furthermore, it is allowed that* $\delta_l, \delta_r \in \{-\infty, \infty\}$.

Proof: From the convexity of F in $\delta \in \mathbb{R}$, the connectivity of the open set Δ is trivial. In addition, since Δ is a connected convex set, δ_l, δ_r are the left- and right-most values in the closed set $\bar{\Delta}$. □

The computation of the largest generalized eigenvalue can be written in terms of an optimization problem with LMI-like constraints. Consider that the positive definiteness of B implies that for sufficiently large λ, $A - \lambda B < 0$. As λ is reduced from some sufficiently high value, at some point the matrix $A - \lambda B$ will lose rank, at which point there exists a nonzero vector y that solves (1.80), implying that this value of λ is the largest generalized eigenvalue. Hence

$$\lambda_{max} = \min_{A-\lambda B \leq 0} \lambda \ = \inf_{A-\lambda B < 0} \lambda. \tag{1.81}$$

Often, our objective is to minimize the largest generalized eigenvalue of two symmetric matrices which depend affinely on a variable x, subject to an LMI constraint on x:

$$\begin{array}{cc} \inf & \lambda_{max}(A(x), B(x)), \\ B(x) > 0 & \\ C(x) > 0 & \end{array} \tag{1.82}$$

where $\lambda_{max}(A(x), B(x))$ is the largest generalized eigenvalue of two matrices, A and B, each of which depend affinely on x. From (1.81) this optimization problem is equivalent to

$$\begin{array}{cc} \inf & \lambda. \\ A(x) - \lambda B(x) < 0 & \\ B(x) > 0 & \\ C(x) > 0 & \end{array} \tag{1.83}$$

1.3.2 The Bilinear Matrix Inequality and Optimization Problems

A bilinear matrix inequality (BMI) is of the form:

$$F(x, y) = F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{j=1}^{n} y_j G_j + \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j H_{ij} > 0 \tag{1.84}$$

where G_j and H_{ij} are symmetric matrices of the same dimension as F_i, and $y \in \mathcal{R}^n$. Bilinear matrix inequalities were popularized by Safonov and co-workers in a series of proceedings papers [50, 51, 52, 72], and first applied to a nontrivial process description (i.e., a chemical reactive ion etcher) by VanAntwerp and Braatz [44], and was later applied to paper machines [45]. A BMI is an LMI in x for fixed y and an LMI in y for fixed x, and so is convex in x and convex in y. The bilinear terms make the set not *jointly convex* in x and y. Besides bilinear and general quadratic inequalities

$$x^T Q x + c^T x + p > 0, \tag{1.85}$$

general polynomial inequalities can also be written as BMIs.

Since a BMI describes sets that are not necessarily convex, they can describe much wider classes of constraint sets than LMIs, and can be used to represent more types of optimization and control problems. The main drawback of BMIs is that they are much more difficult to handle computationally than LMIs.

An optimization over BMI constraints is called a BMI problem:

$$\begin{aligned} \inf_{x, y} \quad & c^T x + d^T y \\ A([x^T \ y^T]^T) &> 0 \\ F(x, y) &> 0 \end{aligned} \tag{1.86}$$

where $F(x, y)$ is defined in (1.84).

Many important problems in control that cannot be stated in terms of LMIs can be stated in terms of BMIs. Examples include robustness analysis [27, 93], a large number of robust controller synthesis problems including low order and decentralized control [72, 50], bilinear programming, and linear complementarity problems [3, 4, 95]. It has been shown that there is a corresponding optimization over BMI constraints called the BMI eigenvalue problem (BEVP) [113]:

$$\begin{aligned} \inf_{x, y, \gamma} \quad & \gamma \\ A([x^T \ y^T]^T) &> 0 \\ \lambda_{\max}(F(x, y)) &< \gamma \end{aligned} \tag{1.87}$$

where $\lambda_{\max}$ is the maximum eigenvalue of $F(x, y)$. It is easy to see that the LMI eigenvalue problem is a special case of the BMI optimization problem.

1.3.3 The S-Procedure

The S-procedure greatly extends the usefulness of LMIs by allowing non-LMI conditions that commonly arise in nonlinear and/or uncertain systems analysis to be represented as LMIs, with the introduction of some conservatism. The original question in the S-procedure is whether a quadratic inequality is satisfied over a domain that is generally represented as a set of constraints and when a quadratic inequality is a consequence of other quadratic inequalities, that is, does the nonempty set $\mathcal{C} \triangleq \{x \in \mathbb{R}^n \,|\, g_i(x) \geq 0,\, i = 1, \ldots, l\}$ imply that $f_0(x) \geq 0$?, where the functionals $f_0, g_i : \mathbb{R}^n \to \mathbb{R}$ are convex for $i = 1, \ldots, l$.

I. The Lagrange Dual Problem in Convex Optimization Context

Lagrangian duality is an promising approach in convex optimization and its development in theory and practice are far reaching. The main idea in Lagrangian duality is to associate an equivalent or weaker *dual problem* with a given constrained *primal problem*. A dual problem may be much easier to solve than the original problem while there might be some potential conservatism. First, we will see that any feasibility problem can be transformed as a corresponding equivalent optimization problem. The S-procedure is a special case of the Lagrange dual problem.

Proposition 6. *(The equivalence of a feasibility problem and a linear objective optimization problem) The feasibility problem*

$$f(x) < q \quad \textit{such that} \quad x \in \mathcal{X}$$

can be reformulated as the LP

$$\begin{array}{cc} \inf & t \\ f(x) - tI \leq q & \\ x \in \mathcal{X} & \end{array}$$

It is easy to see that the optimal solution t^* of the above LP is less than zero if and only if the inequality $f(x) < q$ holds for all $x \in \mathcal{X}$.

We should note that the reformulation of a convex min-max optimization problem may not unique, in general. Often, different reformulations have certain advantages from the point of solution approaches. For the details and examples, see [17]. Let us consider the following general (not necessarily convex) optimization problem:

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{subject to} \quad f_i(x) \le 0, \quad i = 1, \dots, l \text{ and } x \in \mathcal{X}, \end{aligned} \tag{1.88}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ and $f_i : \mathcal{X} \to \mathbb{R}$, $i = 1, \dots, l$. The equivalence of the S-procedure and the Lagrange dual problem is due to their basic idea which is to relax the constraints by augmenting the objective function with a weighted sum of the constraints.

Definition 8. *(The Lagrangian and the Lagrangian dual function [17]) Let us introduce the so-called Lagrange multiplier vector $\lambda \in \mathbb{R}^l$. Then the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ associated with the optimization problem (1.88) is defined as*

$$L(x, \lambda) = f_0(x) + \sum_{i=1}^{l} \lambda_i f_i(x)$$

and the corresponding Lagrange dual function $g : \mathbb{R}^l \to \mathbb{R}$ is defined as

$$g(\lambda) = \inf_{x \in \mathcal{X}} L(x, \lambda) = \inf_{x \in \mathcal{X}} \left(f_0(x) + \sum_{i=1}^{l} \lambda_i f_i(x) \right).$$

The Lagrange dual function g is concave, even if the primal optimization problem (1.88) is not convex.

It is well known that the dual function yields lower bounds on the optimal value of the primal problem (1.88), i.e., $g(\lambda) \le f_0(x^*)$ where x^* is an optimal solution of the problem (1.88). Now, it is natural to ask what the closest approximation based on the Lagrange dual function for the optimal value $f_0(x^*)$ is. Thus, we are led to consider the other optimization problem which is called the dual problem:

$$\begin{aligned} &\text{maximize} \quad g(\lambda) \\ &\text{subject to} \quad \lambda_i \ge 0, \quad i = 1, \dots, l. \end{aligned} \tag{1.89}$$

The main advantage to consider the Lagrange dual problem (1.89) is the fact that the Lagrange dual problem (1.89) is a convex optimization problem whether or not the primal problem (1.88) is convex. The difference between the optimal values of the problems (1.88) and (1.89) is called the dual gap and when there is no dual gap, the Lagrange dual problem (1.89) is said to be strongly dual and the strong duality corresponds to *lossless* in the S-procedure. The following lemma deals under which conditions the strong duality is guaranteed.

Lemma 8. *(Farkas' lemma and Slater's constraint qualifications [87, 17]) Let $f_0, f_i : \mathbb{R}^n \to \mathbb{R}$ be convex functionals for $i = 1, \dots, l$ and $\mathcal{X} \in \mathbb{R}^n$ ba a convex set. In addition, let us assume that the so-called Slater's* *condition [17], which is a simple constraint qualification, holds for the set* $\mathcal{C}_1 \triangleq \{x \in \mathbb{R}^n \,|\, f_i(x) \leq 0,\, i = 1, \ldots, l\}$. *That is, there exists a vector* $\tilde{x} \in$ *rel int* $\mathcal{C}_1 \cap$ *int* $\mathcal{C}_2$ *where rel int indicates the relative interior of a set and for its definition, the readers are recommended to see [99, 17]. Then the following two statements are equivalent:*

1. *The feasibility problem*

$$f_0(x) < 0 \quad \text{such that} \quad x \in \mathcal{C}_1 \cap \mathcal{C}_2$$

   is not solvable.

2. *There exists a Lagrange multiplier vector* $\lambda := (\lambda_1, \cdots, \lambda_l) \geq 0$ *such that*

$$f_0(x) + \sum_{i=1}^{l} \lambda_i f_i(x) \geq 0,\, \forall x \in \mathcal{C}_1.$$

Note that the constraint $\mathcal{C}_2$ *is relaxed in the second statement by the introduction of the Lagrange multiplier vector* λ.

II. The S-Procedure in Control Theory Context

We will mainly consider the use of the S-proceure in finite dimensional spaces and the S-procedure can applied to quadratic functions as easily as to quadratic forms. Let the real valued functionals $f_j(x)$, $j = 0, \ldots, l$ be quadratic of $x \in \mathcal{X} \subset \mathbb{R}^n$:

$$f_j(x) = x^T T_i x + 2u_i^T x + v_i, \tag{1.90}$$

where T_i's are symmetric matrices, u_i's are vectors, and v_i's are scalars. It can be easily seen that if there exists $\lambda \geq 0$ such that

$$f_0(x) - \sum_{i=1}^{l} \lambda_i f_i(x) \geq 0 \quad \forall x \in \mathcal{X}, \tag{1.91}$$

then

$$f_0(x) \geq 0 \quad \forall x \in \mathcal{X} \cap \mathcal{C}, \tag{1.92}$$

where $\mathcal{C} \triangleq \{x \in \mathbb{R}^n \,|\, f_j(x) \geq 0,\, j = 1, \ldots, l\}$. It is well known that the S-procedure is also necessary when $l = 1$ and is called *lossless*.

Theorem 19. *(Yakubovich [128]) Let* $f_j : \mathcal{X} \to \mathbb{R}$, $j = 0, \ldots, l$ *and assume the constraints* $f_j(x) \geq 0$ *for all*

$j = 1, \ldots, l$ are regular, i.e., there exists $\tilde{x} \in \mathcal{X}$ such that $f_j(x) > 0$. In addition, let us define sets

$$\mathcal{C}_f \triangleq \{f(x) \in \mathbb{R}^{l+1} \mid f(x) = (f_0(x), \cdots, f_l(x))^T,\, x \in \mathcal{X}\},$$
$$\mathcal{L} \triangleq \{\lambda \in \mathbb{R}^{l+1} \mid \lambda = (\lambda_0, \cdots, \lambda_l)^T\}.$$

Then if $\mathcal{C}_f \cap \mathcal{L} = \emptyset$ implies that $\boldsymbol{Co}(\mathcal{C}_f) \cap \mathcal{L} = \emptyset$ then the S-procedure is lossless.

Proof: The separating hyperplane theorem [69, 17] plays a crucial role in the proof. The details of the proof are in [128]. □

Remark 5. *As a special case, if the set $\mathcal{C}_f$ is convex set, which is guaranteed when the functionals f_j's are convex in $x \in \mathcal{X}$, then the S-procedure is lossless.*

Sometimes it is very important to know whether the S-procedure is lossless for a certain class of functionals f_j.

Proposition 7. *(Lossless S-procedure [75]) There are some special classes of functionals with which the S-procedure is lossless.*

1. *If $f_j(x)$'s are linear functionals in $x \in \mathcal{X}$, $\mathcal{X}$ is a convex set, and there exists a vector $\tilde{x} \in$ rel int $\mathcal{X}$ such that $\tilde{x} \in \mathcal{C}$, then the S-procedure is lossless.*

2. *The S-procedure is lossless in the case when $l = 1$ and the functionals are all quadratic forms on a real linear space $\mathcal{X}$.*

3. *The S-procedure is lossless in the case when $l = 2$ and the functionals are all Hermitian forms on a complex linear space $\mathcal{X}$.*

We should note that the first statement is nothing but Slater's constraint qualification.

Proof: The proof is given in [75]. □

Example: The feasibility problem (1.92) is equivalent to the following LMI condition in the Lagrange multiplier vector λ:

$$\begin{bmatrix} T_0 & u_0 \\ u_0^T & v_0 \end{bmatrix} - \sum_{i=1}^{l} \lambda_i \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix} \geq 0. \tag{1.93}$$

Comment 3. *The S-procedure also holds for the case of a strict inequality.*

Remark 6. *(Multidimensional Circle and Popov criteria) Consider the set $\bar{\Phi}_{sb}^{[K_1,K_2]}$ of sector-bounded static nonlinear functions:*

$$\bar{\Phi}_{sb}^{[K_1,K_2]} \triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p} \,|\, (\phi(\sigma) - K_1\sigma)^T(\phi(\sigma) - K_2\sigma) \leq 0, \, \forall \sigma \in \mathbb{R}^{n_q}\},$$

where $K_i \in \mathbb{R}^{n_p \times n_q}$ for each index i and the nonlinear function $\phi(\sigma)$ is Lebesgue measurable for all $\sigma \in \mathbb{R}^{n_q}$. Then the relation in the definition of the set $\bar{\Phi}_{sb}^{[K_1,K_2]}$ can be represented as an equivalent LMI

$$f_1(x,\phi) \triangleq \begin{bmatrix} x \\ \phi \end{bmatrix}^T \begin{bmatrix} C_q^T(K_1^TK_2 + K_2^TK_1^T)C_q & \begin{pmatrix} C_q^T(K_1+K_2)^T \\ -C_q^T(K_1^TK_2 + K_2^TK_1)D_{qp} \end{pmatrix} \\ \begin{pmatrix} (K_1+K_2)C_q \\ -D_{qp}^T(K_1^TK_2 + K_2^TK_1)C_q \end{pmatrix} & \begin{pmatrix} 2I - (K_1+K_2)D_{qp} - D_{qp}^T(K_1+K_2)^T \\ +D_{qp}^T(K_1^TK_2 + K_2^TK_1)D_{qp} \end{pmatrix} \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} \leq 0$$

With a predefined Lyapunov function $V(x)$ and its time-derivative (or time-difference) function which is represented by the inequality $f_0(x,\phi) \triangleq \frac{d}{dt}V(x) < 0$ for the continuous-time systems or $\Delta V(x) < 0$ for the discrete-time systems, if there exists a scalar $\lambda \geq 0$ such that the inequality $f_0(x,\phi) - \lambda f_1(x,\phi) < 0$ holds for all $(x,\phi) \in \mathbb{R}^n \times \mathbb{R}^{n_p}$ then the functional $\frac{d}{dt}V(x)$ (or $\Delta V(x)$) is negative for all $(x,\phi) \in \mathbb{R}^n \times \bar{\Phi}_{sb}^{[K_1,K_2]}$. Now, consider the set $\bar{\Phi}_{sr}^{[M_1,M_2]}$ of slope-restricted static nonlinear functions:

$$\bar{\Phi}_{sr}^{[M_1,M_2]} \triangleq \{\phi : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p} \,|\, (\phi'(\sigma) - M_1\sigma)^T(\phi'(\sigma) - M_2\sigma) \leq 0, \, \forall \sigma \in \mathbb{R}^{n_q}\},$$

where $M_i \in \mathbb{R}^{n_p \times n_q}$ for each index i and the derivative of the nonlinear function $\phi'(\sigma)$ is also Lebesgue measurable for all $\sigma \in \mathbb{R}^{n_q}$. Similarly to $\bar{\Phi}_{sb}^{[K_1,K_2]}$, the relation in the definition of the set $\bar{\Phi}_{sr}^{[M_1,M_2]}$ can be represented as an equivalent LMI

$$f_2(x,\phi,\phi') \triangleq \begin{bmatrix} x \\ \phi \\ \phi' \end{bmatrix}^T \begin{bmatrix} C_q^T(M_1^TM_2 + M_2^TM_1^T)C_q & C_q^T(M_1^TM_2 + M_2^TM_1)D_{qp} & -C_q^T(M_1+M_2)^T \\ D_{qp}^T(M_1^TM_2 + M_2^TM_1)C_q & D_{qp}^T(M_1^TM_2 + M_2^TM_1)D_{qp} & -D_{qp}^T(M_1+M_2)^T \\ -(M_1+M_2)C_q & -(M_1+M_2)D_{qp} & I \end{bmatrix} \begin{bmatrix} x \\ \phi \\ \phi' \end{bmatrix} \leq 0$$

for the continuous-time case. Therefore, if there exists a scalar $\lambda_1, \lambda_2 \geq 0$ such that the inequality $f_0(x,\phi) - \lambda_1 f_1(x,\phi) - \lambda_2 f_2(x,\phi,\phi') < 0$ holds for all $(x,\phi,\phi') \in \mathbb{R}^n \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_p}$, then the functional $\frac{d}{dt}V(x)$ is negative for all $(x,\phi) \in \mathbb{R}^n \times \left(\bar{\Phi}_{sb}^{[K_1,K_2]} \cap \bar{\Phi}_{sr}^{[M_1,M_2]}\right)$.

Proof: The proof is just simple algebraic computation using the S-procedure and omitted due to the limited space. □

Chapter 2

Stability and Performance Analysis for Lur'e Systems

This chapter considers the analysis of stability and performance of Lur'e systems. Sufficient conditions for the existence of a certain type of Lyapunov function, a modified Lur'e-Postnikov function, are given in terms of linear matrix inequalities (LMIs). The derived critera for the analysis of stability and performance are shown to be less conservative than criteria published in the literature. Extensions to cover the performance analysis for a specific class of uncertain nonlinearities are also included.

2.1 Introduction

Absolute stability theory considers the stability of a nominal linear time-invariant system interconnected with a static nonlinearity, that is, a real function of real variables with no internal states. The literature on absolute stability theory can be categorized in terms of the properties of static nonlinearities that are considered. This chapter develops new sufficient conditions for the stability analysis of discrete-time systems with the advance being LMI conditions that exploit the static, sector-bounded, and globally slope-restricted nature of the componentwise nonlinear operators. The results are extended to address performance analysis for discrete-time systems [98].

Lyapunov methods provide a simple but powerful way to analyze nonlinear systems and design stabilizing controllers [100, 22, 55, 57, 23]. Many well-known results on absolute stability were developed for a benchmark problem called the Lur'e problem [76, 53, 84, 105]. The Popov and Circle criteria are frequency-domain conditions for absolute stability of the feedback interconnection of a continuous linear time-invariant system with a sector-bounded nonlinearity [32, 35, 56, 62, 106, 132]. Its counterpart for a discrete-time system is known as the Tsypkin criterion [60, 85] and the Jury-Lee criterion [49, 48]. Consider the nonlinear system:

$$x(k+1) = Ax(k) + Bp(k); \quad q(k) = Cx(k) + Dp(k); \quad p(k) = -\phi(q(k)), \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_p}$, $C \in \mathbb{R}^{n_q\times n}$, $D \in \mathbb{R}^{n_q\times n_p}$, the nonlinear operator is given by $\phi \in \Phi$, and Φ is a set of static functions that satisfy $\phi(0) \equiv 0$ and have some specified input-output characteristics. This chapter considers several classes of nonlinear operator $\phi(\cdot)$ that are applied componentwise to the elements of the vectors, defined in Definition 7. Suppose that the state-space system representation (A, B, C, D) is a minimal realization of the transfer function $G(z) = C(zI - A)^{-1}B + D$ such that the triplet (A, B, C) is controllable and observable. [Alternatively, the controllability and observability conditions can be removed because strict inequalities will be used in the stability criteria (see Section 2 in [96]). Instead, the condition $\text{Ker}A \cap \text{Ker}B = \{0\}$, which implies that the origin is the one equilibria, is assumed to hold.] As explained in Chapter 1, the frequency-domain conditions—the Circle, Popov and Tsypkin criteria—for the absolute stability of the system (2.1) are simply represented by linear matrix inequalities (LMIs) written in terms of time-domain systems representation, (A, B, C, D), and some matrix variables whose existence corresponds to the feasibility of LMIs such that determine the stability of the system via the celebrated positive real lemma and KYP lemma [100, 85, 23].

The S-procedure discussed in Chapter **??** converts constraints over the nonlinearities into LMI conditions, which is a key step in the direct derivation of LMI-based tests for absolute stability [100, 75, 84, 129]. Related results have been derived based on integral quadratic constraints (IQCs) [30, 46, 47, 74, 75, 96] of systems with repeated monotonic or slope-restricted nonlinearities in continuous-time cases. Since a quadratic Lyapunov function yields substantial conservatism, we introduce a modified Lur'e-Postnikov function for less conservative analysis of stability, state performance, and input-output performance. The resulting LMI problems are solved using off-the-shelf software [83, 108, 66].

2.2 Sufficient Conditions for Asymptotic Stability–Lyapunov Stability Analysis

2.2.1 Modified Lur'e-Postnikov Stability of a Discrete Time Lur'e System

Due to the conservativeness inherent to quadratic Lyapunov functions when applied to nonlinear systems, this section considers the use of a modified Lur'e-Postnikov function that includes integral terms whose sign definiteness are implied by the sector-bounded property of the nonlinear operators ($\phi \in \Phi_{sb}^{[0,\xi]}$). To derive sufficient conditions for the origin of (2.1) to be globally asymptotically stable, consider the Lyapunov function:

$$V(x_k) = \bar{x}_k^T P \bar{x}_k + 2\sum_{i=1}^{n_q} Q_{ii} \int_0^{q_{k,i}} \phi_i(\sigma) d\sigma + 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_0^{q_{k,i}} [\xi_i \sigma - \phi_i(\sigma)]\, d\sigma, \tag{2.2}$$

where

$$\bar{x}_k \triangleq \begin{bmatrix} x_k \\ p_k \\ q_k \end{bmatrix}, \quad P^T = P \triangleq \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix} \geq 0, \quad P_{11} > 0, \quad Q_{ii} \geq 0, \quad \tilde{Q}_{ii} \geq 0, \quad \forall i = 1, \ldots, n_q$$

and the subscript k indicates a sampling instance. Both p_k and q_k are functions of the state variable vector x_k, and the above Lyapunov function is radially unbounded and positive for all nonzero $x_k \in \mathbb{R}^n$.

For $P_{12} = 0, P_{13} = 0, P_{22} = 0, P_{23} = 0, P_{33} = 0, \tilde{Q} \triangleq \operatorname{diag}\{\tilde{Q}_{ii}\} = 0$, the Lyapunov function (2.2) reduces to the Lur'e-Postnikov Lyapunov function used in the derivation of the Popov criterion. All other Lyapunov functions considered in the Lur'e problem literature are subsets of (2.2). Some papers in the Lur'e stability literature that appear to have some additional terms than (2.2) are actually just introducing the S-procedure in a different way. The dependence of the Lyapunov function (2.2) on the nonlinearities and the states provide additional degrees of freedom for reducing conservatism during stability and performance analysis. The application of the S-procedure provides a standard way to consider more restrictive classes of nonlinearities without having modify the Lyapunov function. The degree of conservatism will be compared for sub-problems with different subsets of the Lyapunov function (2.2), to assess the reduction in conservatism posed by each term.

2.2.2 Lagrange Relaxations

Yakubovich showed that the positiveness of a quadratic function $f_0(x)$ in a constraint set expressed in terms of quadratic functions, say $f_i(x)$ $i = 1, \ldots, m$, can be implied by the relaxed form with (Lagrange) multipliers. The S-procedure is a special case of Lagrange relaxation in which the constraints are represented in terms of quadratic functions, so that the multipliers can be combined into an LMI inequality. The application of the S-procedure appears not only in control theory but also in the more generic optimization literature. This chapter applies the S-procedure in finite-dimensional spaces and especially for quadratic inequalities (see Chapter **??** for the details).

Lemma 9. *(S-procedure for Quadratic Inequalities) For the Hermitian matrices Θ_i, $i = 0, \ldots, m$, consider the two sets:*

(S1) $\zeta^* \Theta_0 \zeta < 0, \quad \forall \zeta \in \Phi \triangleq \{\zeta \in \mathbb{F}^n | \zeta^* \Theta_i \zeta \leq 0, \ \forall i = 1, \ldots, m\}$*;*

(S2) $\exists \tau_i \geq 0$, $i = 1, \ldots, m$ *such that* $\Theta_0 - \sum_{i=1}^m \tau_i \Theta_i < 0$.

The feasibility of (S2) implies (S1).

2.2.3 Discrete-Time Lur'e Systems with Slope-restricted Nonlinearities

While conic-sector bounded nonlinearities, which bound the global slope of the nonlinear functions $\phi(\cdot)$, have been heavily studied in the literature, such a description allows the local slope of the nonlinear functions $\phi(\cdot)$ to vary arbitrarily from one time instance to another. That is, such nonlinearities do not imposes any *local* slope restriction. Most nonlinearities in practice have a local slope restriction, in which case a less conservative analysis condition may be achieved if these constraints on the nonlinearities are included in the analysis via the S-procedure. A local slope restriction also provides an upper bound on the change of the integral term in the Lyapunov function, provided that $\phi(\cdot)$ is continuous almost everywhere (a.e.).

Theorem 20. *Consider a system of the form (2.1) with the memoryless nonlinearity $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$ that is continuous almost everywhere. A sufficient condition for global asymptotic stability is the existence of a positive semidefinite matrix $P = P^T$ with a positive definite submatrix $P_{11} = P_{11}^T$ and diagonal positive semidefinite matrices Q, $\tilde{Q}$, T, $\tilde{T}$, $N \in \mathbb{R}^{n_q \times n_q}$ such that*

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}^T & G_{22} & G_{23} \\ G_{13}^T & G_{23}^T & G_{33} \end{bmatrix} < 0 \tag{2.3}$$

where

$$G_{11} = A^T(P_{11} + P_{13}C + C^T P_{13}^T + C^T P_{33}C)A - P_{11} - P_{13}C - C^T P_{13}^T - C^T P_{33}C + A^T C^T \tilde{Q}\xi CA - C^T \tilde{Q}\xi C$$

$$G_{12} = A^T(P_{11} + P_{13}C + C^T P_{13}^T + C^T P_{33}C)B - P_{12} - P_{13}D - C^T P_{23}^T - C^T P_{33}D - C^T T + (CA - C)^T Q$$
$$+ A^T C^T \tilde{Q}\xi CB + (CA - C)^T \tilde{Q} - C^T \tilde{Q}\xi D + (CA - C)^T \mu N$$

$$G_{13} = A^T P_{12} + A^T P_{13}D + A^T C^T P_{23}^T + A^T C^T P_{33}D - A^T C^T \tilde{T} - (CA - C)^T Q + A^T C^T \tilde{Q}\xi D - (CA - C)^T \mu N$$

$$G_{22} = B^T(P_{11} + P_{13}C + C^T P_{13}^T + C^T P_{33}C)B - P_{22} - P_{23}D - D^T P_{23}^T - D^T P_{33}D - 2\xi^{-1}T - TD - D^T T$$
$$+ Q(CB - D) + (CB - D)^T Q - \mu^{-1}Q + B^T C^T \tilde{Q}\xi CB - D^T \tilde{Q}\xi D + (CB - D)^T \tilde{Q} + \tilde{Q}(CB - D) - \mu^{-1}\tilde{Q}$$
$$- 2N + N\mu(CB - D) + (CB - D)^T \mu N$$

$$G_{23} = B^T P_{12} + B^T P_{13}D + B^T C^T P_{23}^T + B^T C^T P_{33}D - B^T C^T \tilde{T} + QD + Q\mu^{-1} + \mu^{-1}\tilde{Q} + \tilde{Q}D + B^T C^T \tilde{Q}\xi D$$
$$+ 2N - (CB - D)^T \mu N + N\mu D$$

$$G_{33} = P_{22} + P_{23}D + D^T P_{23}^T + D^T P_{33}D - 2\xi^{-1}\tilde{T} - \tilde{T}D - D^T \tilde{T} - QD - D^T Q - Q\mu^{-1} - \mu^{-1}\tilde{Q} + D^T \tilde{Q}\xi D$$
$$- 2N - N\mu D - D^T \mu N$$

Proof: The Lyapunov function (2.2) is used to derive the sufficient condition for global asymptotic stability. The difference in the Lyapunov function between the $k+1$ and k sampling instances is

$$\Delta V(x_k) = \zeta_k^T (A_a^T P A_a - E_a^T P E_a)\zeta_k + 2\sum_{i=1}^{n_q} Q_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma) d\sigma + 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \left[\xi_i \sigma - \phi_i(\sigma)\right] d\sigma, \quad (2.4)$$

where

$$\zeta_k \triangleq \begin{bmatrix} x_k \\ p_k \\ p_{k+1} \end{bmatrix}, \quad A_a \triangleq \begin{bmatrix} A & B & 0 \\ 0 & 0 & I \\ CA & CB & D \end{bmatrix}, \quad E_a \triangleq \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ C & D & 0 \end{bmatrix}$$

Slope restrictions on the nonlinearities place an upper bound on the first integral:

$$\begin{aligned} 2\sum_{i=1}^{n_q} Q_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \phi(\sigma) d\sigma &\leq 2\sum_{i=1}^{n_q} Q_{ii} \left\{ (\phi_{k+1,i} - \phi_{k,i})(q_{k+1,i} - q_{k,i}) - \frac{1}{2\mu_i}(\phi_{k+1,i} - \phi_{k,i})^2 \right\} \\ &= \zeta_k^T U_1 \zeta_k, \end{aligned}$$

where

$$U_1 \triangleq \begin{bmatrix} 0 & (CA - C)^T Q & -(CA - C)^T Q \\ * & Q(CB - D) + (CB - D)^T Q - Q\mu^{-1} & QD + Q\mu^{-1} \\ * & * & -QD - D^T Q - Q\mu^{-1} \end{bmatrix}.$$

Similarly, an upper bound can be derived on the second integral:

$$\begin{aligned} 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \left[\xi_i \sigma - \phi(\sigma)\right] d\sigma &= -2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \phi(\sigma) d\sigma + 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \xi_i \sigma d\sigma \\ &\leq -2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \left\{ \frac{1}{2\mu_i}(\phi_{k+1,i} - \phi_{k,i})^2 + \phi_{k,i}(q_{k+1,i} - q_{k,i}) \right\} + \sum_{i=1}^{n_q} \tilde{Q}_{ii}\xi_i \left[q_{k+1,i}^2 - q_{k,i}^2\right] \\ &= \zeta_k^T U_2 \zeta_k, \end{aligned}$$

where

$$U_2 \triangleq \begin{bmatrix} A^T C^T \tilde{Q}\xi CA - C^T \tilde{Q}\xi C & A^T C^T \tilde{Q}\xi CB + (CA - C)^T \tilde{Q} - C^T \tilde{Q}\xi D & A^T C^T \tilde{Q}\xi D \\ * & \begin{pmatrix} B^T C^T \tilde{Q}\xi CB - D^T \tilde{Q}\xi D \\ +(CB - D)^T \tilde{Q} + \tilde{Q}(CB - D) - \mu^{-1}\tilde{Q} \end{pmatrix} & \mu^{-1}\tilde{Q} + \tilde{Q}D + B^T C^T \tilde{Q}\xi D \\ * & * & -\mu^{-1}\tilde{Q} + D^T \tilde{Q}\xi D \end{bmatrix}.$$

Since the (negative) feedback-connected nonlinear function is monotonic with slope restriction in addition to being $[0,\xi]$ sector bounded, i.e., $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$, it can be shown that the following inequalities are satisfied at each sampling instance k and all indices $i = 1, \dots, n_q$:

$$\phi_{k,i}[\xi_i^{-1}\phi_{k,i} - q_{k,i}] \le 0, \tag{2.5}$$

$$(\phi_{k+1,i} - \phi_{k,i})[\mu_i^{-1}(\phi_{k+1,i} - \phi_{k,i}) - (q_{k+1,i} - q_{k,i})] \le 0. \tag{2.6}$$

The following notations based on (2.5) are useful when applying the S-procedure:

$$\sum_{i=1}^{n_q} 2\tau_i\phi_{k,i}[\xi_i^{-1}\phi_{k,i} - q_{k,i}] = \zeta_k^T S_1 \zeta_k, \quad \sum_{i=1}^{n_q} 2\tilde{\tau}_i\phi_{k+1,i}[\xi_i^{-1}\phi_{k+1,i} - q_{k+1,i}] = \zeta_k^T S_2 \zeta_k, \tag{2.7}$$

where

$$S_1 \triangleq \begin{bmatrix} 0 & C^T T & 0 \\ * & 2\xi^{-1}T + TD + D^T T & 0 \\ * & * & 0 \end{bmatrix}, \quad S_2 \triangleq \begin{bmatrix} 0 & 0 & A^T C^T \tilde{T} \\ * & 0 & B^T C^T \tilde{T} \\ * & * & 2\xi^{-1}\tilde{T} + \tilde{T}D + D^T\tilde{T} \end{bmatrix}.$$

A similar notation based on the inequality (2.6) is:

$$\sum_{i=1}^{n_q} 2N_{ii}(\phi_{k+1,i} - \phi_{k,i})\left[\mu_i^{-1}(\phi_{k+1,i} - \phi_{k,i}) - (q_{k+1,i} - q_{k,i})\right] = \zeta_k^T S_3 \zeta_k, \tag{2.8}$$

where

$$S_3 \triangleq \begin{bmatrix} 0 & -(CA - C)^T N & (CA - C)^T N \\ * & 2N\mu^{-1} - (CB - D)^T N - N(CB - D) & -2N\mu^{-1} + (CB - D)^T N - ND \\ * & * & 2N\mu^{-1} + D^T N + ND \end{bmatrix}.$$

Applying the S-procedure, if the LMI $G \triangleq A_a^T P A_a - E_a^T P E_a + U_1 + U_2 - S_1 - S_2 - S_3 < 0$ is feasible then $\Delta V(x_k) < 0$ is satisfied for the specific class of feedback-connected nonlinearities $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$. All of introduced matrix (decision) variables are of compatible dimensions. □

Remark 7. *The upper bounds on the integral terms in (2.4) are derived by considering the a.e. continuity of the nonlinearity $\phi(\cdot)$. In those upper bounds the slope-restricted properties of the nonlinear feedback are exploited, which gives sharper bounds than the upper bounds in [36, 37, 55].*

2.2.4 Discrete-Time Lur'e Systems with Slope-restricted and Odd Monotonic Nonlinearities

Theorem 20 exploits more information on the nonlinear operator than the original Lur'e problem. This section derives a less conservative sufficient stability condition for the more restrictive class of nonlinearities that are odd monotonic, by the introduction of additional quadratic constraints.

Theorem 21. *Consider a system of the form (2.1) with the memoryless nonlinearity $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]} \cap \Phi_{odd}$ which is continuous almost everywhere. A sufficient condition for global asymptotic stability is the existence of a positive semidefinite matrix $P = P^T$ with a positive definite submatrix $P_{11} = P_{11}^T$ and diagonal positive semidefinite matrices Q, $\tilde{Q}$, T, $\tilde{T}$, N, L, and $\tilde{L} \in \mathbb{R}^{n_q \times n_q}$ such that*

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}^T & G_{22} & G_{23} \\ G_{13}^T & G_{23}^T & G_{33} \end{bmatrix} < 0 \tag{2.9}$$

where

$$G_{11} = A^T(P_{11} + P_{13}C + C^T P_{13}^T + C^T P_{33}C)A - P_{11} - P_{13}C - C^T P_{13} - C^T P_{33}C + A^T C^T \tilde{Q}\xi CA - C^T \tilde{Q}\xi C$$

$$G_{12} = A^T(P_{11} + P_{13}C + C^T P_{13}^T + C^T P_{33}C)B - P_{12} - P_{13}D - C^T P_{23}^T - C^T P_{33}D - C^T T$$
$$+ (CA - C)^T Q + A^T C^T \tilde{Q}\xi CB + (CA - C)^T \tilde{Q} - C^T \tilde{Q}\xi D + (CA - C)^T \mu N - (CA - C)^T L - (CA + C)^T \tilde{L}$$

$$G_{13} = A^T P_{12} + A^T P_{13}D + A^T C^T P_{23}^T + A^T C^T P_{33}D - A^T C^T \tilde{T} - (CA - C)^T Q + A^T C^T \tilde{Q}\xi D$$
$$- (CA - C)^T \mu N - (CA - C)^T L - (CA - C)^T \tilde{L}$$

$$G_{22} = B^T(P_{11} + P_{13}C + C^T P_{13}^T + C^T P_{33}C)B - P_{22} - P_{23}D - D^T P_{23}^T - D^T P_{33}D - 2\xi^{-1}T - TD - D^T T$$
$$+ Q(CB - D) + (CB - D)^T Q - \mu^{-1}Q + B^T C^T \tilde{Q}\xi CB - D^T \tilde{Q}\xi D + (CB - D)^T \tilde{Q} + \tilde{Q}(CB - D) - \mu^{-1}\tilde{Q}$$
$$- 2N + N\mu(CB - D) + (CB - D)^T \mu N - (CB - D)^T L - L(CB - D) - (CB + D)^T \tilde{L} - \tilde{L}(CB + D)$$

$$G_{23} = B^T P_{12} + B^T P_{13}D + B^T C^T P_{23}^T + B^T C^T P_{33}D - B^T C^T \tilde{T} + QD + Q\mu^{-1} + \mu^{-1}\tilde{Q} + \tilde{Q}D + B^T C^T \tilde{Q}\xi D$$
$$+ 2N - (CB - D)^T \mu N + N\mu D - (CB - D)^T L - LD - \xi^{-1}L - (CB - D)^T \tilde{L} - \tilde{L}D + \mu^{-1}\tilde{L}$$

$$G_{33} = P_{22} + P_{23}D + D^T P_{23}^T + D^T P_{33}D - 2\xi^{-1}\tilde{T} - \tilde{T}D - D^T \tilde{T} - QD - D^T Q - Q\mu^{-1} - \mu^{-1}\tilde{Q} + D^T \tilde{Q}\xi D$$
$$- 2N - N\mu D - D^T \mu N - D^T L - LD - \tilde{L}D - D^T \tilde{L}$$

Proof: The difference in the Lyapunov function (2.2) between the $k+1$ and k sampling instances is

$$\Delta V(x_k) = \zeta_k^T (A_a^T P A_a - E_a^T P E_a)\zeta_k + \ 2\sum_{i=1}^{n_q} Q_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \phi_i(\sigma) d\sigma + \ 2\sum_{i=1}^{n_q} \tilde{Q}_{ii} \int_{q_{k,i}}^{q_{k+1,i}} \left[\xi_i \sigma - \phi_i(\sigma)\right] d\sigma, \quad (2.10)$$

where ζ_k, A_a, and E_a are the same as in Theorem 20. Sector-bounded, slope-restricted, odd-monotonic feedback-connected nonlinearities satisfy (2.5) and (2.6) at each sampling time k and all indices $i = 1, \ldots, n_q$ for $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]} \cap \Phi_{odd}$ and satisfy

$$(q_{k+1,i} - q_{k,i})[\phi_{k+1,i} + \phi_{k,i}] - \phi_{k,i} \frac{1}{\xi_i} \phi_{k+1,i} \geq 0, \quad (2.11)$$

$$(q_{k+1,i} - q_{k,i})[\phi_{k+1,i} + \phi_{k,i}] - \phi_{k,i} \frac{1}{\xi_i} \phi_{k+1,i} \leq 2 q_{k+1,i} \phi_{k+1,i}. \quad (2.12)$$

The following notations are motivated by (2.11) and (2.12):

$$\sum_{i=1}^{n_q} 2 L_{ii} \left[\phi_{k,i} \xi_i^{-1} \phi_{k+1,i} - (q_{k+1,i} - q_{k,i})(\phi_{k+1,i} + \phi_{k,i})\right] = \zeta_k^T S_4 \zeta_k, \quad (2.13)$$

where

$$S_4 \triangleq \begin{bmatrix} 0 & (CA - C)^T L & (CA - C)^T L \\ * & (CB - D)^T L + L(CB - D) & (CB - D)^T L + LD + \xi^{-1} L \\ * & * & D^T L + LD \end{bmatrix},$$

and

$$\sum_{i=1}^{n_q} 2 \tilde{L}_{ii} \left[q_{k,i}(\phi_{k+1,i} - \phi_{k,i}) - q_{k+1,i}(\phi_{k+1,i} + \phi_{k,i}) - \phi_{k,i} \frac{1}{\xi_i} \phi_{k+1,i} \right] = \zeta_k^T S_5 \zeta_k, \quad (2.14)$$

where

$$S_5 \triangleq \begin{bmatrix} 0 & (CA + C)^T \tilde{L} & (CA - C)^T \tilde{L} \\ * & (CB + D)^T \tilde{L} + \tilde{L}(CB + D) & (CB - D)^T \tilde{L} + \tilde{L} D - \xi^{-1} \tilde{L} \\ * & * & D^T \tilde{L} + \tilde{L} D \end{bmatrix}.$$

If the LMI $G = A_a^T P A_a - E_a^T P E_a + U_1 + U_2 - S_1 - S_2 - S_3 - S_4 - S_5 < 0$ is feasible then $\Delta V(x_k) < 0$ is satisfied for the class of feedback-connected nonlinearities $\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]} \cap \Phi_{odd}$. All of matrix (decision) variables introduced are of compatible dimensions. □

Remark 8. *The inequality constraints over the odd monotonic nonlinearities (2.11) and (2.12) can be found in [53, 110, 70]. The inequalities (2.11) and (2.12) are written in terms of quadratic functions of ζ_k, so that the S-procedure is applied to combine the constraints with the negative definite condition over $\Delta V(x_k)$.*

Remark 9. *(LMIs and FDI) The well-known KYP lemma replaces an LMI with an equivalent frequency-*

domain inequality (FDI) in terms of the system transfer function [123]. The FDI provides a graphical tool to determine feasibility of the LMI for systems with one nonlinearity, i.e. $n_p = n_q = 1$, such as used in the graphical implementation of the Popov stability criterion. It is straightforward to apply the KYP lemma to transform each LMI (2.3) and (2.9) into an FDI (not shown for space considerations).

Theorem 20 and 21 described LMI feasibility problems. That is, each criterion ask whether the set $\{X \in \mathbb{S}^n | \Psi(X) < 0, X \geq 0\}$ is empty or non-empty. The Lemma below shows the equivalence between two sets of matrix decision variables.

Lemma 10. *(Feasibility issues in the strict and non-strict LMIs) The set of symmetric matrices affine in X, $\{X \in \mathbb{S}^n | \Psi(X) < 0, X \geq 0\}$, is nonempty if and only if the set of symmetric matrices, $\{X \in \mathbb{S}^n | \Psi(X) < 0, X > 0\}$ is nonempty.*

Proof: The (only if) part is obvious. To prove the (if) part, the LMIs in the set can be rewritten as:

$$\Psi(X) = F_0 + \sum_{i=1}^{N} F_i x_i, \tag{2.15}$$

where $X = X^T \in \mathbb{R}^{n \times n}$, $N = \frac{n(n+1)}{2}$, and the F_i are of compatible dimension. Suppose that $X^0 \in \{X \in \mathbb{S}^n | \Psi(X) < 0, X \geq 0\}$. With the definition $X^\delta \triangleq X^0 + \delta I$, $X^\delta > 0$ for any value of positive scalar $\delta > 0$. Now we will show that there exists a scalar $\delta > 0$ such that X^δ is in the set $\{X \in \mathbb{S}^n | \Psi(X) < 0, X > 0\}$. Consider

$$\Psi(X^\delta) = F_0 + \sum_{i=1}^{N} F_i (x_i + \delta_i), \tag{2.16}$$

where $\delta_i \in \{0, \delta\}$ for each $i = 1, \ldots, N$ and the number of nonzero δ_i is n. Since the eigenvalues of $\Psi(X^\delta)$ are continuous for $\delta \to 0$, these eigenvalues approach the eigenvalues of $\Psi(X^0)$ as $\delta \to 0$. In particular, for sufficiently small δ the eigenvalues of $\Psi(X^\delta)$ have negative real part so that $\Psi(X^\delta) < 0$. □

Remark 10. *(Computation for strict and non-strict LMIs in the interior-point algorithm) From the results of Lemma 10, any LMI solver that is guaranteed to converge for an LMI feasibility problem with strict inequalities will converge for the above LMI feasibility problems.*

2.3 Numerical Examples for Stability Analysis

This section provides numerical examples to illustrate the analysis results in Section 2.2. All LMI computations were performed with *MATLAB's LMI Control Toolbox*. The sector bounds ξ_i and slope

Stability Criterion	Ex 1	Ex 2	Ex 3	Ex 4	Ex 5	Ex 6
Circle ($\phi \in \Phi_{sb}^{[0,\xi]}$)	1.0273	0.18358	0.21792	2.91387	0.03660	0.03716
Tsypkin ($\phi \in \Phi_{sb}^{[0,\xi]}$)	1.0273	0.18358	0.21792	2.91387	0.03660	0.03716
Haddad et al. ($\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$)	1.0273	0.18358	0.21792	2.91387	0.03660	0.03716
Kapila et al. ($\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\infty]}$)	1.0273	0.18358	0.21792	2.91387	0.03660	0.03716
Park et al. ($\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\infty]}$)	1.7252	0.18358	0.21792	2.91387	0.03660	0.03716
Theorem 1 ($\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]}$)	2.4475	0.73082	0.30203	43.40412	19.18289	0.04613
Theorem 2 ($\phi \in \Phi_{sb}^{[0,\xi]} \cap \Phi_{sr}^{[0,\mu]} \cap \Phi_{odd}$)	2.5576	0.73082	0.83686	43.40412	19.18289	0.18975

Table 2.1: The maximal upper bound on the sector bound

restrictions μ_i in each element ϕ_i of the nonlinearities were taken to be the same. The maximum upper bound on the ξ was computed, where μ was taken to be linear in ξ, as shown in each example. A large enough value for ξ makes the LMI in Theorem 20 (or 21) infeasible. A small enough value for ξ makes the LMI feasible provided that the system is nominally stable, that is, the system without nonlinearities is stable. These two values provide upper and lower bounds for the value of ξ for which the LMI switches from being feasible to being feasible. The maximum ξ for which the LMI in each criterion is feasible were computed by the bisection method.

As discussed in the theory section, the new stability criteria were derived for a Lyapunov function that has more degrees of freedom than past stability criteria and so have the potential for being less conservative. Table 1 shows that the criteria in Theorems 1 and 2 are less conservative for the six numerical examples. As expected, the stability margins are larger when more information is provided on the nonlinearities connected with the LTI system, and Theorems 20 and 21 provide more accurate estimates of the upper bound on ξ or μ. Although Theorems 20 and 21 apply to semiproper and strictly proper systems, the numerical examples only include strictly proper systems to allow comparison to the other criteria in the literature, which assume $D \equiv 0$.

1. The following example is from [85]:

   $\mu = 2\xi$:

$$G(z) = \frac{-0.5z + 0.1}{(z^2 - z + 0.89)(z + 0.1)}$$

2. The following numerical example has 5 states and 2 nonlinearities:

$\mu = \xi$:

$$A = \begin{bmatrix} 0.2948 & 0 & 0 & 0 & 0 \\ 0 & 0.4568 & 0 & 0 & 0 \\ 0 & 0 & 0.0226 & 0 & 0 \\ 0 & 0 & 0 & 0.3801 & 0 \\ 0 & 0 & 0 & 0 & -0.3270 \end{bmatrix}, \quad B = \begin{bmatrix} -1.1878 & 0.2341 \\ -2.2023 & 0.0215 \\ 0.9863 & -1.0039 \\ -0.5186 & -0.9471 \\ 0.3274 & -0.3744 \end{bmatrix},$$

$$C = \begin{bmatrix} -1.1859 & 1.4725 & -1.2173 & -1.1283 & -0.2611 \\ -1.0559 & 0.0557 & -0.0412 & -1.3493 & 0.9535 \end{bmatrix}, \quad D = 0_{2\times 2}.$$

3. The following numerical example has a dense A-matrix and 2 nonlinearities:

   $\mu = \xi$:

$$A = \begin{bmatrix} 0.0469 & -0.3992 & -0.0835 \\ 0.3902 & -0.5363 & -0.2744 \\ 0.4378 & -1.3576 & 0.4651 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5673 & -0.2785 \\ 0.1155 & -0.0649 \\ -2.1849 & -0.5976 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.3587 & -1.0802 & -0.6802 \\ -1.3833 & -1.0677 & 1.1497 \end{bmatrix}, \quad D = 0_{2\times 2}.$$

4. The following numerical example has two poles at the same location:

   $\mu = 2\xi$:

$$A = \begin{bmatrix} 0.4030 & 0 & 0 \\ 0 & -0.1502 & 0 \\ 0 & 0 & -0.1502 \end{bmatrix}, \quad B = \begin{bmatrix} -0.2494 \\ 0.2542 \\ -0.2036 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.9894 & 0.6649 & 0.4339 \end{bmatrix}, \quad D = 0.$$

5. The following numerical example has three poles at the same location:

$\mu = 2\xi:$

$$A = \begin{bmatrix} 0.4783 & 0 & 0 & 0 \\ 0 & 0.7871 & 0 & 0 \\ 0 & 0 & 0.7871 & 1 \\ 0 & 0 & 0 & 0.7871 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5174 \\ 1.2181 \\ 0.2496 \\ -0.5181 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.8457 & -2.0885 & 1.2190 & 0.1683 \end{bmatrix}, \quad D = 0.$$

6. The following numerical example has a wide range of pole locations including two poles near 1:

$\mu = \xi:$

$$A = \begin{bmatrix} 0.5359 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9417 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9802 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5777 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1227 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0034 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5721 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2870 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3599 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad D = 0_{4\times 4}.$$

The stability margins for Example 5 indicate that Theorem 1 can reduce the conservatism compared to literature results by more than 53,200%. The stability margins for Example 6 indicate that, for odd nonlinearities, Theorem 2 can reduce the conservatism by more than 400% compared to Theorem 1 and even more when compared to literature results.

2.4 Performance Analysis for Discrete-Time Lur'e Systems

To quantify input-output properties, consider the system

$$\begin{aligned} x(k+1) &= Ax(k) + B_p p(k) + B_w w(k), \\ q(k) &= C_q x(k) + D_{qp} p(k) + D_{qw} w(k), \\ z(k) &= C_z x(k) + D_{zp} p(k) + D_{zw} w(k), \end{aligned} \tag{2.17}$$

which has (2.1) as a subsystem and $p(k) = -\phi(q(k))$ in the classes of nonlinearities defined previously. The time difference of Lyapunov function, i.e. $\Delta V(x(k))$, derived in Theorems 20 and 21 provides more flexibility in computing estimates for input-output properties than a quadratic Lyapunov function. The more decision variables, the sharper the bounds computed for analyzing performance of the system. The sufficient conditions for guaranteed performance are found in many papers. The general frame of optimization problems for three types of performance analysis of the system (2.17) are given below:

1. *Decay rate*

   If the condition $\Delta V(x(k)) \leq -2\alpha V(x(k))$ for all trajectories of the solution in (2.1), then the largest lower bound on the decay rate is found by solving a generalized eigenvalue problem (GEVP) in the decision variables X and α:

   $$\text{maximize} \;\; \alpha \;\; \text{subject to} \;\; A(X) - \alpha B(X) \leq 0, \tag{2.18}$$

   where X is the composition of the (matrix) decision variables introduced in Theorems 20 and 21, and $A(X) \triangleq G < 0$ and $B(X) > 0$ are affine functions of X. The actual construction of $B(X)$ is omitted due to limited space.

2. *l_2 and RMS gains*

   If the condition $\Delta V(x(k)) + z_k^T z_k - \gamma^2 w_k^T w_k \leq 0$ for all trajectories of the solution x_k and w_k in (2.17), then the l_2 and RMS gains of the system are less than γ, i.e. $\frac{||z||_2^2}{||w||_2^2} \leq \gamma$ and $\frac{RMS(z)}{RMS(w)} \leq \gamma$. An upper bound on the l_2 and RMS gains is computed by solving the following eigenvalue problem (EVP):

   $$\text{minimize} \;\; \gamma^2 \;\; \text{subject to} \;\; A(X, \gamma^2) \leq 0, \tag{2.19}$$

   where A is a jointly affine function of X and γ^2. The actual construction of $A(X, \gamma^2)$ is omitted due to limited space.

3. *Dissipativity*

   If the condition $\Delta V(x(k)) - 2w_k^T z_k + 2\eta w_k^T w_k \le 0$ for all trajectories of the solution x_k and w_k in (2.17), then system has dissipation η, i.e., $\sum_{k=0}^{\tau}(w_k^T z_k - \eta w_k^T w_k) \ge 0$ holds for all trajectories of the solution with the zero initial condition and all $\tau \ge 0$. The largest such η satisfying the inequality is called its dissipativity. A lower bound on the dissipativity of the system (2.17) is computed by solving by the following EVP:

$$\text{maximize} \;\; \eta \;\; \text{subject to} \;\; A(X,\eta) \le 0, \tag{2.20}$$

   where A is a jointly affine function of X and η. The actual construction of $A(X,\eta)$ is omitted due to limited space.

2.5 Summary

This chapter considers the analysis of stability and performance for the so-called Lur'e systems with multiple nonlinearities, as well as various extensions. Even though the history of analysis of Lur'e problem can be traced back to 1940s, this is the first publication to consider the Lyapunov function (2.2), for which other Lyapunov functions in literature are special cases. The S-procedure was applied in the standard way to construct sufficient conditions for globally asymptotic stability in term of LMI feasibility problems. If the nonlinear operator, which is connected with a linear time invariant plant in a negative feedback loop, satisfies other constraints in its input and output behavior, then conservatism can be further reduced. Future work should be to investigate more general forms of constraints on the nonlinearities, such as ellipsoidal constraints that contain linear and constant terms, and how these constraints can be imposed into the LMI to imply negative definiteness of the time-difference Lyapunov function. Numerical algorithms could be developed to consider rank conditions or reduced forms for the LMIs or methods to exploit structure within the LMI. The numerical examples are consistent with theory that indicates that the sufficient conditions are less conservative than the the other criteria in literature. The chapter also briefly presented sufficient conditions for state and input-output performance. Since the new sufficient conditions for stability and performance of Lur'e problem and its variants have more degrees of freedom in the LMI feasibility and optimization problems, the conditions can be considered as supersets of the other conditions reported in the literature.

Chapter 3

Controller Synthesis Problems for Lur'e Systems

A popular representation in robust control design is the linear fractional transformation (LFT), which involves the interconnection of a nominal plant G, a controller K and an uncertainty Δ. Much of the robust control literature considers analysis and controller synthesis for problems in which the nominal plant G is linear time-invariant and the perturbations on the nominal plant are norm-bounded (either assumed to be linear or nonlinear). Given the nominal plant G, the robust control problem is to design a controller K that guarantees closed-loop stability and some specified performance in the presence of the perturbations Δ belonging to a predetermined family of uncertainties Φ_Δ. The controller optimization is usually for the worst-case or expected value for the performance objective.

Within the worst-case robust control literature, most of the literature assumes very little about the perturbations other than being norm-bounded. This is not much of a limitation for linear time-invariant perturbations, as the dynamic effect of the perturbations on the nominal plant can be shaped by using transfer function weights. On the other hand, for nonlinear perturbations, there is usually a lot of information is available that is not readily described by transfer function weights. For such systems, constraining the nonlinear perturbations using only bounds on their norm can be very conservative.

This chapter considers systems described in terms of the standard nonlinear operator form (SNOF), which is very similar to the LFT. The results of this section are very closely related to standard results in the robust control literature, but presented in a manner that can be easily extended for more specific classes of nonlinear operators that correspond to perturbations. These extensions are treated in Chapters 4 and **??**.

This chapter has two objectives. The first objective is to derive LMI feasibility problems for the analysis of a general description of Lur'e systems. The second objective is to formulate optimization problems for controller design and to suggest numerical methods to solve these problems. The proposed feasibility and optimization problems provide only sufficient conditions for quadratic stability, due to use of the S-procedure. The optimization procedures generate control laws that simultaneously stabilize the overall system and maximize bounds on the nonlinear perturbations.

3.1 Introduction

Consider the nonlinear discrete-time system

$$x(k+1) = f(x(k)) + g(k, x(k)) \tag{3.1}$$

where $x(k) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state at time $k \in \mathbb{Z}_+$, $f : \mathcal{X} \to \mathbb{R}^n$ is a time-invariant nominal part of the system and $g : \mathbb{Z}_+ \times D \to \mathbb{R}^n$ is a nonlinear time-varying perturbation function. Suppose that $f(x)$ is twice continuously differentiable. Equation (3.1) can be represented as

$$x(k+1) = Ax(k) + \left[f(x(k)) - \frac{\partial f}{\partial x}(0)x(k)\right] + g(k, x(k)) \tag{3.2}$$

$$= Ax(k) + \tilde{f}(x(k)) + g(k, x(k)) \tag{3.3}$$

$$= Ax(k) + B_p\phi(k, x(k)) \tag{3.4}$$

where

$$\tilde{f}(x(k)) = f(x(k)) - \frac{\partial f}{\partial x}(0)x(k) \tag{3.5}$$

is twice continuously differentiable and

$$\tilde{f}(0) = 0; \qquad \frac{\partial \tilde{f}}{\partial x}(0) = 0. \tag{3.6}$$

The nonlinear function $\phi(k, x(k))$ is assumed to be uncertain and satisfy the quadratic inequality for all $(k, x(k)) \in \mathbb{Z}_+ \times D$

$$\phi^T(k, x(k))\phi(k, x(k)) \leq \alpha^2 x^T C_q^T C_q x, \tag{3.7}$$

which can be written as

$$\begin{bmatrix} x \\ \phi \end{bmatrix}^T \begin{bmatrix} -\alpha^2 C_q^T C_q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} \leq 0 \tag{3.8}$$

where $\alpha > 0$ is the bounding parameter on the uncertain perturbation function ϕ, and C_q is a constant matrix with the compatible dimension. Suppose the nominal system has a uniformly asymptotically stable equilibrium point at the origin. The natural approach to address the stability condition is to use a Lyapunov function for the nominal system as a Lyapunov function candidate for the perturbed system. As a matter of fact, the conclusions we can arrive at depend critically on whether the perturbation term vanishes at the origin. If $\phi(k, 0) = 0$ for all $k \in \mathbb{Z}_+$ then the perturbed system has an equilibrium point at the origin. In this case, we analyze the stability behavior of the origin as an equilibrium point of the perturbed system. If $\phi(k,0) \neq 0$ for some $k \in \mathbb{Z}_+$ then the origin will not be an equilibrium point of the perturbed system. In this case, we study ultimate boundedness of the solutions of the perturbed system. We note that for any given C_q, the inequality (3.7) defines a set of functions

$$\bar{\Phi}_{sb}^{\alpha} \triangleq \{\phi : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^n \,|\, \phi^T\phi \leq \alpha^2 x^T C_q^T C_q x, \text{ for all } (k,x) \in \mathbb{Z}_+ \times \mathbb{R}^n\} \tag{3.9}$$

Since the class $\bar{\Phi}_{sb}^{\alpha}$ is a subset of the set of vanishing perturbations we can analyze the stability of the system around the origin as an equilibrium point of the perturbed system. By introducing the following definition of robust stability we have a tool for measuring the degree of stability of a system.

Definition 9. *System (4) is robustly stable with degree α if the equilibrium $x = 0$ is globally asymptotically stable for all $\phi(k, x(k)) \in \bar{\Phi}_{sb}^{\alpha}$.*

Let us consider a quadratic Lyapunov function

$$V(x) = x^T P x \tag{3.10}$$

with a symmetric positive definite matrix P denoted $P > 0$. By computing the time difference of the Lyapunov function, a sufficient condition for stability is

$$\Delta V(x_k) = V(x_{k+1}) - V(x_k) \tag{3.11}$$

$$= x_k^T(A^TPA - P)x_k + x_k^T A^T P B_p \phi_k + \phi_k^T B_p^T P A x_k + \phi_k^T B_p^T P B_p \phi_k \tag{3.12}$$

$$= \begin{bmatrix} x_k \\ \phi_k \end{bmatrix}^T \begin{bmatrix} A^TPA - P & A^TPB_p \\ B_p^TPA & B_p^TPB_p \end{bmatrix} \begin{bmatrix} x_k \\ \phi_k \end{bmatrix} < 0. \tag{3.13}$$

Using the S-procedure, the inequality (3.13) with the quadratic constraint (3.7) can be rewritten as:

$$\begin{bmatrix} A^TPA - P + \tau\alpha^2 C_q^T C_q & A^TPB_p \\ B_p^TPA & B_p^TPB_p - \tau I \end{bmatrix} < 0, \tag{3.14}$$

$$P > 0, \tag{3.15}$$

$$\tau \geq 0. \tag{3.16}$$

Recalling that the minimization under non-strict LMI constraints produces the same result as minimization under strict LMI constraints when both strict and non-strict LMI constraints are feasible, replacing $\tau \geq 0$ by $\tau > 0$ allows the optimization problem to be rewritten in the equivalent form:

$$\begin{bmatrix} A^T\tilde{P}A - \tilde{P} + \alpha^2 C_q^T C_q & A^T\tilde{P}B_p \\ B_p^T\tilde{P}A & B_p^T\tilde{P}B_p - I \end{bmatrix} < 0, \qquad \tilde{P} > 0, \tag{3.17}$$

where $\tilde{P} = \frac{1}{\tau}P$. Now apply the Schur complement lemma to obtain the equivalent LMI:

$$\begin{bmatrix} -P & 0 & A^T P & C_q^T \\ 0 & -I & B_p^T P & 0 \\ PA & PB_p & -P & 0 \\ C_q & 0 & 0 & -\gamma I \end{bmatrix} < 0, \qquad P > 0, \tag{3.18}$$

which is an LMI in P and $\gamma \triangleq 1/\alpha^2$ and where $\tilde{P}$ in (3.17) is replaced by P without loss of generality. To establish robust stability under constraint (3.7) with maximal α, propose the eigenvalue problem:

$$\text{minimize} \quad \gamma \tag{3.19}$$

$$\text{subject to} \quad P > 0 \tag{3.20}$$

$$\begin{bmatrix} -P & 0 & A^T P & C_q^T \\ 0 & -I & B_p^T P & 0 \\ PA & PB_p & -P & 0 \\ C_q & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{3.21}$$

With $Q = P^{-1}$, the LMI constraints (3.21) are also equivalent to the LMI condition

$$\begin{bmatrix} -Q & 0 & QA^T & QC_q^T \\ 0 & -I & B_p^T & 0 \\ AQ & B_p & -Q & 0 \\ C_q Q & 0 & 0 & -\gamma I \end{bmatrix} < 0. \tag{3.22}$$

To derive a feasibility problem for the sufficient condition for stability of the closed-loop system, consider the more general class of nonlinear functions: $\phi \in \Phi_{sb}^{|\alpha|}$. This component-wise nonlinear mapping satisfies the inequality

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix}^T \begin{bmatrix} -\alpha_i^2 C_{q,i}^T C_{q,i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \leq 0 \tag{3.23}$$

where the subscript i indicates the ith row of a matrix.

Proposition 8. *(Sufficient condition for the stability of Lur'e system with $\phi \in \Phi_{sb}^{|\alpha|}$) Consider the system of the form (3.4) with the memoryless nonlinear mapping $\phi \in \Phi_{sb}^{|\alpha|}$ which is continuous almost everywhere. A sufficient condition for global asymptotic stabiltiy is the existence of a positive definite matrix $Q = Q^T$ and diagonal positive definite matrix T such that the following LMI is feasible:*

$$\begin{bmatrix} -Q & 0 & QA^T & QC_q^T \\ 0 & -T & TB_p^T & 0 \\ AQ & B_pT & -Q & 0 \\ C_qQ & 0 & 0 & -S_\alpha T \end{bmatrix} < 0, \tag{3.24}$$

where $S_\alpha = diag\{1/\alpha_1^2, \cdots, 1/\alpha_{n_p}^2\}$.

Proof: To apply the S-procedure, let us consider the sector condition given by (3.23) and rewrite it as

$$\sum_{i=1}^{n_p} \tau_i (3.23) = \begin{bmatrix} x \\ \phi \end{bmatrix}^T \underbrace{\begin{bmatrix} -C_{q,i}^T S_\alpha T C_{q,i} & 0 \\ 0 & T \end{bmatrix}}_{U_1} \begin{bmatrix} x \\ \phi \end{bmatrix} \leq 0. \tag{3.25}$$

Now, the matrix inequality $G \triangleq \Delta V(x_k) - U_1$ yields

$$\begin{bmatrix} A^TPA - \tilde{P} + C_q^T S_\alpha T C_q & A^TPB_p \\ B_p^TPA & B_p^TPB_p - T \end{bmatrix} < 0, \tag{3.26}$$

where $P = P^T > 0$ and $T > 0$ is diagonal. When applying the Schur complement lemma, we have

$$\begin{bmatrix} -P & 0 & A^TP & C_q^TT \\ 0 & -T & B_p^TP & 0 \\ PA & PB_p & -P & 0 \\ TC_q & 0 & 0 & -S_\alpha T \end{bmatrix} < 0 \tag{3.27}$$

and equivalently

$$\begin{bmatrix} -Q & 0 & QA^T & QC_q^T \\ 0 & -T^{-1} & T^{-1}B_p^T & 0 \\ AQ & B_pT^{-1} & -Q & 0 \\ C_qQ & 0 & 0 & -S_\alpha T^{-1} \end{bmatrix} < 0. \tag{3.28}$$

Now, replacing T^{-1} by T we have the inequality (3.24). □

Similar to the discrete-time system, let us consider the continuous-time system

$$\dot{x}(t) = Ax(t) + B_p\phi(t, x(t)) \tag{3.29}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state at time $t \in \mathbb{R}_+$ and $\phi : \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}^n$ is a time-varying perturbation function. It is assumed that the nonlinearity $\phi(t, x(t))$ is a piecewise continuous function in both t and x in the right-hand side of (3.29). Consider a quadratic Lyapunov function

$$V(x) = x^T Px \tag{3.30}$$

with a symmetric positive-definite matrix P denoted $P > 0$ and the sector-bounded nonlinear function $\phi \in \bar{\Phi}_{sb}^{\alpha}$. By computing the time derivative of the Lyapunov function (3.30), a sufficient condition for stability is

$$\dot{V}(x) = x^T(A^TP + PA)x + \phi^T B_p^T Px + x^T PB_p\phi \tag{3.31}$$

$$= \begin{bmatrix} x \\ \phi \end{bmatrix}^T \begin{bmatrix} A^TP + PA & PB_p \\ B_p^TP & 0 \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} < 0. \tag{3.32}$$

Using the S-procedure, the inequality (3.32) with the quadratic constraint (3.7) can be rewritten as:

$$\begin{bmatrix} A^TP + PA + \tau\alpha^2 C_q^T C_q & PB_p \\ B_p^TP & -\tau I \end{bmatrix} < 0, \tag{3.33}$$

$$P > 0, \tag{3.34}$$

$$\tau \geq 0. \tag{3.35}$$

This is further equivalent to the existence of a matrix Q so that

$$Q > 0, \tag{3.36}$$

$$\begin{bmatrix} AQ + QA^T + \alpha^2 QC_q^T C_q Q & B_p \\ B_p^T & -I \end{bmatrix} < 0 \tag{3.37}$$

where $Q = \tau P^{-1}$ and $\tau > 0$ without loss of generality.

From the Schur complement formula, (3.36) and (3.37) can be rewritten as

$$Q > 0, \tag{3.38}$$

$$\begin{bmatrix} AQ + QA^T & B_p & QC_q^T \\ B_p^T & -I & 0 \\ C_q Q & 0 & -\gamma I \end{bmatrix} < 0 \tag{3.39}$$

where $\gamma = 1/\alpha^2$. Robust stability under constraint (3.7) with maximal α is established for the eigenvalue problem:

$$\text{minimize} \quad \gamma \tag{3.40}$$

$$\text{subject to} \quad Q > 0, \tag{3.41}$$

$$\begin{bmatrix} AQ + QA^T & B_p & QC_q^T \\ B_p^T & -I & 0 \\ C_q Q & 0 & -\gamma I \end{bmatrix} < 0. \tag{3.42}$$

To derive a feasibility problem for the sufficient condition for the stability of the closed-loop system, consider the more general class of nonlinear functions $\phi \in \Phi_{sb}^{|\alpha|}$ defined in (3.23).

Proposition 9. *(Sufficient condition for the stability of Lur'e system with $\phi \in \Phi_{sb}^{|\alpha|}$) Consider the system of the form (3.29) with the memoryless nonlinear mapping $\phi \in \Phi_{sb}^{|\alpha|}$ that is continuous almost everywhere. A sufficient condition for global asymptotic stabiltiy is the existence of a positive definite matrix $Q = Q^T$ and diagonal positive definite matrix T such that the LMI*

$$\begin{bmatrix} AQ + QA^T & B_p T & QC_q^T \\ TB_p^T & -T & 0 \\ C_q Q & 0 & -S_\alpha T \end{bmatrix} < 0, \tag{3.43}$$

is feasible, where $S_\alpha = diag\{1/\alpha_1^2, \cdots, 1/\alpha_{n_p}^2\}$.

Proof: Similar to the discrete-time case, the matrix inequality $G \triangleq \dot{V}(x(t)) - U_1$ yields

$$\begin{bmatrix} A^T P + PA + C_q^T S_\alpha T C_q & PB_p \\ B_p^T P & -T \end{bmatrix} < 0, \tag{3.44}$$

where $P = P^T > 0$ and $T > 0$ is diagonal. Applying the Schur complement lemma and taking a congruence transformation results in

$$\begin{bmatrix} AQ + QA^T & B_p T^{-1} & QC_q^T \\ T^{-1} B_p^T & -T^{-1} & 0 \\ C_q Q & 0 & -S_\alpha T^{-1} \end{bmatrix} < 0, \tag{3.45}$$

Replacing T^{-1} by T gives the inequality (3.43). □

3.2 State-Feedback Control via LMI Optimization

3.2.1 Continuous-Time Lur'e Systems

To introduce a stabilizing state feedback controller to the system (3.29) such that closed-loop stability is achieved is a feasibility problem and its maximum tolerance to the uncertain nonlinear perturbations is achieved in an optimization problem, let us consider the system with affine control input described by

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_p \phi(t, x(t)) \tag{3.46}$$

$$= (A + B_u K)x(t) + B_p \phi(t, x(t)) \tag{3.47}$$

where $B_u \in \mathbb{R}^{n \times n_u}$ is a constant matrix, $u : \mathbb{R}^n \to \mathbb{R}^{n_u}$ is the linear feedback control law

$$u(t) = Kx(t) \tag{3.48}$$

and $K \in \mathbb{R}^{n_u \times n}$ is a control gain matrix. That is, it is assumed that full information about the system states are available–even through this assumption is not generally true in practical cases it gives a tool for potential extensions of its developments. It can be further assumed that the pair (A, B_u) is controllable or at least stabilizable. Now, let us consider the optimization problem in which the control objective is to achieve the maximum sector-bounds of the nonlinear function $\phi \in \bar{\Phi}_{sb}^{\alpha}$ such that the closed-loop system is stable.

Theorem 22. *Using quadratic function $V(x)$ and computing its derivative $\dot{V}(x)$, the design of the state feedback controller $u(t) = u(t) = Kx(t)$ that maximizes the quadratic robust stability margin α of the closed-loop system (3.47) can be formulated as the EVP:*

$$\text{minimize} \quad \gamma \tag{3.49}$$

$$\text{subject to} \quad Q > 0 \tag{3.50}$$

$$\begin{bmatrix} AQ + QA^T + B_u L + L^T B_u^T & B_p & QC_q^T \\ B_p^T & -I & 0 \\ C_q Q & 0 & -\gamma I \end{bmatrix} < 0 \tag{3.51}$$

where $\gamma = 1/\alpha^2$ and $L = KQ$ such that $K = LQ^{-1}$.

Proof: Replacing the system matrix A by $A + B_u K$ in (3.42) and defining $L \triangleq KQ$, we have the inequality (3.51) for Q and L. □

In addition, let us consider a sufficient condition for the stability of the closed-loop system (3.47) for a general Lur'e system with the nonlinear function perturbation $\phi \in \Phi_{sb}^{|\alpha|}$ where the component-wise sector condition is assumed to be known.

Theorem 23. *Using quadratic function $V(x)$ and computing its derivative $\dot{V}(x)$, a sufficient condition for the stability of the closed-loop system (3.47) with known sector condition S_α can be formulated as a feasibility problem for $Q = Q^T > 0$ and $T > 0$:*

$$\begin{bmatrix} AQ + QA^T + B_u L + L^T B_u^T & B_p T & QC_q^T \\ T B_p^T & -T & 0 \\ C_q Q & 0 & -S_\alpha T \end{bmatrix} < 0, \tag{3.52}$$

where $S_\alpha = diag\{1/\alpha_1^2, \cdots, 1/\alpha_{n_p}^2\}$ and $L = KQ$ such that $K = LQ^{-1}$.

Proof: The proof is trivial from the previous theorem. □

3.2.2 Discrete-Time Lur'e Systems

In order to apply the stabilizing state feedback to the perturbed system, and to solve a problem using convex programming tools for LMIs, we consider the system with a control affine term

$$x(k+1) = Ax(k) + B_u u(k) + B_p \phi(k, x(k)) \tag{3.53}$$

where $x(k) \in \mathbb{R}^n$ is the state variable and $u(k) \in \mathbb{R}^m$ is the control input at sampling time $k \in \mathbb{Z}_+$. $A \in \mathbb{R}^{n\times n}$ and $B_u \in \mathbb{R}^{n\times m}$ are constant matrices. It can be further assumed that the pair (A, B_u) is controllable. Considering the linear state feedback control law

$$u(k) = Kx(k) \tag{3.54}$$

where K is a control gain matrix of the compatible dimensions. Applying the feedback control law (3.54) to the system (3.53) results in the closed-loop system:

$$x(k+1) = A_u x(k) + B_p \phi(k, x(k)) \tag{3.55}$$

where

$$A_u \triangleq A + B_u K$$

is the closed-loop system matrix. The system (3.53) is robustly stabilized by the state feedback control law (3.54) if the closed-loop system (3.55) is robustly stable with degree α.

Theorem 24. *Using quadratic function $V(x_k)$ and computing its derivative $\Delta V(x_k)$, The design of the state feedback controller that maximizes the quadratic robust stability margin α of the closed-loop system (3.53) can be formulated as the eigenvalue problem (EVP):*

$$\textit{minimize} \quad \gamma \tag{3.56}$$

$$\textit{subject to} \quad Q > 0, \tag{3.57}$$

$$\begin{bmatrix} -Q & 0 & QA^T + L^T B_u^T & QC_q^T \\ 0 & -I & B_p^T & 0 \\ AQ + B_u L & B_p & -Q & 0 \\ C_q Q & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{3.58}$$

where $\gamma = \frac{1}{\alpha^2}$ and $L = KQ$ such that $K = LQ^{-1}$.

Proof: Replacing the system matrix A by $A + B_u K$ in (3.42) and defining $L \triangleq KQ$, we have the inequality (3.71) for Q and L. □

In addition, let us consider a sufficient condition for the stability of the closed-loop system (3.55) for a general Lur'e system with the nonlinear function perturbation $\phi \in \Phi_{sb}^{|\alpha|}$ where the component-wise section condition is assumed to be known.

Theorem 25. *Using quadratic function $V(x_k)$ and computing its derivative $\Delta V(x_k)$, a sufficient condition for the stability of the closed-loop system (3.55) with known section condition S_α can be formulated as a feasibility problem for $Q = Q^T > 0$ and $T > 0$:*

$$\begin{bmatrix} -Q & 0 & QA^T + L^T B_u^T & QC_q^T \\ 0 & -T & TB_p^T & 0 \\ AQ + B_u L & B_p T & -Q & 0 \\ C_q Q & 0 & 0 & -S_\alpha T \end{bmatrix} < 0, \tag{3.59}$$

where $S_\alpha = diag\{1/\alpha_1^2, \cdots, 1/\alpha_{n_p}^2\}$ and $L = KQ$ such that $K = LQ^{-1}$.

Proof: The proof is trivial from the previous theorem. □

3.2.3 Illustrative Examples

We consider some numerical examples to show the applicability of the previously defined LMI synthesis problems for a certain case of Lur'e systems where the feedback connected nonlinear mapping is classified as $\phi \in \bar{\Phi}_{sb}^{\alpha}$ or $\phi \in \Phi_{sb}^{|\alpha|}$.

Example 1. *(Optimal state-feedback control for the system (3.53) with $\phi \in \bar{\Phi}_{sb}^{\alpha}$) Consider the system (3.53) whose state space realization is given by*

$$A = \begin{bmatrix} 0.8000 & -0.2500 & 0 & 1.0000 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0.2000 & 0.3000 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_q = \begin{bmatrix} 0.8000 & -0.5000 & 0 & 1.0000 \end{bmatrix}.$$

Let us consider the optimization problem where the control objective is to maximize the upper bound α on the sector condition such that the closed-loop system (3.53) is stabilized by the state-feedback control law $u(k) = Kx(k)$. Then, the optimal solution for Q^* and K^* of the EVP in Theorem 24 is obtained as

$$Q^* = \begin{bmatrix} 8.3243 & 6.8006 & -0.8894 & -3.2592 \\ 6.8006 & 10.5381 & -0.9147 & -0.1714 \\ -0.8894 & -0.9147 & 1.8296 & 0.2542 \\ -3.2592 & -0.1714 & 0.2542 & 2.5216 \end{bmatrix}, \quad K^* = \begin{bmatrix} 0.1568 & 0.0426 & -1.2500 & -0.5852 \end{bmatrix},$$

such that the maximum upper sector bound is given as $\alpha^* = 2.4601 \times 10^4$. *The simulation below is for the system (3.53) with* $\phi(\cdot) = \alpha^* \tanh(\cdot)$.

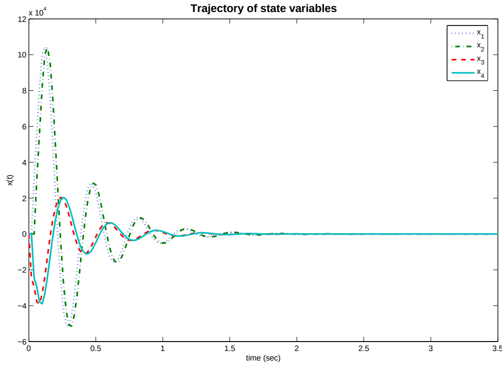


Figure 3.1: Trajectory of solution for the closed-loop system

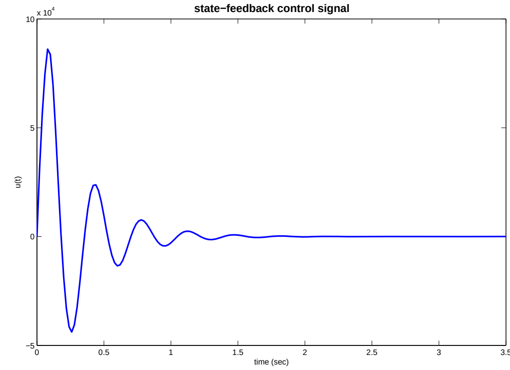


Figure 3.2: State-feedback control law $u(k) = Kx(k)$

Example 2. *(Feasible state-feedback control for the system (3.53) with* $\phi \in \Phi_{sb}^{|\alpha|}$*) Consider the system (3.53) whose state space realization is given by*

$$A = \begin{bmatrix} 0.0469 & -0.3992 & -0.0835 \\ 0.3902 & -0.5363 & -0.2744 \\ 0.4378 & -1.3576 & 0.4651 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_p = \begin{bmatrix} -0.5673 & -0.2785 \\ 0.1155 & -0.0649 \\ -2.1849 & -0.5976 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0.3587 & -1.0802 & -0.6802 \\ -1.3833 & -1.0677 & 1.1497 \end{bmatrix}.$$

Let us consider the optimization problem where the objective is to design a state-feedback control law $u(k) = Kx(k)$ *that stabilizes the closed-loop system (3.53) where* $\phi \in \Phi_{sb}^{|\alpha|}$ *with* $\alpha = [0.1, 0.2]^T$. *Then, a solution for* Q *and* K *of the EVP in Theorem 25 is obtained as*

$$Q = \begin{bmatrix} 0.6089 & -0.1635 & 0.5807 \\ -0.1635 & 0.0439 & -0.1560 \\ 0.5807 & -0.1560 & 0.5539 \end{bmatrix}, \quad K = \begin{bmatrix} -0.1361 & -0.5379 & 0.7482 \\ -0.4108 & 0.4270 & 0.0410 \end{bmatrix}.$$

The simulation below is for the system (3.53) with $\phi(\cdot) = \alpha \tanh(\cdot)$.

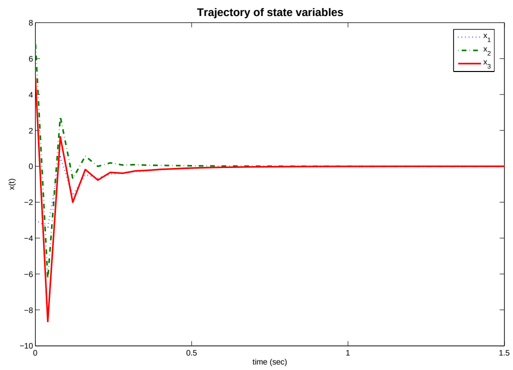


Figure 3.3: Trajectory of solution for the closed-loop system

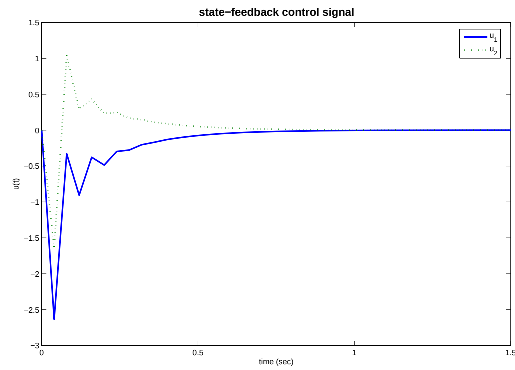


Figure 3.4: State-feedback control law $u(k) = Kx(k)$

3.3 Observer-Based State-Feedback Control via LMI Optimization

In the previous section, it has been assumed that all state variables of the system are available for feedback. However, only part of the state variables are measurable or measured for feedback control in practice. In such a case, to apply state-feedback control, one needs to construct a scheme to estimate the actual state based on the available outputs, which is called an (closed) state observer (or estimator), so that the output from the observer would give an estimate for the states of the system. Our objective, therefore, is to design a state observer where the estimated state $\hat{x}$ converges to the true state variable x such that the state-feedback control with the estimated states, $u(k) = K\hat{x}(k)$, stabilizes the actual system which is a special class of Lur'e systems with the feedback connected nonlinear mapping $\phi \in \bar{\Phi}_{sb}^{\alpha} \cap \bar{\Phi}_{sr}^{\mu}$.

3.3.1 Discrete-Time Lur'e Systems

The system dynamics for the plant is given by

$$\begin{aligned} x(k+1) &= Ax(k) - B_p\phi(q(k)) \\ y(k) &= C_y x(k) \\ q(k) &= C_q x(k). \end{aligned} \tag{3.60}$$

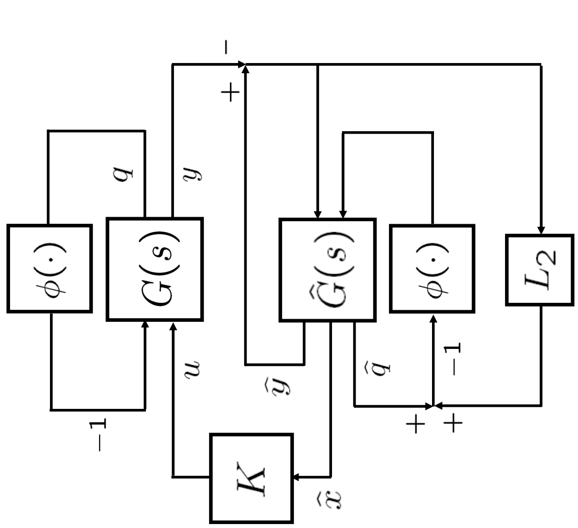


Figure 3.5: Observer-Based State-Feedback Control

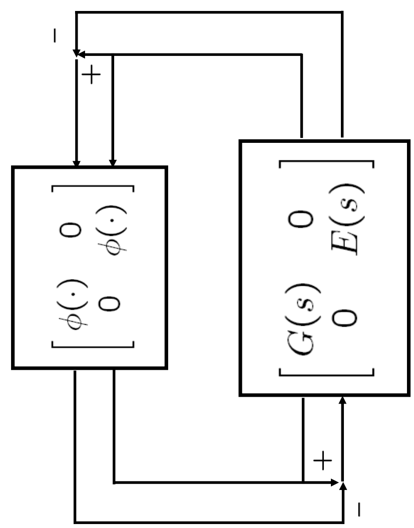


Figure 3.6: Error Dynamics

Consider the following estimator which is introduced in [11]:

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + L_1(\hat{y} - y) - B_p\phi(\hat{q}(k) + L_2(\hat{y} - y)) \\ \hat{y}(k) &= C_y\hat{x}(k) \\ \hat{q}(k) &= C_q\hat{x}(k).\end{aligned} \tag{3.61}$$

Then, the estimation error dynamics with the estimation error $e \triangleq x - \hat{x}$ becomes

$$\begin{aligned}e(k+1) &= (A + L_1C_y)e(k) - B_p\hat{\phi}(z(k); q(k)) \\ z(k) &= (C_q + L_2C_y)e(k),\end{aligned} \tag{3.62}$$

where $\hat{\phi}(z(k); q(k)) \triangleq \phi(q(k)) - \phi(\hat{q}(k) + L_2(\hat{y} - y))$ and $z(k) \triangleq q(k) - \hat{q}(k) - L_2(\hat{y} - y)$. A block diagram for the error dynamics is given in Figure 3.6. Now, the error dynamics can be considered as a special class of Lur'e systems.

Corollary 2. *The feedback connected nonlinear function $\hat{\phi}$ has the following properties:*

1. *for any $q(k)$, $\hat{\phi}$ vanishes at $z(k) \equiv 0$, i.e., $\hat{\phi}(0; q(k)) \equiv 0$ for all $q(k) \in \mathbb{R}^{n_q}$;*
2. *if $\phi \in \bar{\Phi}^{\alpha}_{sb} \cap \bar{\Phi}^{\mu}_{sr}$ then $\hat{\phi} \in \bar{\Phi}^{\mu}_{sb}$.*

Proof: From the definition of $\hat{\phi}$ and $z(k)$, proof for the first property is trivial. For the second property, since $\phi \in \bar{\Phi}^{\mu}_{sr}$ one can derive

$$0 \le \hat{\phi}^T(z(k); q(k))\hat{\phi}(z(k); q(k)) = \phi(q(k)) - \phi(q(k) - z(k))^T\phi(q(k)) - \phi(q(k) - z(k)) \le \mu^2 z^T(k)z(k) \tag{3.63}$$

for all $z(k) \in \mathcal{Z} \subset \mathbb{R}^{n_q}$ with any fixed $q(k) \in \mathbb{R}^{n_q}$ for each sampling time $k \in \mathbb{Z}_+$. Therefore, $\hat{\phi} \in \Phi^{\mu}_{sb}$ in this sense. □

Now, from the results of Section 3.1, a sufficient condition for the asymptotic stability of the error dynamics can be written as the following LMIs:

$$P > 0, \tag{3.64}$$

$$\begin{bmatrix} -P & 0 & A^TP + C_y^TY_1^T & C_q^T + C_y^TL_2^T \\ 0 & -I & -B_p^TP & 0 \\ PA + Y_1C_y & -PB_p & -P & 0 \\ C_q + L_2C_y & 0 & 0 & -\frac{1}{\mu^2}I \end{bmatrix} < 0, \tag{3.65}$$

where $Y_1 = PL_1$.

Theorem 26. *(Maximization of the decay-rate of the estimation error dynamics) If the condition* $\Delta V(x(k)) < -(1-\lambda)V(x(k))$ *for all trajectories of the solution in the estimation error dynamics, then the largest lower bound on the decay rate is found by solving a general eigenvalue problem (GEVP) in decision variables* P*,* Y_1*,* L_2*, and* λ *for a fixed value of* μ*:*

$$\begin{aligned} &\textit{minimize} \quad && \lambda && (3.66)\\ &\textit{subject to} \quad && P > 0, && (3.67)\\ & && \begin{bmatrix} -\lambda P & 0 & A^T P + C_y^T Y_1^T & C_q^T + C_y^T L_2^T \\ 0 & -I & -B_p^T P & 0 \\ PA + Y_1 C_y & -PB_p & -P & 0 \\ C_q + L_2 C_y & 0 & 0 & -\frac{1}{\mu^2} I \end{bmatrix} < 0, && (3.68) \end{aligned}$$

where $Y_1 = PL_1$ *such that* $L_1 = P^{-1}Y_1$*. One can easily see that the smaller value of* λ *such that* $\lambda \in (0,1]$*, the faster rate of convergence.*

Proof: The proof follows from the inequality (3.65) and $\lambda \equiv 1$ corresponds the asymptotic stability of the error dynamics. □

Proposition 10. *(Design process for observer-based state-feedback controller)*

- Step 1: *To determine the control gain* K_s*, which guarantees the optimal robust stability of the limiting dynamics of the discrete-time linear time-invariant system interconnected with a certain class of perturbations* $\phi \in \bar{\Phi}_{sb}^{\alpha} \cap \bar{\Phi}_{sr}^{\mu}$*, solve the EVP*

$$\begin{aligned} &\textit{minimize} \quad && \gamma && (3.69)\\ &\textit{subject to} \quad && Q > 0, && (3.70)\\ & && \begin{bmatrix} -Q & 0 & QA^T + L^T B_u^T & QC_q^T \\ 0 & -I & -B_p^T & 0 \\ AQ + B_u L & -B_p & -Q & 0 \\ C_q Q & 0 & 0 & -\gamma I \end{bmatrix} < 0 && (3.71) \end{aligned}$$

  where $\gamma = \frac{1}{\alpha^2}$ *and* $L = K_s Q$ *such that* $K_s = LQ^{-1}$.

- Step 2: *To determine the estimation gains* L_1 *and* L_2*, which maximize the decay-rate of the estimation* *error dynamics, solve the GEVP*

$$\text{minimize} \quad \lambda \tag{3.72}$$

$$\text{subject to} \quad P > 0, \tag{3.73}$$

$$\begin{bmatrix} -\lambda P & 0 & A^T P + C_y^T Y_1^T & C_q^T + C_y^T L_2^T \\ 0 & -I & -B_p^T P & 0 \\ PA + Y_1 C_y & -PB_p & -P & 0 \\ C_q + L_2 C_y & 0 & 0 & -\frac{1}{\mu^2} I \end{bmatrix} < 0, \tag{3.74}$$

where $Y_1 = PL_1$ *such that* $L_1 = P^{-1} Y_1$ *and* $\mu = \alpha$ *is the optimal solution of the problem in* Step 1 *provided that the upper bound on the slope is the same as the maximum sector bound. Moreover,* $\mu \geq \alpha$ *in general.*

Proof: Using the properties of the feedback connected nonlinear function in the error dynamics and from controller synthesis problem for the so-called Lur'e systems, the proof is trivial. □

Now, instead of using the design process in Proposition 10 which consists of successive EVP and GEVP, consider an EVP with a fixed decay-rate for the closed-loop system. Then, the closed-loop system with the observer-based state-feedback becomes

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A + B_u K_s & B_u K_s \\ 0 & A + L_1 C_y \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} - \underbrace{\begin{bmatrix} B_p & 0 \\ 0 & B_p \end{bmatrix}}_{B_{p,cl}} \underbrace{\begin{bmatrix} \phi(q(k)) \\ \bar{\phi}(z(k); q(k)) \end{bmatrix}}_{\phi_{cl}} \tag{3.75}$$

$$\begin{bmatrix} q(k) \\ z(k) \end{bmatrix} = \underbrace{\begin{bmatrix} C_q & 0 \\ 0 & C_q + L_2 C_y \end{bmatrix}}_{C_{q,cl}} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}. \tag{3.76}$$

Therefore, the stability of the closed-loop system which is the feedback interconnection of the system, whose transfer function is $G_{cl}(s) \triangleq C_{cl}(sI - A_{cl})^{-1} B_{cl}$ and the feedback connected set-valued nonlinear function is characterized by $\phi_{cl} \in \bar{\Phi}_{sb}^{\mu}$, can be seen as the feasibility of the following matrix inequality for some

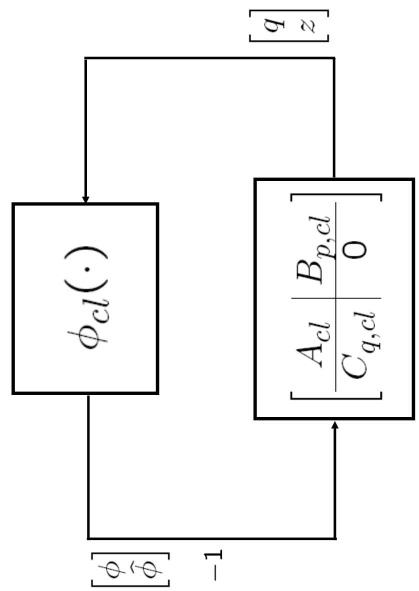


Figure 3.7: Closed-Loop System with State Observer

$X = X^T > 0$ provided $\alpha = \mu$:

$$\begin{bmatrix} -X & 0 & A_{cl}^T X & C_{q,cl}^T \\ 0 & -I & -B_{p,cl}^T X & 0 \\ XA_{cl} & -XB_{p,cl} & -X & 0 \\ C_{q,cl} & 0 & 0 & -\frac{1}{\mu^2} I \end{bmatrix} < 0. \tag{3.77}$$

We should note that this is not a convex optimization (or feasibility) problem since there are bilinear product terms of decision matrix variables. The following theorem shows that the non-LMI condition (3.77) reduces to two LMI conditions with a conservative Lyapunov matrix.

Proposition 11. *(Design scheme for observer-based state-feedback controller) Let consider the diagonal block Lyapunov matrix* $X = diag\{X_1, X_2\} = X^T > 0$*. Then the BMI (3.77) is feasible for a diagonal block Lyapunov matrix* X*,* L_1*,* K_s*, and* L_2 *if and only if the following two LMI conditions for* $Y_1 \triangleq X_1^{-1}$*,* $W_1 \triangleq K_s X_1^{-1}$*,* X_2*,* $W_2 \triangleq X_2 L_1$*, and* L_2*:*

$$\bar{B}_u^{\perp} \Pi (\bar{B}_u^{\perp})^T < 0 \quad and \quad (\bar{E}^T)^{\perp} \Pi ((\bar{E}^T)^{\perp})^T < 0, \tag{3.78}$$

where

$$\Pi \triangleq \begin{bmatrix} -Y_1 & 0 & 0 & 0 & \begin{pmatrix} Y_1A^T+ \\ W_1^TB_u^T \end{pmatrix} & 0 & Y_1C_q^T & 0 \\ 0 & -X_2 & 0 & 0 & 0 & \begin{pmatrix} A^TX_2+ \\ C_y^TW_2^T \end{pmatrix} & 0 & \begin{pmatrix} C_q^T+ \\ C_y^TL_2^T \end{pmatrix} \\ 0 & 0 & -I & 0 & -B_p^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & -B_p^TX_2 & 0 & 0 \\ \begin{pmatrix} AY_1+ \\ B_uW_1 \end{pmatrix} & 0 & -B_p & 0 & -Y_1 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} X_2A+ \\ W_2C_y \end{pmatrix} & 0 & -X_2B_p & 0 & -X_2 & 0 & 0 \\ C_qY_1 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\mu^2}I & 0 \\ 0 & \begin{pmatrix} C_q+ \\ L_2C_y \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\mu^2}I \end{bmatrix},$$

and

$$\bar{B}_u^T \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 & B_u^T & 0 & 0 & 0 \end{bmatrix}, \quad \bar{E} \triangleq \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof: Let us take a congruence transformation to (3.77) with transformation matrix $T = \text{diag}\{Y_1, I, I, I, Y_1, I, I, I, I\}$. Then we have the equivalent LMI condition

$$\Pi + \bar{B}_uK_s\bar{E} + \bar{E}^TK_s^T\bar{B}_u^T < 0.$$

From the Finsler's lemma or the elimination lemma for an unstructured, unknown variable K_s, one can conclude that the feasibility of the BMI (3.77) is equivalent to the feasibilities of two LMIs in (3.78). □

3.3.2 Illustrative Examples

We consider some numerical examples to show the applicability of the previously defined LMI synthesis problems for a certain case of Lur'e systems where the feedback connected nonlinear mapping is classified as $\phi \in \bar{\Phi}_{sb}^{\alpha}$.

Example 3. *(Design scheme for observer-based state-feedback controller in Proposition 10) Consider the* *system (3.60) whose state space realization is given by*

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ -0.2703 & -0.0124 & 0.2703 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0.1075 & 0 & 0.0743 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 0.0216 \\ 0 \\ 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 & 0 \\ 0.2703 & 0 \\ 0 & 0 \\ -0.1075 & 0.0332 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_q = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let us consider the optimization problem where the control objective is to maximize the upper bound α *on the sector condition and to maximize the decay-rate of the error dynamics (3.62) such that the closed-loop system (3.60) is stabilized by the control law* $u(k) = K\hat{x}(k)$*. Then, the optimal solution for* Q^**,* P^**,* K_s^**,* L_1^**, and* L_2^* *of the successive EVP and GEVP in Proposition 10 is obtained as*

$$Q^* = \begin{bmatrix} 0.2988 & 0.0000 & 0.1476 & 0.0000 \\ 0.0000 & 0.2988 & 0.0000 & 0.1476 \\ 0.1476 & 0.0000 & 0.1509 & 0.0000 \\ 0.0000 & 0.1476 & 0.0000 & 0.1509 \end{bmatrix}, \quad P^* = \begin{bmatrix} 109.3211 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 55.8375 & 0.0000 & 126.2874 \\ 0.0000 & 0.0000 & 103.2043 & 0.0000 \\ 0.0000 & 126.2874 & 0.0000 & 348.8080 \end{bmatrix},$$

$$K_s^* = \begin{bmatrix} 18.8558 & 0.5750 & -8.1083 & 0.0000 \end{bmatrix}, \quad L_1^* = \begin{bmatrix} 0.0000 & -1.0000 \\ 0.2703 & 0.0124 \\ 0.0000 & 0.3621 \\ -0.1075 & 0.0000 \end{bmatrix}, \quad L_2^* = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix},$$

such that the maximum upper bound on the sector and slope for the nonlinear function ϕ *is given as* $\alpha^* = \mu^* = 2.5281$ *and the optimal decay rate* $\lambda^* = 0.2959$ *is achieved. The simulation in Figure 3.8 and Figure 3.9 is for the system (3.60) with* $\phi(\cdot) = \alpha^* \tanh(\cdot)$*.*

Example 4. *(Design scheme for observer-based state-feedback controller in Proposition 11) Consider the system (3.60) whose state space realization is the same as in Example 3 Let us consider the problem where the control objective is to maximize the upper bound* α *on the sector condition such that the closed-loop system (3.60) is stabilized by the control law* $u(k) = K\hat{x}(k)$*. Then, the optimal solution for* X_1^**,* X_2^**,* K_s^**,*

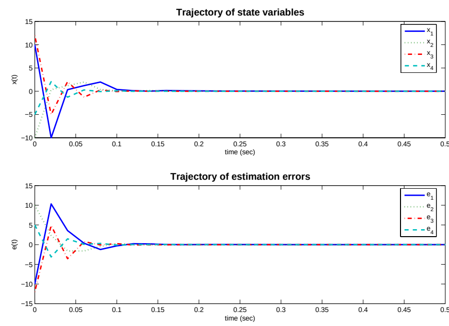


Figure 3.8: Trajectory of solution for the closed-loop system and the error dynamics

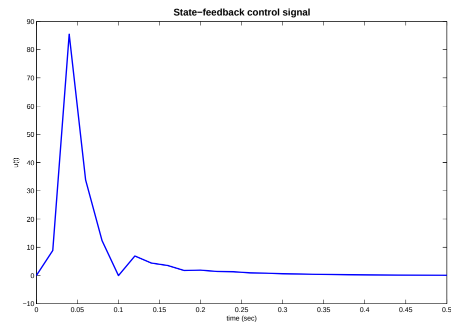


Figure 3.9: State-feedback control law $u(k) = K_s \hat{x}(k)$

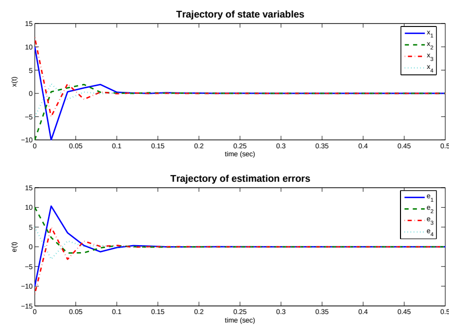


Figure 3.10: Trajectory of solution for the closed-loop system and the error dynamics

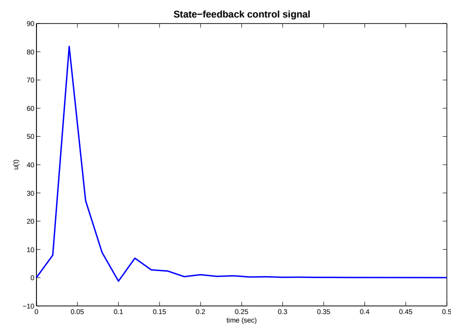


Figure 3.11: State-feedback control law $u(k) = K_s \hat{x}(k)$

L_1^*, and L_2^* of the EVP in Proposition 11 is obtained as

$$X_1^* = \begin{bmatrix} 6.4747 & 0.0000 & -6.3332 & 0.0000 \\ 0.0000 & 6.4747 & 0.0000 & -6.3332 \\ -6.3332 & 0.0000 & 12.8220 & 0.0000 \\ 0.0000 & -6.3332 & 0.0000 & 12.8220 \end{bmatrix}, \quad X_2^* = \begin{bmatrix} 3.2485 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 2.5393 & 0.0000 & 0.4391 \\ 0.0000 & 0.0000 & 13.6026 & 0.0000 \\ 0.0000 & 0.4391 & 0.0000 & 14.5500 \end{bmatrix},$$

$$K_s^* = \begin{bmatrix} 18.8558 & 0.5750 & -8.1083 & 0.0000 \end{bmatrix}, \quad L_1^* = \begin{bmatrix} 0.0000 & -1.0000 \\ 0.2703 & 0.0124 \\ 0.0000 & 0.0301 \\ -0.1075 & 0.0000 \end{bmatrix}, \quad L_2^* = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix},$$

such that the maximum upper bound on the sector and slope for the nonlinear function ϕ is given as $\alpha^* = \mu^* = 2.5281$. The simulation in Figure 3.10 and Figure 3.11 is for the system (3.60) with $\phi(\cdot) = \alpha^* \tanh(\cdot)$.

3.4 Output-Feedback Control via LMI Optimization

While the state observer-based state-feedback control gives an easy way to handle the so-called measurement deficiency problems in feedback control, it is generally not applicable in the presence of uncertainties, especially with the unmodeled dynamics. For robustness purpose, the static output-feedback (SOF) and dynamic output-feedback (DOF) control problems are among the most important classes of questions in control context. Moreover, the state-feedback control problem is a special case of SOF problem with $C_y \equiv I$. In this section, the main objectives are to suggest LMI conditions for the existence of a stabilizing SOF and DOF control laws and to give a tool to design such controllers for a certain class of Lur'e systems which have the nonlinear mapping characterized by $\phi \in \bar{\Phi}_{sb}^{\alpha}$ or $\phi \in \Phi_{sb}^{|\alpha|}$ as a feedback connection. The results obtained in SOF control problems can be trivially extended for the so-called differentiator-free control model which will be introduced in Section 3.4.3.

3.4.1 Static Output-Feedback Control via LMI Optimization

Let us consider the SOF control problems for a certain class of Lur'e systems which is given by

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) + B_p \phi(q(k)) \\ y(k) &= C_y x(k) \\ q(k) &= C_q x(k), \end{aligned} \tag{3.79}$$

where the feedback connected nonlinear function $\phi(\cdot)$ is known to be in a specific class $\bar{\Phi}_{sb}^{\alpha}$ or $\Phi_{sb}^{|\alpha|}$. Furthermore, the triplet realization (A, Bu, Cy) is assumed to be stabilizable and detectable without loss of generality.

LMI Optimization Problems for Discrete-Time Lur'e Systems with $\phi \in \bar{\Phi}_{sb}^{\alpha}$

Necessary and sufficient conditions for static output-feedback can be formulated in terms of coupled Lyapunov matrix inequalities which follows a quadratic Lyapuonv function approach. From Lyapunov stability theory, it has been known that the closed-loop linear time-invariant continuous-time system which is given by

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) \\ y(k) &= C_y x(k) \end{aligned} \tag{3.80}$$

is GUAS with an output-feedback control law $u(t) = K_o y(t)$ if and only if the matrix $A + B_u K_o C_y$ is Hurwitz, or equivalently, there exists a gain matrix K_o such that the following matrix inequality holds for some $X = X^T > 0$:

$$(A + B_u K_o C_y)X + X(A + B_u K_o C_y)^T < 0. \tag{3.81}$$

We should note that in the LMI (3.81), there are bilinear terms which are composed of the multiplications of the unknown (decision) matrices X and K_o and constant system matrices and so, to check the solvability or feasibility of the LMI (3.81) is a non-convex problem and known to be NP-hard [41, 16] in general.

Lemma 11. *(Coupled linear matrix inequality formulation [41]) There exists a stabilizing static output-feedback control gain matrix K_o for $X > 0$ if and only if X satisfies the following two matrix inequalities:*

$$B_u^{\perp}(AX + XA^T)(B_u^{\perp})^T < 0, \tag{3.82}$$

$$(C_y^T)^{\perp}(A^T Y + YA)((C_y^T)^{\perp})^T < 0, \tag{3.83}$$

where $XY = YX = I$ and $(\cdot)^{\perp}$ indicates a full rank matrix that is orthogonal to $(\cdot)$.

Proposition 12. *(Parameterization of static output-feedback control gains [41]) All stabilizing static output control gain matrices are parameterized by*

$$K_o = -R^{-1}B_u^T XWC_y^T(C_y WC_y^T)^{-1} + S^{1/2}Z(C_y WC_y^T)^{-1/2}, \tag{3.84}$$

where

$$\Psi = X^{-1}A^T + AX^{-1} \tag{3.85}$$

$$R^{-1} > B_u^{\dagger}[\Psi - \Psi(B_u^{\perp})^T(B_u^{\perp}\Psi(B_u^{\perp})^T)^{-1}B_u^{\perp}\Psi](B_u^{\dagger})^T \tag{3.86}$$

$$W^{-1} = XB_u R^{-1}B_u^T X - XA - A^T X \tag{3.87}$$

$$S = R^{-1} - R^{-1}B_u^T XW[W^{-1} - C_y^T(C_y WC_y^T)^{-1}C_y]^{-1}W^{-1}XB_u R^{-1} > 0, \tag{3.88}$$

in which X is any positive definite matrix satisfying two LMIs (3.82) and (3.83), and Z is any matrix with $||Z|| < 1$. In addition, $||\cdot||$ can be any matrix norms and $(\cdot)^{\dagger}$ indicates the pseudo-inverse of $(\cdot)$. This control gain parameterization is based on the solution of the Riccati equation for LQR problem. For more details, see [41].

Finding $X = X^T > 0$ (or $Y = Y^T > 0$) which satisfies the two matrix inequalities (3.82) and (3.83) is also a non-convex problem, since these are not convex in X (or Y). Some alternative computational methods based on iterative sequential solutions of the two LMI problems with respect to X and Y have been proposed to handle this non-convex problem, where the purpose of the feasibility problem to find a stabilizing static output-feedback gain matrix K_o. However, those approaches do not guarantee the convergence of the algorithms.

For a certain class of state-space representation used for describing the system, a sufficient LMI condition for the original non-convex feasibility problem (3.176)(3.81) has been derived in [24].

Proposition 13. *Let us consider a particular state-space representation for the system where the system matrices A, B_u, C_y ($D_{yu} \equiv 0$) are given and C_y is full row rank. Then if there exist the matrices X, M, N such that*

$$\begin{aligned} AX + XA^T + B_u N C_y + C_y^T N^T B_u^T &< 0 \\ X &> 0 \\ M C_y &= C_y X \end{aligned} \tag{3.89}$$

hold for some X, M, N, then a stabilizing static output-feedback control gain K_o is NM^{-1}, i.e., the feedback control signal

$$u(k) = NM^{-1} y(k)$$

stabilizes the discrete-time linear time-invariant system whose realization is the triplet (A, B_u, C_y). In other words, the existence of the matrices satisfying (3.89) is sufficient for the feasibility of (3.81) with the same Lyapunov matrix X, where the quadratic Lyapunov function is $V(\xi) = \xi^T X \xi$, and $K_o = NM^{-1}$.

Proposition 14. *Let us consider a particular state-space representation for the system where the system matrices A, B_u, C_y ($D_{yu} \equiv 0$) are given and B_u is full column rank. Then if there exist the matrices Y, M, N such that*

$$\begin{aligned} YA + A^T Y + B_u N C_y + C_y^T N^T B_u^T &< 0 \\ Y &> 0 \\ B_u M &= Y B_u \end{aligned} \tag{3.90}$$

hold for some Y, M, N, then a stabilizing static output-feedback control gain K_o is $M^{-1}N$, i.e., the feedback control signal

$$u(k) = M^{-1} N y(k)$$

stabilizes the discrete-time linear time-invariant system whose realization is the triplet (A, B_u, C_y). In other *words, the existence of the matrices satisfying (3.90) is sufficient for the feasibility of (3.81) with the same Lyapunov matrix X, where the quadratic Lyapunov function is $V(\xi) = \xi^T Y \xi$, and $K_o = M^{-1}N$.*

The proofs for the previous two propositions are given in [24], which requires simple linear algebra.

Replacing A by $A + B_u K_o C_y$ in the LMI constraint (3.22), the following optimization problem is constructed for a robust static output-feedback controller synthesis:

$$\text{minimize} \quad \gamma \tag{3.91}$$

$$\text{subject to} \quad Q > 0, \tag{3.92}$$

$$\begin{bmatrix} -Q & 0 & QA^T + QC_y^T K_o^T B_u^T & QC_q^T \\ 0 & -I & B_p^T & 0 \\ AQ + B_u K_o C_y Q & B_p & -Q & 0 \\ C_q Q & 0 & 0 & -\gamma I \end{bmatrix} < 0. \tag{3.93}$$

The LMI constraint (3.93) can be rewritten in the same form as (3.81):

$$(\bar{A} + \bar{B}_u K_o \bar{C}_y)\bar{Q} + \bar{Q}(\bar{A} + \bar{B}_u K_o \bar{C}_y)^T < 0, \tag{3.94}$$

where

$$\bar{A} \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 & 0 \\ A & B_p & -\frac{1}{2}I & 0 \\ C_q & 0 & 0 & -\frac{1}{2}\gamma I \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} 0 \\ 0 \\ B_u \\ 0 \end{bmatrix}, \quad \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 & 0 & 0 \end{bmatrix}, \tag{3.95}$$

$$\bar{Q} \triangleq \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{3.96}$$

Now, the next theorem follows Lemma 11.

Theorem 27. *There exists a stabilizing static output-feedback control gain matrix K_o for linear time-invariant discrete-time plant interconnected with a nonlinear mapping $\phi \in \bar{\Phi}_{sb}^{\alpha}$ with the upper sector bound*

$\alpha \triangleq \frac{1}{\sqrt{\gamma}}$ *if and only if there exists* $Q = Q^T > 0$ *such that*

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & B_u^{\perp} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} -Q & 0 & QA^T & QC_q^T \\ 0 & -I & B_p^T & 0 \\ AQ & B_p & -Q & 0 \\ C_qQ & 0 & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & (B_u^{\perp})^T & 0 \\ 0 & 0 & 0 & I \end{bmatrix} < 0, \tag{3.97}$$

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} -P & 0 & A^TP & C_q^T \\ 0 & -I & B_p^TP & 0 \\ PA & PB_p & -P & 0 \\ C_q & 0 & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} < 0, \tag{3.98}$$

where $PQ = QP = I$*.*

From the results in Proposition 12, we can conclude the following theorem.

Theorem 28. *All stabilizing static output-feedback control gain matrices* K_o *for linear time-invariant discrete-time plant interconnected with a nonlinear mapping* $\phi \in \bar{\Phi}_{sb}^{\alpha}$ *with the upper sector bound* $\alpha \triangleq \frac{1}{\sqrt{\gamma}}$ *are parameterized by*

$$K_o = -R^{-1}\bar{B}_u^T\bar{Q}W\bar{C}_y^T(\bar{C}_yW\bar{C}_y^T)^{-1} + S^{1/2}Z(\bar{C}_yW\bar{C}_y^T)^{-1/2}, \tag{3.99}$$

where

$$\Psi = \bar{Q}^{-1}\bar{A}^T + \bar{A}\bar{Q}^{-1} \tag{3.100}$$

$$R^{-1} > \bar{B}_u^{\dagger}[\Psi - \Psi(\bar{B}_u^{\perp})^T(\bar{B}_u^{\perp}\Psi(\bar{B}_u^{\perp})^T)^{-1}\bar{B}_u^{\perp}\Psi](\bar{B}_u^{\dagger})^T \tag{3.101}$$

$$W^{-1} = \bar{Q}\bar{B}_uR^{-1}\bar{B}_u^T\bar{Q} - \bar{Q}\bar{A} - \bar{A}^T\bar{Q} \tag{3.102}$$

$$S = R^{-1} - R^{-1}\bar{B}_u^T\bar{Q}W[W^{-1} - \bar{C}_y^T(\bar{C}_yW\bar{C}_y^T)^{-1}\bar{C}_y]^{-1}W^{-1}\bar{Q}\bar{B}_uR^{-1} > 0, \tag{3.103}$$

in which X *is any positive definite matrix satisfying two LMIs (3.97) and (3.98), and* Z *is any matrix with* $||Z|| < 1$*. In addition,* $||\cdot||$ *can be any matrix norm and* $(\cdot)^{\dagger}$ *indicates the pseudo-inverse of* $(\cdot)$*.*

For a certain class of state-space representation used for describing the system, a sufficient LMI condition for the original non-convex feasibility problem (4.20) can be derived and a suboptimal robust static output-feedback controller synthesis also can be constructed.

Theorem 29. *Let us consider a particular state-space representation for the system where the system matrices A, B_u, C_y ($D_{yu} \equiv 0$) are given and C_y is full row rank which implies that $\bar{C}_y$ is also full row rank. Then the following convex optimization problem gives a tool for designing a robust static output-feedback controller to maximize the robustness of the system under a certain class of perturbation (or uncertainty) $\phi \in \Phi_{sb}^{\alpha}$:*

$$\begin{aligned} \text{minimize} \quad & \gamma && (3.104) \\ \text{subject to} \quad & Q > 0, && (3.105) \\ & \bar{A}\bar{Q} + \bar{Q}\bar{A}^T + \bar{B}_u N \bar{C}_y + \bar{C}_y^T N^T \bar{B}_u^T < 0, && (3.106) \end{aligned}$$

where $\bar{A}, \bar{B}_u, \bar{C}_y$ are the ones defined earlier and $\gamma = \frac{1}{\alpha^2}$. Moreover, a static output-feedback control gain K_o is given by NM^{-1} with the full rank matrix M satisfying $M\bar{C}_y = \bar{C}_y\bar{Q}$, or equivalently $MC_y = C_yQ$. Therefore, the static output-feedback control signal

$$u(k) = NM^{-1}y(t)$$

stabilizes the system (3.79) with the degree of robustness $\alpha^* = \frac{1}{\sqrt{\gamma^*}}$ where γ^* indicates the optimal solution of the above optimization problem.

Theorem 30. *Let us consider a particular state-space representation for the system where the system matrices A, B_u, C_y ($D_{yu} \equiv 0$) are given and B_u is full column rank which implies that $\bar{B}_u$ is also full row rank. Then the following convex optimization problem gives a tool for designing a robust static output-feedback controller to maximize the robustness of the system under a certain class of perturbation (or uncertainty) $\phi \in \Phi_{sb}^{\alpha}$:*

$$\begin{aligned} \text{minimize} \quad & \gamma && (3.107) \\ \text{subject to} \quad & P > 0, && (3.108) \\ & \bar{P}\bar{A} + \bar{A}^T\bar{P} + \bar{B}_u N \bar{C}_y + \bar{C}_y^T N^T \bar{B}_u^T < 0, && (3.109) \end{aligned}$$

where $\bar{P} = diag\{P, I, P, I\}$ and $\bar{A}, \bar{B}_u, \bar{C}_y$ are the ones defined earlier, and $\gamma = \frac{1}{\alpha^2}$. Moreover, a static output-feedback control gain K_o is given by $M^{-1}N$ with the full rank matrix M satisfying $\bar{B}_u M = \bar{P}\bar{B}_u$, or equivalently $B_u M = PB_u$. Therefore, the static output-feedback control signal

$$u(k) = M^{-1}Ny(t)$$

stabilizes the system (3.79) with the degree of robustness $\alpha^* = \frac{1}{\sqrt{\gamma^*}}$ *where* γ^* *indicates the optimal solution of the above optimization problem.*

LMI Optimization Problems for Discrete-Time Lur'e Systems with $\phi \in \Phi_{sb}^{|\alpha|}$

All of the results in the previous section can be reformulated for the Lur'e system whose feedback interconnected nonlinear function is a more general class of sector conditions, $\phi \in \Phi_{sb}^{|\alpha|}$, which is supposed to be defined in componentwise mapping sense. Replacing A by $A + B_u K_o C_y$ in the LMI constraint (3.24), the following feasibility problem is constructed for a static output-feedback controller synthesis: there exist a positive definite matrix $Q = Q^T > 0$, a positive definite and diagonal matrix T, and K_o such that

$$\begin{bmatrix} -Q & 0 & QA^T + QC_y^T K_o^T B_u^T & QC_q^T \\ 0 & -T & TB_p^T & 0 \\ AQ + B_u K_o C_y Q & B_p T & -Q & 0 \\ C_q Q & 0 & 0 & -S_\alpha T \end{bmatrix} < 0. \tag{3.110}$$

The LMI constraint (3.110) can be rewritten in the same form as (3.81):

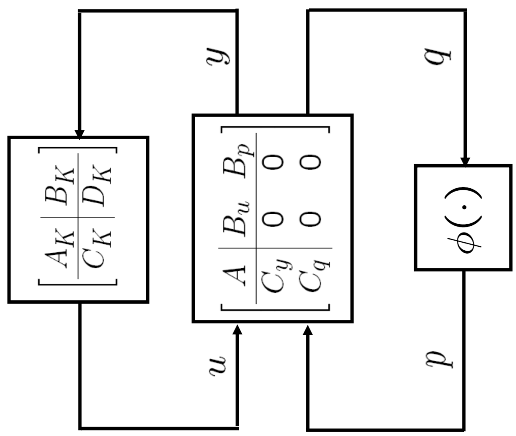
$$(\bar{A} + \bar{B}_u K_o \bar{C}_y)\bar{Q} + \bar{Q}(\bar{A} + \bar{B}_u K_o \bar{C}_y)^T < 0, \tag{3.111}$$

where

$$\bar{A} \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 & 0 \\ A & B_p & -\frac{1}{2}I & 0 \\ C_q & 0 & 0 & -\frac{1}{2}S_\alpha \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} 0 \\ 0 \\ B_u \\ 0 \end{bmatrix}, \quad \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 & 0 & 0 \end{bmatrix}, \tag{3.112}$$

$$\bar{Q} \triangleq \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & T \end{bmatrix}. \tag{3.113}$$

With these definitions for $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{Q}$, all of the LMI conditions for the Lur'e systems with its feedback connected nonlinear function $\phi \in \bar{\Phi}_{sb}^{\alpha}$ can be applied to the stability of the Lur'e systems with its feedback connected nonlinear function $\phi \in \Phi_{sb}^{|\alpha|}$. The detailed algebraic computations and derivations of the LMIs are left for the readers to derive.



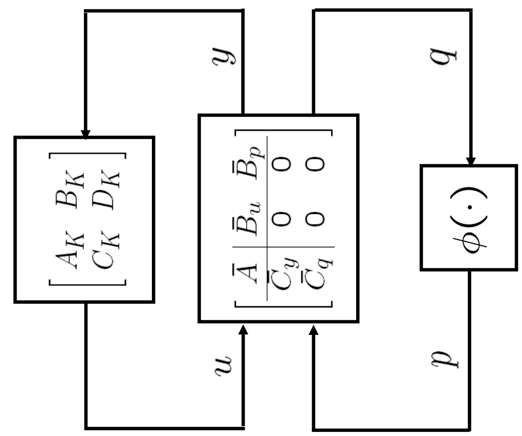


Figure 3.12: LFT representation for DOF Control

Figure 3.13: LFT representation for Equivalent SOF Control

3.4.2 Extension for Dynamic Output-Feedback Control

For the case where the order of the dynamic output-feedback controller is less than or equal to the order of the nominal system, i.e., $n_K \leq n$, the design problem for such DOF controls can be reformulated as an equivalent SOF problem. Let us consider a state space realization of the dynamical output-feedback controller given by

$$\begin{aligned} x_K(k+1) &= A_K x_K(k) + B_K y(k) \\ u(k) &= C_K x_K(k) + D_K y(k), \end{aligned} \tag{3.114}$$

and whose transfer function $K(z) = C_K(zI - A_K)^{-1}B_K + D_K$. Then an augmented closed-loop Lur'e system with the output feedback control law given by

$$u(z) = K(z)y(z), \tag{3.115}$$

where $u(z)$ and $y(z)$ are the z-transformations of $u(k)$ and $y(k)$, respectively, becomes

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}_u u(k) + \bar{B}_p \phi(k, x(k)) \\ \bar{y}(k) &= \bar{C}_y \bar{x}(k), \end{aligned} \tag{3.116}$$

where the output-feedback control law is given as

$$u(k) = K_{dof}\bar{y}(k), \tag{3.117}$$

where

$$K_{dof} \triangleq \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}$$

and

$$\bar{x} \triangleq \begin{bmatrix} x \\ x_K \end{bmatrix}, \quad \bar{y} \triangleq \begin{bmatrix} y \\ x_K \end{bmatrix}, \quad \bar{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} B_u & 0 \\ 0 & I \end{bmatrix}, \quad \bar{B}_p \triangleq \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 \\ 0 & I \end{bmatrix}, \quad \bar{C}_q \triangleq \begin{bmatrix} C_q & 0 \end{bmatrix}. \tag{3.118}$$

Therefore, all of the results for SOF control problems can be extended for DOF control problems when $n_K \leq n$ with transformed state space realization,

$$\bar{G}(z) \triangleq \left[\begin{array}{c|cc} \bar{A} & \bar{B}_u & \bar{B}_p \\ \hline \bar{C}_y & 0 & 0 \\ \bar{C}_q & 0 & 0 \end{array}\right].$$

3.4.3 Static Output-Feedback Control Based on Differentiator-Free Control Model

In the adaptive output-feedback control literature [78], it is well known that there exists k^* such that the transfer function of the plant

$$y_p(s) = W_P(s)u(s) = k_p \frac{D_p(s)}{N_p(s)} u(s) \tag{3.119}$$

together with a so-called differentiator-free controller of ideal adaptive and control parameters, which is given by

$$\begin{array}{c}\zeta_1(k+1) = F\zeta_1(k) + gu(k);\ \zeta_1(0) = 0, \quad \zeta_2(k+1) = F\zeta_2(k) + gy_p(k);\ \zeta_2(0) = 0 \\ u(k) = K_{o,f}\zeta(k); \quad \zeta(k) \triangleq \begin{bmatrix} y_p(k) & \zeta_1^T(k) & \zeta_2^T(k) \end{bmatrix}^T \text{ and } K_{o,f} \triangleq \begin{bmatrix} K_o & K_1 & K_2 \end{bmatrix}\end{array} \tag{3.120}$$

matches the reference model

$$y_m(s) = W_m(s)r(s) = k_m \frac{D_m(s)}{N_m(s)} r(s). \tag{3.121}$$

For a stabilizing controller design, the reference input $r(t)$ is a constant, the origin, and the ideal control gain k_r^* becomes the unit. Motivated by this output-feedback control structure, our control objective is to find a constant control gain vector $k = \begin{bmatrix} k_o & k_1^T & k_2^T \end{bmatrix}^T$ such that the origin of the closed-loop system with the output-feedback controller $u(t) = k^T \zeta(t)$ is globally asymptotically stable.

Augmented Controller Synthesis for Discrete-Time Systems

Introducing the differentiator-free control model is promising for the controller synthesis problem to optimize the performance of the closed-loop system. Let us consider the following nominal system dynamics

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) \\ y(k) &= C_y x(k) \end{aligned} \tag{3.122}$$

with a differentiator-free control model given by

$$\zeta_1(k+1) = F\zeta_1(k) + gu(k); \quad \zeta_1(0) = 0 \tag{3.123}$$

$$\zeta_2(k+1) = F\zeta_2(k) + gy_p(k); \quad \zeta_2(0) = 0 \tag{3.124}$$

$$\zeta(k) \triangleq \begin{bmatrix} y_p(k) & \zeta_1^T(k) & \zeta_2^T(k) \end{bmatrix}^T \tag{3.125}$$

$$K_{o,f} \triangleq \begin{bmatrix} K_o & K_1 & K_2 \end{bmatrix} \tag{3.126}$$

$$u(k) = K_{o,f}\zeta(k), \tag{3.127}$$

where $\zeta_1, \zeta_2 \in \mathbb{R}^{n-1}$ and the realization (F, g) is the minimal state-space realization of $\frac{\varsigma(z)}{\Lambda(z)}$ with the Schur polynomial $\Lambda(z) = z^{n-1} + \lambda_{n-2}z^{n-2} + \cdots + \lambda_1 z + \lambda_0 = \det(zI - F)$.

Comment 4. *(Flexibility of the differentiator-free control model) For two pseudo state variables ζ_1, ζ_2 of the differentiator-free control model, the same system matrices (F, g) are introduced. If different realizations* *are introduced for each pseudo dynamics, i.e.,*

$$(sI - F_1)^{-1} g_1 = \frac{\varsigma_1(z)}{\Lambda_1(z)}, \quad (sI - F_2)^{-1} g_2 = \frac{\varsigma_2(z)}{\Lambda_2(z)} \tag{3.128}$$

then there is more flexibility in output-feedback controller design. However, for convenience, $F = F_1 = F_2$ *and* $g = g_1 = g_2$ *are assumed in the remainder.*

Now, the overall system can be represented by the following augmented system dynamics:

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_{u,f} u(k) \\ y_f(k) &= C_f x_f(k) \\ y(k) &= C_{y,f} x_f(k), \end{aligned} \tag{3.129}$$

where

$$A_f \triangleq \begin{bmatrix} A & 0 & 0 \\ 0 & F & 0 \\ gC_y & 0 & F \end{bmatrix}, \quad B_{u,f} \triangleq \begin{bmatrix} B_u \\ g \\ 0 \end{bmatrix}, \quad C_f \triangleq \begin{bmatrix} C_y & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad C_{y,f} \triangleq \begin{bmatrix} C_y & 0 & 0 \end{bmatrix}. \tag{3.130}$$

Proposition 15. *(Controllability and Observability) The triplet realization* $(A_f, B_{u,f}, C_{y,f})$ *of the augmented system (3.129) is controllable and/or observable if and only if the triplet realization* (A, B_u, C_y) *of the original system (3.129) is controllable and/or observable.*

Proof: The triplet realization (A, B_u) of a system is controllable if and only if

$$\text{rank}\,[A - \lambda I \;\; B_u] = n \quad \forall \lambda \in \mathbb{C}$$

This controllability test is called Popov-Belevitch-Hautus (PBH) tests [54, 10]. It is easy to see that the matrix

$$[A_f - \lambda I \;\; B_{u,f}]$$

has full row rank for all $\lambda \in \mathbb{C}$ if and only if the matrix

$$[A - \lambda I \;\; B_u]$$

does for all $\lambda \in \mathbb{C}$. The proof for observability is the dual part of the proof for controllability. □

Corollary 3. *(Stability Analysis) The augmented system is stable if there exists $Q = Q^T > 0$ such that the following matrix inequality holds:*

$$A_f^T Q^{-1} A_f - Q^{-1} < 0. \tag{3.131}$$

Further, the original system is stable if and only if the augmented system is stable.

Proof: The stability condition (3.131) is from the Lyapunov function $V(\xi) = \xi^T Q^{-1} \xi$. Moreover, since the eigenvalues of A_f are the union of the eigenvectors of A and F, A_f is Shur stable if and only if is A, provided that F is also Schur. □

Stabilizing controller design: convex optimization (feasibility) problems

Lemma 12. *(Stabilizing Static Output-Feedback Controller Synthesis) The static output-feedback control law $u(k) = K_{o,f} y_f(k)$ (or $u(k) = K_{o,f}\zeta(k)$) stabilizes the system (3.122) if $K_{o,f}$ satisfies the matrix inequality*

$$(A_f + B_{u,f} K_{o,f} C_f)^T Q^{-1} (A_f + B_{u,f} K_{o,f} C_f) - Q^{-1} < 0 \tag{3.132}$$

for some Lyapunov matrix $Q^{-1} > 0$. Further, the matrix inequality (3.132) is equivalent with the following inequality:

$$\begin{bmatrix} -Q & Q(A_f + B_{u,f} K_{o,f} C_f)^T \\ (A_f + B_{u,f} K_{o,f} C_f) Q & -Q \end{bmatrix} < 0. \tag{3.133}$$

Proof: The Lyapunov function $V(\xi) = \xi^T Q^{-1} \xi$ is considered to analyze the stability of the origin of the system (3.129). Further, using the Schur complement lemma and congruence transformation one has the equivalent matrix inequality in $K_{o,f}$ and $Q = Q^T > 0$. □

Extensions to Lur'e Systems with the feedback interconnected nonlinear function $\phi \in \bar{\Phi}_{sb}^{\alpha}$

Introducing the differentiator-free control model is promising for the controller synthesis problem to optimize the performance of the closed-loop system. Consider the nominal system dynamics

$$\begin{aligned} x(k+1) &= Ax(k) + B_p \phi(k, x(k)) + B_u u(k) \\ y(k) &= C_y x(k) \end{aligned} \tag{3.134}$$

with a differentiator-free control model given by

$$\zeta_1(k+1) = F\zeta_1(k) + gu(k) \tag{3.135}$$

$$\zeta_2(k+1) = F\zeta_2(k) + gy_p(k) \tag{3.136}$$

$$\zeta(k) \triangleq \begin{bmatrix} y_p(k) & \zeta_1^T(k) & \zeta_2^T(k) \end{bmatrix}^T \tag{3.137}$$

$$K_f \triangleq \begin{bmatrix} K_o & K_1 & K_2 \end{bmatrix} \tag{3.138}$$

$$u(k) = K_f\zeta(k). \tag{3.139}$$

Now, the overall system can be represented by the following augmented system dynamics:

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_{p,f}\phi(k, x(k)) + B_{u,f}u(k) \\ q(k) &= C_{q,f}x_f(k) \\ y(k) &= C_{y,f}x_f(k) \\ y_f(k) &= C_f x_f(k), \end{aligned} \tag{3.140}$$

where

$$A_f \triangleq \begin{bmatrix} A & 0 & 0 \\ 0 & F & 0 \\ gC_y & 0 & F \end{bmatrix}, B_{p,f} \triangleq \begin{bmatrix} B_p \\ 0 \\ 0 \end{bmatrix}, B_{u,f} \triangleq \begin{bmatrix} B_u \\ g \\ 0 \end{bmatrix}, C_{q,f} \triangleq \begin{bmatrix} C_q & 0 & 0 \end{bmatrix}, C_{y,f} \triangleq \begin{bmatrix} C_y & 0 & 0 \end{bmatrix}, C_f \triangleq \begin{bmatrix} C_y & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{3.141}$$

Corollary 4. *(Stability Analysis) The stability condition for the system (3.140) can be written as a feasibility condition for a Lyapunov matrix Q^{-1} which is an eigenvalue problem:*

$$Q = Q^T > 0, \quad \begin{bmatrix} -Q & 0 & QA_f^T & QC_{q,f}^T \\ 0 & -I & B_{p,f}^T & 0 \\ A_fQ & B_{p,f} & -Q & 0 \\ C_{q,f}Q & 0 & 0 & -\gamma I \end{bmatrix} < 0, \tag{3.142}$$

where $\gamma \triangleq 1/\alpha^2$. As linear time-invariant systems, the original system is stable if and only if the augmented system is stable.

Proof: The proof is nothing but an extension of the stability condition for the original Lur'e system (3.134) to the augmented Lur'e system (3.140). □

Stabilizing controller design: convex optimization (feasibility) problems

Since state-feedback control is a special case of static output-feedback control, we will only focus on finding static output-feedback control law without loss of generality.

Lemma 13. *(Stabilizing Static Output-Feedback Controller Synthesis) Replacing A_f by $A_f + B_{u,f}K_{o,f}C_f$ in the LMI constraint (3.142), if there exists a static output-feedback gain matrix $K_{o,f}$ satisfying (3.142) for some $Q = Q^T > 0$ then the closed-loop system is stabilized by that control law $u(k) = K_{o,f}y_f(k)$. The LMI constraint can be rewritten in the canonical form for the static output-feedback control problem:*

$$(\bar{A}_f + \bar{B}_{u,f}K_{o,f}\bar{C}_{y,f})\bar{Q} + \bar{Q}(\bar{A}_f + \bar{B}_{u,f}K_{o,f}\bar{C}_{y,f})^T < 0, \tag{3.143}$$

where

$$\bar{A}_f \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 & 0 \\ A_f & B_{p,f} & -\frac{1}{2}I & 0 \\ C_{q,f} & 0 & 0 & -\frac{1}{2}\gamma I \end{bmatrix}, \ \bar{B}_{u,f} \triangleq \begin{bmatrix} 0 \\ 0 \\ B_{u,f} \\ 0 \end{bmatrix}, \ \bar{C}_{y,f} \triangleq \begin{bmatrix} C_f & 0 & 0 & 0 \end{bmatrix}, \ \bar{Q} \triangleq \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{3.144}$$

Proof: The proof is nothing but an extension of the existence condition for a stabilizing static output-feedback control law of the original Lur'e system (3.134) to the augmented Lur'e system (3.140). □

Remark 11. *(Extensions to Lur'e Systems with the feedback interconnected nonlinear function $\phi \in \Phi_{sb}^{|\alpha|}$) The results for the Lur'e system with the feedback interconnected nonlinear function $\phi \in \bar{\Phi}_{sb}^{\alpha}$ can be trivially extended to Lur'e Systems with the feedback interconnected nonlinear function $\phi \in \Phi_{sb}^{|\alpha|}$. The detailed algebraic computations and derivations of the LMIs are left for the readers to derive.*

3.4.4 Dynamic Output-Feedback Control via LMI Optimization

Discrete-Time Lur'e Systems with $\phi \in \bar{\Phi}_{sb}^{\alpha}$

The critical change of variables defined in the continuous-time cases [21] can be applied to the discrete-time cases. The goal is to design a dynamical output-feedback controller

$$K(z) \triangleq C_K(zI - A_K)^{-1}B_K + D_K \tag{3.145}$$

that stabilizes the closed-loop system and is described by

$$\begin{aligned} x_K(k+1) &= A_K x_K(k) + B_K y(k) \\ u(k) &= C_K x_K(k) + D_K y(k). \end{aligned} \tag{3.146}$$

Consider the nominal system

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) + B_p \phi(k, x(k)) \\ y(k) &= C_y x(k), \end{aligned} \tag{3.147}$$

where $x(k) \in \mathcal{X} \subset \mathbb{R}^n$ is the state variable of the system at time $k \in \mathbb{Z}_+$ and $\phi : \mathbb{Z}_+ \times \mathcal{X} \to \mathbb{R}^n_p$ is a time varying perturbation function. With the plant and controller defined above, the closed-loop system admits the realization

$$\bar{x}(k+1) = A_{cl}\bar{x}(k) + B_{cl}\bar{\phi}(k, \bar{x}(k)) \tag{3.148}$$

where

$$\bar{x} \triangleq \begin{bmatrix} x \\ x_K \end{bmatrix}, \quad A_{cl} \triangleq \begin{bmatrix} A + B_u D_K C_y & B_u C_K \\ B_K C_y & A_K \end{bmatrix}, \quad B_{cl} \triangleq \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad \bar{\phi}(k, \bar{x}(k)) \equiv \phi(k, x(k)) \; \forall k \in \mathbb{Z}_+. \tag{3.149}$$

To establish robust stability under the constraint

$$\begin{bmatrix} \bar{x} \\ \bar{\phi} \end{bmatrix}^T \begin{bmatrix} -\alpha^2 \bar{C}_q^T \bar{C}_q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{\phi} \end{bmatrix} \leq 0 \tag{3.150}$$

with maximal α, the following EVP is solved:

$$\text{minimize} \quad \gamma \tag{3.151}$$

$$\text{subject to} \quad Q > 0 \tag{3.152}$$

$$\begin{bmatrix} -Q & 0 & QAcl^T & Q\bar{C}_q^T \\ 0 & -I & B_{cl}^T & 0 \\ A_{cl}Q & B_{cl} & -Q & 0 \\ \bar{C}_q Q & 0 & 0 & -\gamma I \end{bmatrix} < 0, \tag{3.153}$$

where $\alpha \triangleq \frac{1}{\sqrt{\gamma}}$ is the robust stability margin and

$$\bar{C}_q \triangleq \begin{bmatrix} C_q & 0 \end{bmatrix} \tag{3.154}$$

is a constant matrix with the compatible dimensions. Moreover, for any given $\bar{C}_q$, the inequality (3.150) defines a class of piecewise continuous functions

$$\bar{\Phi}_{sb}^{\alpha} \triangleq \{\phi : \mathbb{Z}_+ \times \mathbb{R}^{n+n_K} \to \mathbb{R}^{n+n_K} \mid \phi^T\phi \le \alpha^2 \bar{x}^T \bar{C}_q^T \bar{C}_q \bar{x}, \text{ for all } (t, \bar{x}) \in \mathbb{Z}_+ \times \mathbb{R}^{n+n_K}\}. \tag{3.155}$$

Recall that $P \in \mathbb{R}^{(n+n_K)\times(n+n_K)}$ is a symmetric matrix. Define the matrices X and Y as $n \times n$ submatrices of P and P^{-1}, by

$$P \triangleq \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix}, \qquad P^{-1} \triangleq \begin{bmatrix} X & M \\ M^T & * \end{bmatrix}. \tag{3.156}$$

From the definition of the inverse matrix, we infer

$$P\Pi_1 = \Pi_2 \text{ with} \tag{3.157}$$

$$\Pi_1 \triangleq \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \quad \Pi_2 \triangleq \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}. \tag{3.158}$$

Define the change of controller variables introduced in [21]:

$$\hat{A} \triangleq NA_K M^T + NB_K C_y X + YB_u C_K M^T + Y(A + B_u D_K C_y)X \tag{3.159}$$

$$\hat{B} \triangleq NB_K + YB_u D_K \tag{3.160}$$

$$\hat{C} \triangleq C_K M^T + D_K C_y X \tag{3.161}$$

$$\hat{D} \triangleq D_K. \tag{3.162}$$

Note that if M and N have full low rank and $\hat{A}, \hat{B}, \hat{C}, \hat{D}, X, Y$ are given, we can always compute controller matrices A_K, B_K, C_K, and D_K. If M and N are square ($n_K = n$) and invertible matrices, then A_K, B_K, C_K, and D_K are unique. For full-order design, we can assume that M and N have full row rank. Hence the variables A_K, B_K, C_K, and D_K can be replaced by their one-to-one correspondences $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$ without loss of generality. When performing a congruence transformation with $\mathrm{diag}(\Pi_2, I, \Pi_2, I)$ on the inequality

(3.153), the following inequality is derived:

$$\begin{bmatrix} -\Pi_2^T Q \Pi_2 & 0 & \Pi_2^T Q A_{cl}^T \Pi_2 & \Pi_2^T Q \bar{C}_q^T \\ 0 & -I & B_p^T \Pi_2 & 0 \\ \Pi_2^T A Q \Pi_2 & \Pi_2^T B_p & -\Pi_2^T Q \Pi_2 & 0 \\ HQ\Pi_2 & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{3.163}$$

which is equivalent to

$$\begin{bmatrix} -X & -I & 0 & XA^T + (B_u\hat{C})^T & \hat{A}^T & XC_q^T \\ -I & -Y & 0 & (A + B_u\hat{D}C_y)^T & A^TY + (\hat{B}C_y)^T & C_q^T \\ 0 & 0 & -I & B_p^T & B_p^T Y & 0 \\ AX + B_u\hat{C} & (A + B_u\hat{D}C) & B_p & -X & -I & 0 \\ \hat{A} & YA + (\hat{B}C_y) & YB_p & -I & -Y & 0 \\ C_qX & C_q & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0. \tag{3.164}$$

The LMI (3.164) is clearly affine in $\hat{A}, \hat{B}, \hat{C}, \hat{D}, X$, and Y. Thus we have proved that the solvability or feasibility of this LMI is necessary for the existence of a robust stabilizing controller which resists the sector-bounded (or norm-bounded), possibly time-varying, perturbations. Thus the design of the (dynamic) full-order output feedback controller that maximizes the quadratic robust stability margin α of the closed-loop system can be formulated as the EVP:

$$\text{minimize} \quad \gamma \tag{3.165}$$

$$\text{subject to} \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \tag{3.166}$$

$$\begin{bmatrix} -X & -I & 0 & XA^T + (B_u\hat{C})^T & \hat{A}^T & XC_q^T \\ -I & -Y & 0 & (A + B_u\hat{D}C_y)^T & A^TY + (\hat{B}C_y)^T & C_q^T \\ 0 & 0 & -I & B_p^T & B_p^T Y & 0 \\ AX + B_u\hat{C} & (A + B_u\hat{D}C_y) & B_p & -X & -I & 0 \\ \hat{A} & YA + (\hat{B}C_y) & YB_p & -I & -Y & 0 \\ C_qX & C_q & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{3.167}$$

where since the requirement $P > 0$ must be satisfied, the constraint

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{3.168}$$

should always be included in the list of synthesis LMIs, either explicitly or as part of some other LMI constraints. After solving the synthesis optimization problem above, the controller construction proceeds as follows: find nonsingular matrix M to satisfy

$$MN^T = I - XY \tag{3.169}$$

and define the controller by

$$\begin{aligned} D_K &= \hat{D} \\ C_K &= (\hat{C} - D_K C_y X)M^{-T} \\ B_K &= N^{-1}(\hat{B} - YB_u D_K) \\ A_K &= N^{-1}\left(\hat{A} - NB_K C_y X - YB_u C_K M^T - Y(A + B_u D_K C_y)X\right) M^{-T}. \end{aligned} \tag{3.170}$$

This gives a formal description of all problems to which we can apply the controller parameter transformation in order to obtain the synthesis LMIs. Note that the necessity part of the proof does not restrict the order of the controller and that the construction in the sufficiency part leads to a controller that is of the same order as the plant.

3.4.5 Illustrative Examples

We consider some numerical examples to show the applicability of the previously defined LMI synthesis problems for a certain case of Lur'e systems which is given as (3.79) where the feedback connected nonlinear mapping is classified as $\phi \in \bar{\Phi}_{sb}^{\alpha}$ for the optimization problems previously formulated or $\phi \in \Phi_{sb}^{|\alpha|}$ for the feasibility problems previously formulated.

Example 5. *(Optimal SOF control for the system (3.79) with $\phi \in \bar{\Phi}_{sb}^{\alpha}$) Consider the system (3.79) whose* *state space realization is given by*

$$A = \begin{bmatrix} 0.9996 & 0.0000 & 0.0000 & 0.0000 \\ 0.0002 & 0.9998 & 0.0000 & 0.0000 \\ 0.0002 & 0.0000 & 0.9994 & 0.0000 \\ 0.0000 & 0.0000 & 0.0001 & 0.9999 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.0000 \\ 0.0200 \\ 0.0000 \\ 0.0200 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.0200 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \end{bmatrix}, \quad C_q = \begin{bmatrix} 0 & -0.0100 & 0 & -0.0100 \end{bmatrix}.$$

Let us consider the optimization problem where the control objective is to maximize the upper bound α on the sector condition such that the closed-loop system (3.79) is stabilized by the output-feedback control law $u(k) = K_o y(k)$. Then, the optimal solutions K^* for the optimization problems in Theorem 27, 29, and 30 are obtained as

$$\begin{aligned} \textit{Theorem 27}: \quad K^* &= \begin{bmatrix} -0.00554519341685 & 0.00288799826175 \end{bmatrix}, \\ \textit{Theorem 28}: \quad K^* &= \begin{bmatrix} -0.01302109408254 & -0.29231658319437 \end{bmatrix}, \\ \textit{Theorem 29}: \quad K^* &= \begin{bmatrix} -0.00133373405726 & -0.00799964157676 \end{bmatrix}, \\ \textit{Theorem 30}: \quad K^* &= \begin{bmatrix} -0.00530655880274 & 0.00222723380322 \end{bmatrix}. \end{aligned}$$

Example 6. *(Optimal SOF control for the system (3.79) with $\phi \in \bar{\Phi}_{sb}^{\alpha}$) Consider the system (3.79) whose state space realization is given by*

$$A = \begin{bmatrix} 0.9722 & -0.0103 & -0.0370 \\ -0.0338 & 0.9664 & -0.0233 \\ 0.0183 & 0.0398 & 0.9506 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.0046 \\ 0.0078 \\ 0.0129 \end{bmatrix}, \quad B_p = \begin{bmatrix} -0.0034 & -0.0258 \\ 0.0159 & 0.0192 \\ 0.0083 & 0.0128 \end{bmatrix},$$

$$C_y = \begin{bmatrix} -0.4100 & 0.4400 & 0.6800 \\ -1.7700 & 0.5000 & -0.4000 \end{bmatrix}, \quad C_q = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \end{bmatrix}.$$

Let us consider the optimization problem where the control objective is to maximize the upper bound α on the sector condition such that the closed-loop system (3.79) is stabilized by the output-feedback control law $u(k) = K_o y(k)$. Then, the optimal solutions K^* for the optimization problems in Theorem 27, 29, and 30 *are obtained as*

$$\textit{Theorem 27}: \quad K^* = 10^2 \times \begin{bmatrix} -1.67396832729383 & -0.38942209046374 \end{bmatrix},$$

$$\textit{Theorem 28}: \quad K^* = \begin{bmatrix} -5.62391896169544 & 1.47109250263480 \end{bmatrix},$$

$$\textit{Theorem 29}: \quad K^* = \begin{bmatrix} -64.01393096811263 & 14.73009846418861 \end{bmatrix},$$

$$\textit{Theorem 30}: \quad K^* = \begin{bmatrix} -15.82846011893918 & 3.57952990338737 \end{bmatrix}.$$

Example 7. *(Optimal SOF control for the system (3.79) with $\phi \in \bar{\Phi}_{sb}^{\alpha}$) Consider the system (3.79) whose state space realization is given by*

$$A = \begin{bmatrix} 0.0469 & 0.0000 & 0.0000 \\ 0.0000 & -0.1512 & 0.0000 \\ 0.0000 & 0.0000 & 0.1512 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.2365 & -0.9876 \\ -1.5164 & 1.2181 \\ 0.2496 & -0.5181 \end{bmatrix}, \quad B_p = \begin{bmatrix} -0.2494 & 0.2341 \\ 0.2542 & 0.0215 \\ -0.2036 & -0.3744 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \end{bmatrix}, \quad C_q = \begin{bmatrix} 1.0000 & 0.0000 & 1.0000 \\ 0.0000 & 1.0000 & 1.0000 \end{bmatrix}.$$

Let us consider the optimization problem where the control objective is to maximize the upper bound α on the sector condition such that the closed-loop system (3.79) is stabilized by the output-feedback control law $u(k) = K_o y(k)$. Then, the optimal solutions K^* for the optimization problems in Theorem 27, 29, and 30 are obtained as

$$\textit{Theorem 27}: \quad K^* = \begin{bmatrix} -0.01938448484711 & -0.02871614200279 \\ -0.02871614200279 & -0.01185342501961 \end{bmatrix},$$

$$\textit{Theorem 28}: \quad K^* = \begin{bmatrix} 0.04190432483494 & 0.01148477459598 \\ -0.11549610629164 & -0.05097686400942 \end{bmatrix},$$

$$\textit{Theorem 29}: \quad K^* = \begin{bmatrix} -0.14258812542474 & -0.12686178592710 \\ 0.12820257392364 & 0.27846823294168 \end{bmatrix},$$

$$\textit{Theorem 30}: \quad K^* = \begin{bmatrix} 0.04791671972032 & -0.12540884561053 \\ -0.33039453130135 & -0.23684835212644 \end{bmatrix}.$$

Example 8. *(Optimal DOF control for the system (3.116) or (3.147) with $\phi \in \bar{\Phi}_{sb}^{\alpha}$) Consider the system (3.116) or (3.147) whose state space realization for the nominal plant is the same as in Example 5. Let* *us consider the optimization problem where the control objective is to maximize the upper bound α on the sector condition such that the closed-loop system (3.79) is stabilized by the output-feedback control law $u(z) = K(z)y(z)$ where $\xi(z)$ is the z-transform of $\xi(k)$ for each ξ $in\{u, y, K\}$. Then, the optimal solutions $K_o(z)$ for the optimization problems in Section 3.4.2 and 3.4.4 are obtained as*

$$\textit{Section 3.4.2}: \quad K_o(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_k \end{array}\right];$$

$$A_K = \begin{bmatrix} 0.00000000000004 & -0.00000000000002 & 0.00000000000003 & -0.00000000000006 \\ -0.00000000000004 & 0.00000000000011 & -0.00000000000004 & 0.00000000000008 \\ 0.00000000000001 & -0.00000000000002 & 0.00000000000004 & -0.00000000000004 \\ -0.00000000000003 & 0.00000000000011 & -0.00000000000004 & 0.00000000000005 \end{bmatrix},$$

$$B_K = \begin{bmatrix} -0.00000000482683 & -0.00000004832344 \\ -0.00000005476478 & 0.00000046669972 \\ -0.00000000314110 & -0.00000002877317 \\ 0.00000000405537 & 0.00000018313272 \end{bmatrix},$$

$$C_K = \begin{bmatrix} -0.00000009254012 & 0.00000058061119 & -0.00000005652954 & 0.00000037704576 \end{bmatrix},$$

$$D_K = \begin{bmatrix} -0.12463151573796 & -0.00041376802662 \end{bmatrix}.$$

$$\textit{Section 3.4.4}: \quad K_o(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_k \end{array}\right];$$

$$A_K = \begin{bmatrix} -0.09438308068412 & -0.64758523926930 & -0.00230545695179 & -0.00024376280078 \\ -0.68119470507844 & -0.11276229377833 & -0.00266195332435 & 0.00098217901763 \\ -0.36869938691084 & 0.19774166820556 & 0.00015087641080 & -0.00007797185095 \\ -47.06343894684123 & -10.24014681053870 & -0.14349557136522 & 0.08069214977010 \end{bmatrix},$$

$$B_K = \begin{bmatrix} -0.00117518986784 & -0.11714899136500 \\ 0.06569434370426 & -1.71655962342927 \\ -0.07142219208562 & -0.60188578881433 \\ -56.56650343686050 & -125.16716273829300 \end{bmatrix},$$

$$C_K = \begin{bmatrix} 12.33330341910328 & -3.71599785603426 & 0.01420494573434 & -0.00203461338737 \end{bmatrix},$$

$$D_K = \begin{bmatrix} -0.00144086465809 & -27.27591728963192 \end{bmatrix}.$$

Example 9. *(Optimal DOF control for the system (3.116) or (3.147) with* $\phi \in \bar{\Phi}^{\alpha}_{sb}$*) Consider the system (3.116) or (3.147) whose state space realization for the nominal plant is the same as in Example 6. Let us consider the optimization problem where the control objective is to maximize the upper bound* α *on the sector condition such that the closed-loop system (3.79) is stabilized by the output-feedback control law* $u(z) = K(z)y(z)$ *where* $\xi(z)$ *is the z-transform of* $\xi(k)$ *for each* ξ *in*$\{u, y, K\}$*. Then, the optimal solutions* $K_o(z)$ *for the optimization problems in Section 3.4.2 and 3.4.4 are obtained as*

$$\textit{Section 3.4.2}: \quad K_o(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_k \end{array}\right];$$

$$A_K = 10^2 \times \begin{bmatrix} 0.00000000000548 & 0.00000000000088 & 0.00000000000073 \\ 0.00000000000088 & 0.00000000000014 & 0.00000000000012 \\ 0.00000000000073 & 0.00000000000012 & 0.00000000000010 \end{bmatrix},$$

$$B_K = 10^2 \times \begin{bmatrix} -0.00000202270988 & -0.00000136477133 \\ -0.00000032548647 & -0.00000021910356 \\ -0.00000027105445 & -0.00000018282697 \end{bmatrix},$$

$$C_K = 10^2 \times \begin{bmatrix} 0.00000137154146 & 0.00000021836004 & 0.00000018351858 \end{bmatrix},$$

$$D_K = 10^2 \times \begin{bmatrix} -2.10988117920826 & -0.77746531450058 \end{bmatrix}.$$

$$\text{Section 3.4.4}: \quad K_o(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_k \end{array}\right];$$

$$A_K = \begin{bmatrix} -0.01612395466721 & -0.00479276552508 & 0.00688942417057 \\ 0.95342728765310 & 0.85955992922295 & 0.01944695054919 \\ -7.79414061121223 & -7.16554494648548 & -0.26162587289086 \end{bmatrix},$$

$$B_K = \begin{bmatrix} 0.11521191111547 & 0.08615317152045 \\ 0.43763488357333 & 0.08970208601816 \\ -4.80743529087313 & -1.60981658615619 \end{bmatrix},$$

$$C_K = \begin{bmatrix} 15.59181171535629 & 9.61926447352134 & -3.12719806437947 \end{bmatrix},$$

$$D_K = 10^2 \times \begin{bmatrix} -1.27292064065320 & -0.15203017584326 \end{bmatrix}.$$

Example 10. *(Optimal DOF control for the system (3.116) or (3.147) with $\phi \in \bar{\Phi}^{\alpha}_{sb}$) Consider the system (3.116) or (3.147) whose state space realization for the nominal plant is the same as in Example 7. Let us consider the optimization problem where the control objective is to maximize the upper bound α on the sector condition such that the closed-loop system (3.79) is stabilized by the output-feedback control law $u(z) = K(z)y(z)$ where $\xi(z)$ is the z-transform of $\xi(k)$ for each ξ $in\{u, y, K\}$. Then, the optimal solutions $K_o(z)$ for the optimization problems in Section 3.4.2 and 3.4.4 are obtained as*

$$\text{Section 3.4.2}: \quad K_o(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_k \end{array}\right];$$

$$A_K = \begin{bmatrix} 0.00153264721512 & 0.00739742518440 & -0.00074014941071 \\ 0.00739742518440 & 0.16700774306169 & 0.03209087166099 \\ -0.00074014941071 & 0.03209087166099 & 0.01004390072236 \end{bmatrix},$$

$$B_K = \begin{bmatrix} 0.00133426941877 & 0.00368129857959 \\ 0.04466024627003 & 0.00418820864814 \\ 0.00973663652653 & -0.00546618881788 \end{bmatrix},$$

$$C_K = \begin{bmatrix} 0.00133426941877 & 0.04466024627003 & 0.00973663652653 \\ 0.00368129857959 & 0.00418820864814 & -0.00546618881788 \end{bmatrix},$$

$$D_K = \begin{bmatrix} -0.04365449948212 & -0.08999104228215 \\ -0.08999104228215 & -0.09456809489039 \end{bmatrix}.$$

Control Schemes	Ex 5 & 8	Ex 6 & 9	Ex 7 & 10
Theorem 27 & 28	21.8420	1.5325	2.0607
Theorem 29	4.1535	0.4367	1.2322
Theorem 30	37.4841	0.3882	1.4358
Section 3.4.2	25.4709	1.5325	2.0754
Section 3.4.4	22.1679	1.7502	2.0540

Table 3.1: The maximal upper sector-bound achieved by the output-feedback control schemes

$$\textit{Section 3.4.4}: \quad K_o(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_k \end{array}\right];$$

$$A_K = 10^3 \times \begin{bmatrix} 0.00005539158542 & 0.00000017085965 & -0.00000000122581 \\ 0.02689099890390 & 0.00008548551544 & -0.00000001002758 \\ 9.99905232328686 & 0.03224763285586 & 0.00010649556568 \end{bmatrix},$$

$$B_K = \begin{bmatrix} 0.00020788964672 & 0.00031606307571 \\ 0.10014135234422 & 0.15416936092571 \\ 37.09397281334177 & 57.45846222071164 \end{bmatrix},$$

$$C_K = \begin{bmatrix} 39.89768712841692 & 0.26594096699122 & 0.00027787750491 \\ 31.72975576809898 & 0.20751916077895 & 0.00020206180352 \end{bmatrix},$$

$$D_K = \begin{bmatrix} 0.15523936011026 & 0.14587516460479 \\ 0.14587516460479 & 0.18346177851977 \end{bmatrix}.$$

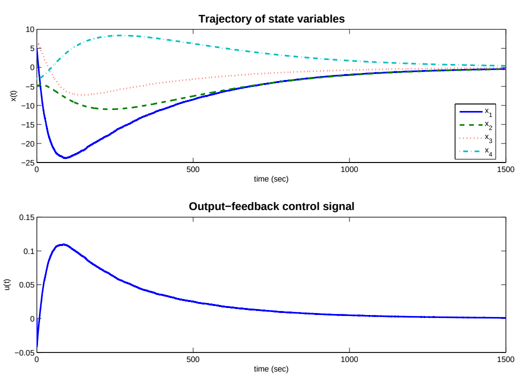


Figure 3.14: Example 5: Theorem 27

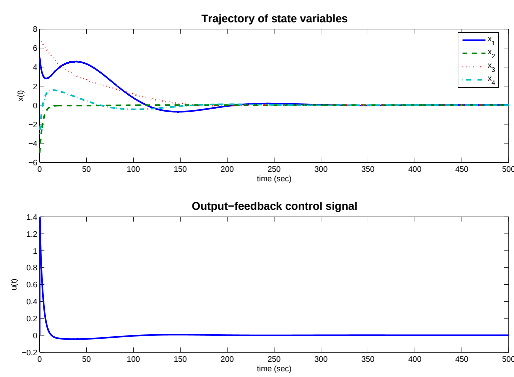


Figure 3.15: Example 5: Theorem 28

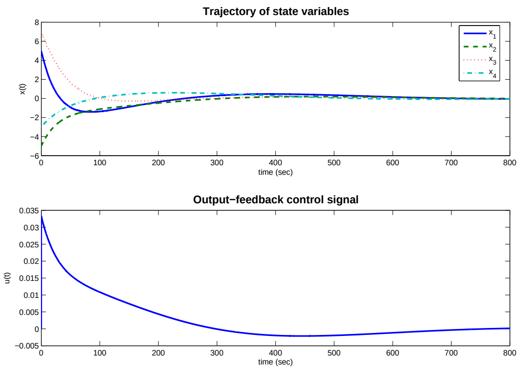


Figure 3.16: Example 5: Theorem 29

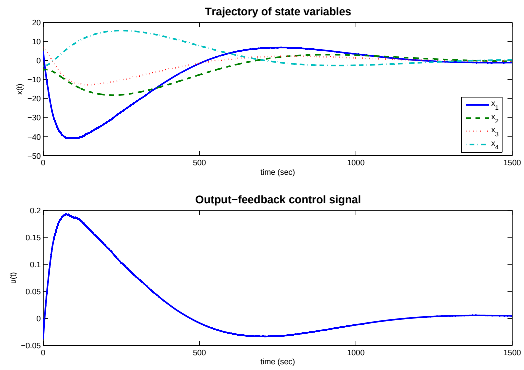


Figure 3.17: Example 5: Theorem 30

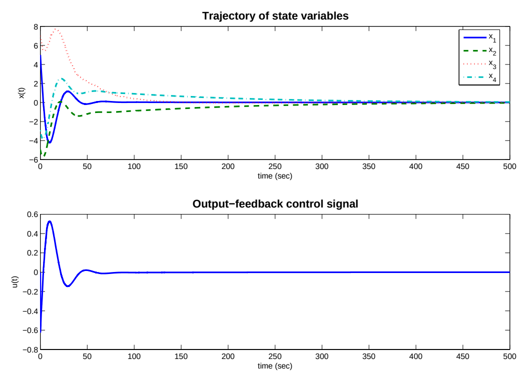


Figure 3.18: Example 8: Section 3.4.2

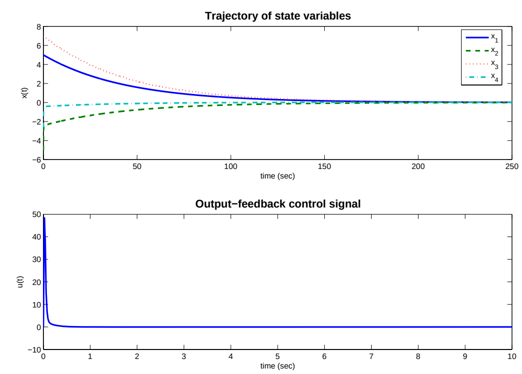


Figure 3.19: Example 8: Section 3.4.4

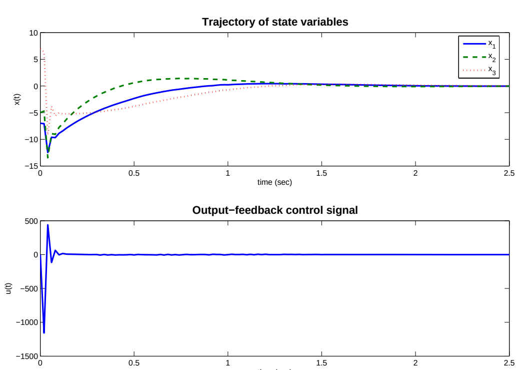


Figure 3.20: Example 6: Theorem 27

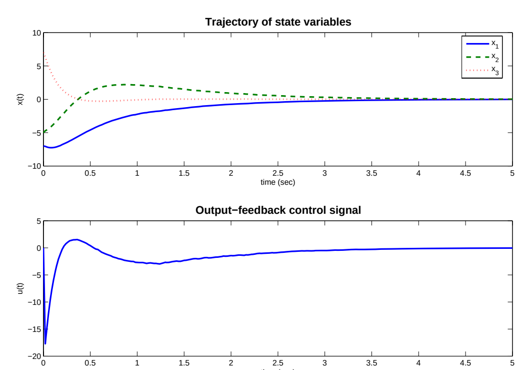


Figure 3.21: Example 6: Theorem 28

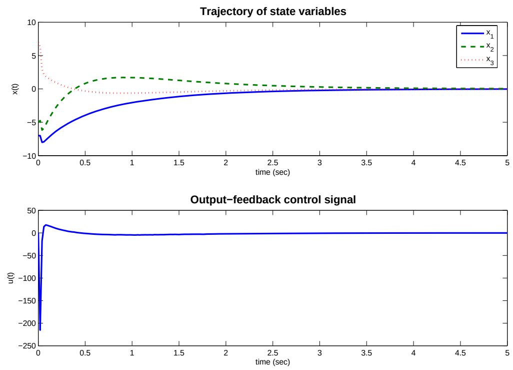


Figure 3.22: Example 6: Theorem 29

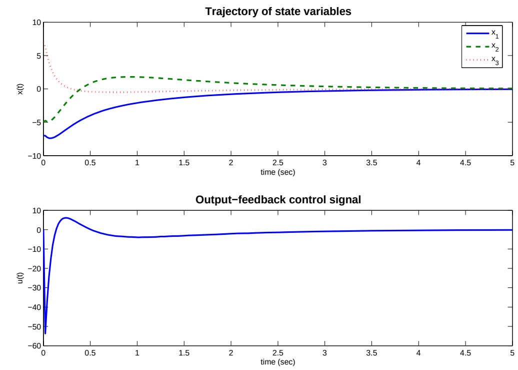


Figure 3.23: Example 6: Theorem 30

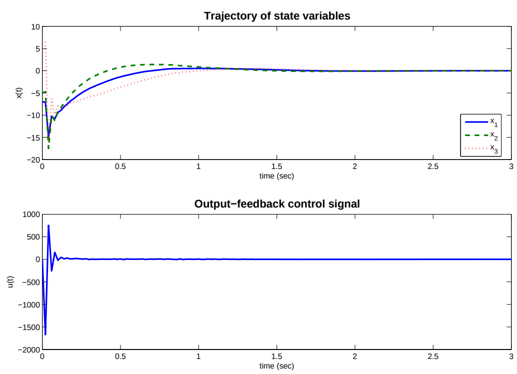


Figure 3.24: Example 9: Section 3.4.2

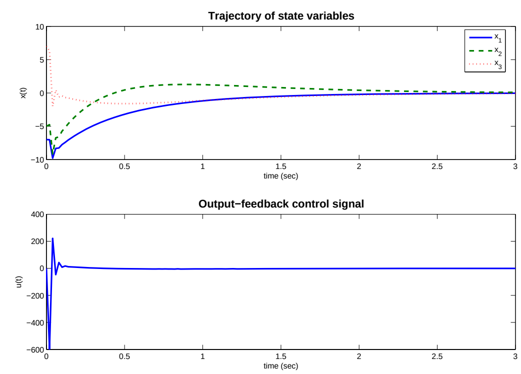


Figure 3.25: Example 9: Section 3.4.4

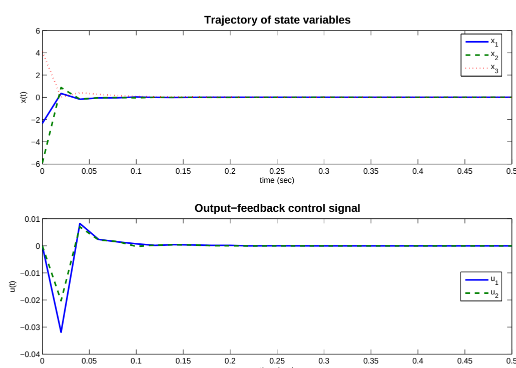


Figure 3.26: Example 7: Theorem 27

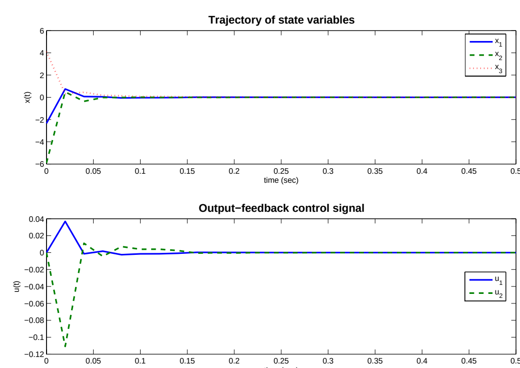


Figure 3.27: Example 7: Theorem 28

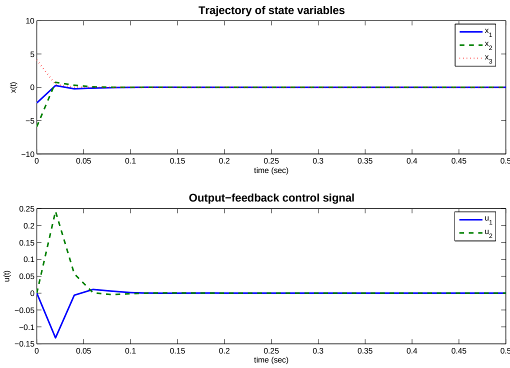


Figure 3.28: Example 7: Theorem 29

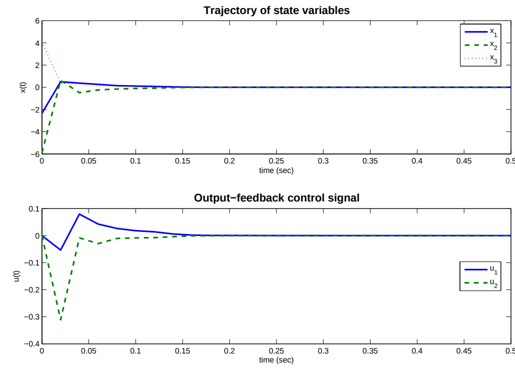


Figure 3.29: Example 7: Theorem 30

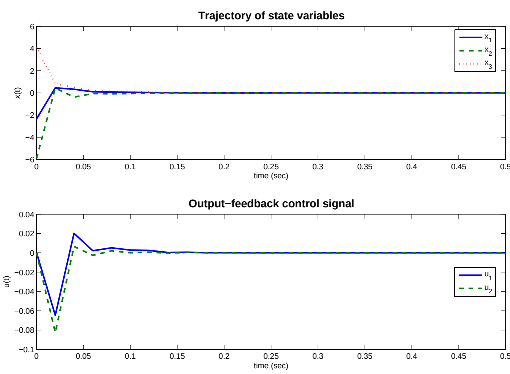


Figure 3.30: Example 10: Section 3.4.2

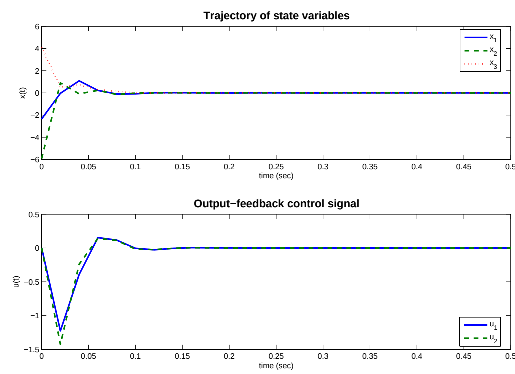


Figure 3.31: Example 10: Section 3.4.4

3.5 Computational Issues in SOF Controller Synthesis Problems

3.5.1 Introduction

The main objective of this section is to introduce the numerical algorithms to solve the stabilizing static output-feedback control problem and to discuss associated computational issues. Given a triplet state-space realization of a system (A, B_u, C_y) with compatible dimensions, an interesting problem is to find a static output-feedback control gain K_o, if any, such that the closed-loop system is asymptotically stable. The existence of such a stabilizing static output-feedback control gain K_o is equivalent to the feasibility condition of two convex constraints involving a positive definite matrix and its inverse [40, 41]. This problem can be formulated as a problem to locate a common point in two convex sets and there are many papers in control and optimization literatures that tackle the problem, where alternating or successive projection (or proximity) mappings are considered with convex constraints that have been specified in different ways at each paper [43, 59, 15].

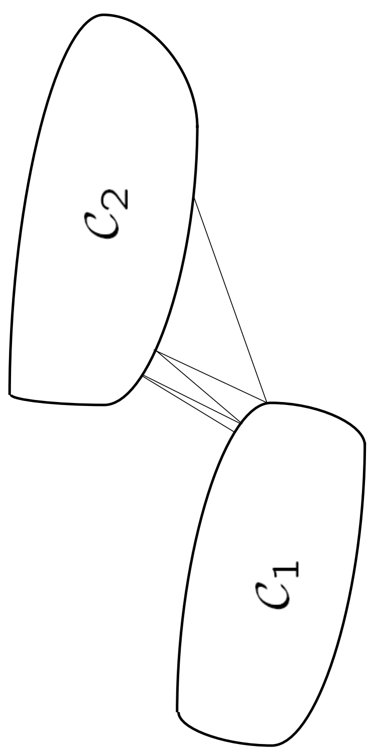


Figure 3.32: Successive Projections: rel int $\mathcal{C}_1 \cap$ rel int $\mathcal{C}_2 = \emptyset$

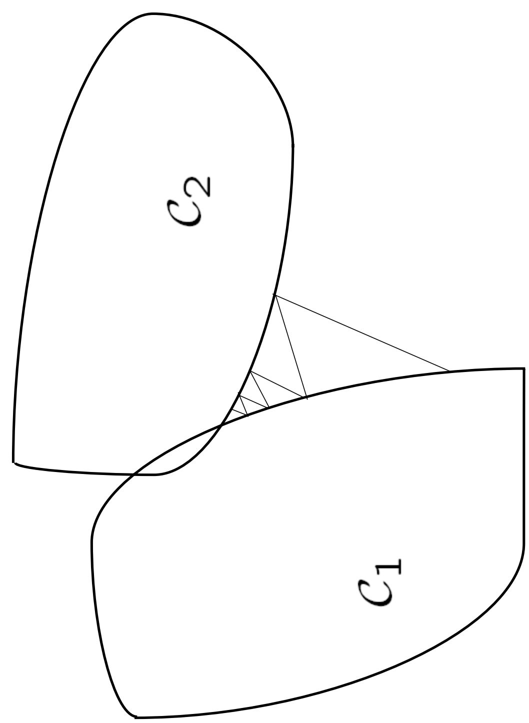


Figure 3.33: Successive Projections: rel int $\mathcal{C}_1 \cap$ rel int $\mathcal{C}_2 \neq \emptyset$

3.5.2 Successive Projection Algorithm

Projection onto A Convex Set

The method of projection is applied to the problem of finding points in a convex set of minimum distance from a point in another convex set, and vice versa.

Definition 10. *(Projection mappings) Given a closed convex set $\mathcal{C} \subset \mathbb{R}^n$, let P be the mapping which projects each point x in another convex set $\mathcal{D} \subset \mathbb{R}^n$ onto $\mathcal{C}$ such that $Px \in \mathcal{D}$ is the closet point in $\mathcal{D}$ to x. Then the mapping P is called the projection mapping or proximity mapping for the set $\mathcal{C}$. Consider two closed convex sets or two convex sets whose interiors are non-empty and one of them is possibly open such that their intersection is non-empty. Then the composition of two successive projection mappings has at least one fixed point and this point is a point of a set closest to the other set.*

Proposition 16. *(Composition of projection mappings [117]) Let S be a projection mapping of a metric space $\mathcal{C} \in \mathbb{R}^n$ onto itself. Then it has the following properties:*

i. $d(Sx_1, Sx_2) \leq d(x_1, x_2)$,

ii. If x is not a fixed point of S, then the mapping S is strictly non-expansive in the following sense: $d(Sx, S^2x) \leq d(x, Sx)$,

iii. For each x the sequence of mappings $S^n x$ has a cluster point. Then for each given x in a matric space, the sequence of mappings $S^n x$ converges to a fixed point of S.

The function $d(\cdot, \cdot)$ on the cross product space $\mathcal{C} \times \mathcal{C}$ indicates the distance between two points (or sets) in a metric space, which is equal to the metric norm of the difference of two points (or the metric norm of the difference of two closest points to each set).

Proof: It can be easily seen that a successive n mapping S^n of a projection operator S has contraction mapping properties [26] such that it has the properties described above. See [117] for more details. □

Now, the successive projection mapping is applied to tackle alternating projection mappings for two convex sets in a matric space.

Lemma 14. *(Projection mappings for two convex sets [117]) Consider (i) two closed convex sets or (ii) two convex sets whose interiors are non-empty and one of them is possibly open such that their intersection is non-empty. Let P_j denote the projection mapping for a convex set $\mathcal{C}_j$ for each $j = 1, 2$. Then any fixed point of their composition mapping P_iP_j is a closest point of $\mathcal{C}_i$ to $\mathcal{C}_j$ for $(i, j) \in \{(1, 2), (2, 1)\}$. Further, such* *fixed point of P_iP_j is on the boundary of $\mathcal{C}_i$ for $(i,j) \in \{(1,2),(2,1)\}$ in the case (i), and in the intersection of two convex sets in the case (ii).*

Proof: Let $S = P_iP_j$ be a projection mapping of a subset $\mathcal{C}_i$ in a metric space onto itself for $(i,j) \in \{(1,2),(2,1)\}$. With this definition for the alternating projection mapping, the proof is trivial, which follows Proposition 16. The details are given in [117]. □

Alternating Projection and Its Convergence to a point in the intersection of some convex sets

Suppose that the interiors of two convex sets $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{R}^n$ are non-empty and let $P_{\mathcal{C}_i}$ denote projection on the set $\mathcal{C}_i$ for each $i = 1, 2$. In addition, it is assumed that a starting point $x_1^0 \in \mathcal{C}_1$ is given. Then the alternating or successive projection mappings are defined as $x_2^{(k)} = P_{\mathcal{C}_2}(x_1^{(k)})$ and $x_1^{(k+1)} = P_{\mathcal{C}_1}(x_2^{(k)})$ for each iteration index $k = 0, 1, \ldots$ such that two sequences of points $\{x_i^{(k)}\}$ are generated for $i = 1, 2$. There have been many research papers that have proven convergence of the alternating or successive projection mappings in the literature and whose basic idea should be considered as von Neumann's work in 1930s [116].

Lemma 15. *(Convergence of alternating projection algorithm in Hilbert Spaces) If two convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$ have the non-empty intersection, i.e., $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, then the sequence of points $\{y^{(k)}\} := \{x_1^{(0)}, x_2^{(1)}, x_1^{(1)}, x_2^{(2)}, \ldots\}$ has a cluster point in $\mathcal{C}_1 \cap \mathcal{C}_2$. In other words, the composition of successive mappings $S := P_{\mathcal{C}_i\mathcal{C}_j}$ for $(i,j) \in \{(1,2),(2,1)\}$ gives a point in the intersection of two convex sets, provided that their intersection is non-empty.*

Proof: Since two closed and convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$ in a Hilbert space have non-empty intersection, there exists a point x^* in their intersection, i.e., $x^* \in \mathcal{C}_1 \cap \mathcal{C}_2$. Further, since $x_2^{(k)} = P_{\mathcal{C}_2}(x_1^{(k)})$ and $\mathcal{C}_2$ is a convex set in a Hilbert space, we have

$$\langle x_1^{(k)} - x_2^{(k)}, x - x_2^{(k)} \rangle \leq 0 \quad \forall x \in \mathcal{C}_2,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two elements in a Hilbert space. Thus, we can observe that

$$||x_1^{(k)} - x^*||^2 = ||x_1^{(k)} - x_2^{(k)} + x_2^{(k)} - x^*||^2 \tag{3.171}$$

$$= ||x_1^{(k)} - x_2^{(k)}||^2 + ||x_2^{(k)} - x^*||^2 + 2\langle x_1^{(k)} - x_2^{(k)}, x_2^{(k)} - x^* \rangle \tag{3.172}$$

$$\geq ||x_1^{(k)} - x_2^{(k)}||^2 + ||x_2^{(k)} - x^*||^2. \tag{3.173}$$

Similarly, we also have

$$\langle x_2^{(k)} - x_1^{(k+1)}, x - x_1^{(k+1)} \rangle \leq 0 \quad \forall x \in \mathcal{C}_1,$$

so that

$$||x_2^{(k)} - x^*||^2 \geq ||x_2^{(k)} - x_1^{(k+1)}||^2 + ||x_1^{(k+1)} - x^*||^2. \tag{3.174}$$

Now, we have the following inequalities:

$$||x_1^{(k)} - x^*||^2 \geq ||x_2^{(k)} - x^*||^2 \geq ||x_1^{(k+1)} - x^*||^2 \geq ||x_2^{(k+1)} - x^*||^2 \geq \cdots \tag{3.175}$$

Therefore, one can conclude that the sequence $\{y^{(k)}\}$ is bounded and monotonic as well as are $\{x_1^{(k)}\}$ and $\{x_2^{(k)}\}$. This implies that there exists an accumulation point $\bar{y}$ such that

$$\lim_{k\to\infty} y^{(k)} = \lim_{k\to\infty} x_1^{(k)} = \lim_{k\to\infty} x_2^{(k)} = \bar{y}.$$

Since the convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed, one can observe that $\bar{y} \in \mathcal{C}_1 \cap \mathcal{C}_2$. $\square$

3.5.3 Computational Issues in Fixed-Order Output-Feedback Controller Design

Necessary and sufficient conditions for static output-feedback can be formulated in terms of coupled Lyapunov matrix inequalities by following a quadratic Lyapuonv function approach. From Lyapunov stability theory, it has been known that the closed-loop linear time-invariant continuous time system is GUAS if and only if the matrix $A + B_u K_o C_y$ is Hurwitz, or equivalently, there exists a gain matrix K_o such that the following matrix inequality holds for some $X = X^T > 0$:

$$(A + B_u K_o C_y)X + X(A + B_u K_o C_y)^T < 0. \tag{3.176}$$

To handle this BMI problem, the so-called coupled linear matrix inequality formulation has been introduced–see Section 3.4.1. That is, introduce the following two matrix inequalities to solve:

$$B_u^{\perp}(AX + XA^T)(B_u^{\perp})^T < 0, \tag{3.177}$$

$$(C_y^T)^{\perp}(A^T Y + YA)((C_y^T)^{\perp})^T < 0, \tag{3.178}$$

where $XY = YX = I$–these are also introduced in (3.82) and (3.83).

Definition 11. *Define the two convex sets for X and Y, respectively:*

$$\mathcal{C}_1 = \{X \in \mathbb{S}^n | X > 0, (3.82)\} \tag{3.179}$$

$$\mathcal{C}_2 = \{Y \in \mathbb{S}^n | Y > 0, (3.83)\}, \tag{3.180}$$

where $XY = YX = I$.

In this section, we will show three numerical algorithms to find a feasible solution X (or Y) such that the inequalities (3.177) and (3.178) hold. The existence of such a feasible solution is equivalent to the existence of a static output-feedback control gain K_o that stabilizes the system whose realization is (A, B_u, C_y). The first algorithm is called *the min/max algorithm* whose properties are not very mathematically clear, but heuristically has shown good convergence, and the other two algorithms are based on the alternating projection mapping in Hilbert space such that their properties including convergence are clearly understood.

The Min/Max Algorithm

Proposition 17. *(The Min/Max Algorithm) Let us consider the following successive optimization problem to approximate the best solution at each step, which has been suggested in [40, 43]:*

$$X_k = arg\ min\ \{l_u : X \in \mathcal{C}_1, I \le Y_k^{1/2} X Y_k^{1/2} \le l_u I\}$$
$$Y_{k+1} = arg\ max\ \{l_l : Y \in \mathcal{C}_2, l_l I \le X_k^{1/2} Y X_k^{1/2} \le I\}$$

Semi-Definite Programming (SDP) Approach

The following lemma shows that the successive projection mappings can be formulated as the alternating two projection problems where one needs to solve a SDP problem at each projection step.

Lemma 16. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the convex sets described by LMIs. Then the projection $X_k = \mathcal{P}_{\mathcal{C}_1}(Y_k)$ and $Y_{k+1} = \mathcal{P}_{\mathcal{C}_2}(X_k)$ can be characterized as the unique solutions to the following SDP problems:*

$$X_k = \mathcal{P}_{\mathcal{C}_1}(Y_k) = arg\ min_{X \in \mathcal{C}_1} ||Y_k - X||_F,$$
$$Y_{k+1} = \mathcal{P}_{\mathcal{C}_2}(X_k) = arg\ min_{Y \in \mathcal{C}_2} ||Y - X_k||_F,$$

where $|| \cdot ||_F$ indicates the Frobenius norm.

The next two algorithms suggest different SDP problem formulations for the alternating projections.

Proposition 18. *([15]) Let $\mathcal{C}_1$ be the convex set described by LMIs. Then the projection $X_k = \mathcal{P}_{\mathcal{C}_1} Y_k$ can be computed as the unique solution to the following SDP problem:*

$$X_k = arg\ min\ \left\{ \boldsymbol{Tr}(S) : X \in \mathcal{C}_1, \begin{bmatrix} S & Y_k^{-1} - X \\ Y_k^{-1} - X & S \end{bmatrix} \geq 0,\ S \in \bar{\mathbb{S}}_+^n \right\},$$

for a given $Y_k \in \mathcal{C}_2$. Similarly, the projection $Y_{k+1} = \mathcal{P}_{\mathcal{C}_2} X_k$ can be computed as the unique solution to the following SDP problem:

$$Y_{k+1} = arg\ min\ \left\{ \boldsymbol{Tr}(S) : Y \in \mathcal{C}_2, \begin{bmatrix} S & Y - X_k^{-1} \\ Y - X_k^{-1} & S \end{bmatrix} \geq 0,\ S \in \bar{\mathbb{S}}_+^n \right\},$$

for a given $X_k \in \mathcal{C}_1$.

Proposition 19. *(A cone complementary problem [59]) The idea is to associate the static output-feedback control problem with the following semi-definite programming problem:*

$$\min \boldsymbol{Tr}(XY) \tag{3.181}$$

$$subject\ to\ (X, Y) \in \mathcal{C}_1 \times \mathcal{C}_2 \cap \mathcal{M}_{X,Y}, \tag{3.182}$$

where

$$\mathcal{M}_{X,Y} \triangleq \{(X, Y) \in S_+^n \times S_+^n | \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0\}.$$

One should notice that the objective function of the above SDP problem is non-convex. To solve this problem, the following Linearization Algorithm has been introduced in [59]:

1. *Choose the initial guess $(X_0, Y_0) \in \mathcal{C}_1 \times \mathcal{C}_2 \cap \mathcal{M}_{X,Y}$ and set the iteration index k as 0.*

2. *Solve the SDP problem to move one step forward:*

$$(X_{k+1}, Y_{k+1}) = \ arg\ min\ \{ \boldsymbol{Tr}(XY_k + X_k Y) : (X, Y) \in \mathcal{C}_1 \times \mathcal{C}_2 \cap \mathcal{M}_{X,Y} \}. \tag{3.183}$$

3. *If the stopping criterion*

$$|t_{k+1} - t_k| < \epsilon \tag{3.184}$$

is satisfied for a specified error tolerance $\epsilon > 0$, then escape the iteration loop. Otherwise, go to step 2 with the increased iteration index $k = k + 1$.

Chapter 4

Robust Controller Synthesis Problems for Lur'e Systems

4.1 Introduction

The previous chapter developed controller synthesis problems in which the nominal plant part of Lur'e system was assumed to be completely known. Since no single fixed model can provide an input-output map exactly like the true plant, a realistic representation requires a set of maps. Such a set of uncertain mappings are characterized by its input-output relations so that the uncertainty Δ in Figures 4.1 and 4.2 can be considered as a set-valued function. The approach in this chapter is to parameterize the uncertain mapping Δ and the unknown nonlinear mapping ϕ separately as shown in Figures 4.1 and 4.2. Note that sometimes these two uncertain feedback connected mappings, which both can be parameterized as set-valued functions, might be combined as block components in an augmented mapping. Two different types of uncertainties Δ are considered–norm-bounded (or matching) and polytopic parameter-dependent uncertainties.

Note that the LMI formulations for the static output-feedback control problems in continuous-and discrete-time systems are equivalent in the sense that the LMIs in continuous-time systems can be transformed as the LMIs in discrete-time systems and vice versa.

Equivalence of LMI formulations for continuous and discrete time systems

Necessary and sufficient conditions for static output-feedback can be formulated in terms of coupled Lyapunov matrix inequalities by using a quadratic Lyapuonv function. From Lyapunov stability theory, it has been known that the closed-loop linear time-invariant continuous-time system is GUAS if and only if the matrix $A + B_u K_o C_y$ is Hurwitz, or equivalently, there exists a gain matrix K_o such that the matrix inequality holds for some $X = X^T > 0$ (or $Y = Y^T > 0$):

$$(A + B_u K_o C_y)X + X(A + B_u K_o C_y)^T < 0 \quad (\text{or, } Y(A + B_u K_o C_y) + (A + B_u K_o C_y)^T Y < 0) \tag{4.1}$$

Corollary 5. *(Equivalent formulation of an LMI for discrete-time systems as continuous-time systems via*

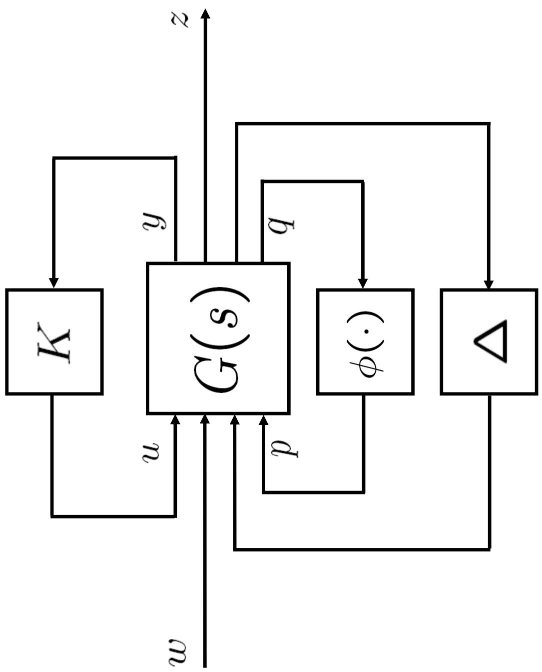


Figure 4.1: LFT and SNOF for Robust Control for Uncertain Lur'e Systems with External Input-Output

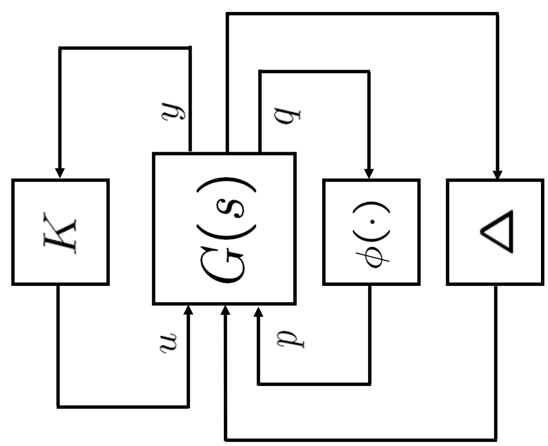


Figure 4.2: LFT and SNOF for Robust Control for Uncertain Lur'e Systems

the Schur complement lemma) Similarly to the stability condition for continuous-time systems, for discrete-time systems the closed-loop linear time-invariant discrete-time system is GUAS if and only if the matrix $A+B_uK_oC_y$ *is Schur, i.e., the eigenvalues of* $A+B_uK_oC_y$ *are in the open unit circle in the complex variable domain. Equivalently, there exists a control gain matrix* K_o *such that*

$$(A + B_uK_oC_y)^T X^{-1}(A + B_uK_oC_y) - X^{-1} < 0 \tag{4.2}$$

for some $X = X^T > 0$*. We have the following two equivalent inequalities with* $X^{-1} = Y$*:*

$$\begin{bmatrix} -Y & (A + B_uK_oC_y)^T Y \\ Y(A + B_uK_oC_y) & -Y \end{bmatrix} < 0 \tag{4.3}$$

and

$$(A_d + B_{u,d}K_oC_{y,d})X_d + X_d(A_d + B_{u,d}K_oC_{y,d})^T < 0, \tag{4.4}$$

where

$$A_d \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 \\ A & -\frac{1}{2}I \end{bmatrix}, B_{u,d} \triangleq \begin{bmatrix} 0 \\ B_u \end{bmatrix}, C_{y,d} \triangleq \begin{bmatrix} C_y & 0 \end{bmatrix}, X_d \triangleq \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}. \tag{4.5}$$

Proof: Using the Schur complement lemma and a congruence transformation, it is easy to see the equivalence between the matrix inequalities. □

Note that the LMI (4.4) has bilinear terms composed of the multiplications of the unknown (decision) matrices X and K_o and constant system matrices, so checking the solvability of the LMI (4.4) is a non-convex problem and known to be NP-hard [41, 16] in general. As shown in Section 3.5, this problem can be handled with the well-known successive or alternating projection methods–see Section 3.5.

4.2 Static Output-Feedback Control via LMI Optimization for a Certain Class of Lur'e Systems under Matching Uncertainties

In this section, the uncertainty satisfying the so-called matching condition is considered. The LMI formulations derived in this section are very flexible in the following senses:

i. With a known norm-bound on the uncertain mapping Δ, say $1/\gamma_\theta$, the problem to maximize the upper bound on the sector-condition for the nonlinear mapping $\phi \in \bar{\Phi}_{sb}^{\alpha}$ can be represented as a convex optimization problem, and vice versa.

ii. The problem to achieve a performance objective for a Lur'e system can be represented as a convex optimization problem, provided that the norm-bound on the uncertain mapping Δ and the upper bound on the sector-condition for the nonlinear mapping $\phi \in \bar{\Phi}_{sb}^{\alpha}$ or $\phi \in \Phi_{sb}^{|\alpha|}$ are known or at least conservatively approximated.

4.2.1 SOF Control via LMI Optimization for a Certain Class of Lur'e Systems under Matching Uncertainties–Case I

In the Absence of Feedback-Connected Nonlinearities

In order to apply the stabilizing SOF to the perturbed system with uncertainties, and to solve SOF controller synthesis problems using LMIs, consider the system with a control affine term

$$\begin{aligned} x(k+1) &= (A + \Delta A(\theta_a(k)))x(k) + (B_u + \Delta B_u(\theta_b(k)))u(k) \\ y(k) &= C_y x(k), \end{aligned} \tag{4.6}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^m$ is the control input at time $k \in \mathbb{Z}_+$, and $A \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times m}$, and $C_y \in \mathbb{R}^{n_y\times n}$ are known constant matrices. In addition, $\theta_r(k) \in \Theta_r$ represents the parametric uncertainty of the system and the subset $\Theta_r \subset \mathbb{R}^{n_{\theta_r}}$ is assumed to be compact for each $r = a, b$. It is assumed that the triplet nominal realization of the system (A, B_u, C_y) is stabilizable and detectable, and both the uncertainty mappings $\Delta A : \Theta_a \to \mathbb{R}^{n\times n}$ and $\Delta B_u : \Theta_b \to \mathbb{R}^{n\times m}$ are continuous in $\theta_r \in \Theta_r$ which is Lebesgue measurable for each $r = a, b$.

Assumption 1. *(Matching condition) Assume that the parametric uncertainty can be represented as*

$$[\Delta A \ \Delta B_u] = E_0 F(\theta) \left[E_A \ E_B\right], \tag{4.7}$$

where $||F(\theta)||_2 \leq 1/\gamma_\theta$ with a fixed value of $\gamma_\theta > 0$ for all $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$.

Note that in a number of papers in literature, matching conditions have different representations where matching conditions are weakened in some degree and/or combined with additional assumptions.

Lemma 17. *The system (4.6) is GUAS if and only if there exists a gain matrix K_o such that the following* *matrix inequality holds for some $X = X^T > 0$:*

$$
\begin{aligned}
&(A_d + B_{u,d}K_oC_{y,d})X_d + X_d(A_d + B_{u,d}K_oC_{y,d})^T + E_{0,d}F(\theta)(E_{A,d} + E_{B,d}K_oC_{y,d})X_d \\
&+X_d(E_{A,d} + E_{B,d}K_oC_{y,d})^T F^T(\theta)E_{0,d}^T < 0 \quad \forall\theta \in \Theta,
\end{aligned}
\tag{4.8}
$$

where $||F(\theta)|| \leq 1/\gamma_\theta$ for all $\theta \in \Theta$. The definitions for A_d, $B_{u,d}$, $C_{y,d}$, X_d are given in (4.4) and

$$
E_{0,d} \triangleq \begin{bmatrix} 0 \\ E_0 \end{bmatrix}, E_{A,d} \triangleq \begin{bmatrix} E_A & 0 \end{bmatrix}, E_{B,d} \triangleq E_B.
$$

Further, (4.8) is feasible for some $X > 0$ and K_o if and only if there exist $X > 0$ (or its inverse Y) and K_o that satisfy the matrix inequality

$$
(\hat{A} + \hat{B}K_o\hat{C}_y)\hat{X} + \hat{X}(\hat{A} + \hat{B}K_o\hat{C}_y)^T < 0, \tag{4.9}
$$

or

$$
\hat{Y}(\hat{A} + \hat{B}K_o\hat{C}_y) + (\hat{A} + \hat{B}K_o\hat{C}_y)^T\hat{Y} < 0, \tag{4.10}
$$

where

$$
\hat{A} \triangleq \begin{bmatrix} A_d & 0 & E_{0,d} \\ E_{A,d} & -\frac{1}{2}\gamma_\theta I & 0 \\ 0 & 0 & -\frac{1}{2}\gamma_\theta I \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} B_{u,d} \\ E_{B,d} \\ 0 \end{bmatrix}, \quad \hat{C}_y \triangleq \begin{bmatrix} C_{y,d} & 0 & 0 \end{bmatrix}, \tag{4.11}
$$

and

$$
\hat{X} \triangleq \begin{bmatrix} X_d & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \hat{Y} \triangleq \begin{bmatrix} Y_d & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{4.12}
$$

Proof: Replacing (A, B_u) in (4.4) by $(A+\Delta A(\theta(k)), B_u+\Delta B_u(\theta(k)))$ and assuming the matched uncertainty (4.7) for $(\Delta A, \Delta B_u)$, Lemma **??** implies the equivalent matrix inequality for K_o, $X = X^T > 0$, and $\epsilon > 0$ to (4.8):

$$
\gamma_\theta \left((A_d + B_{u,d}K_oC_{y,d})X_d + X_d(A_d + B_{u,d}K_oC_{y,d})^T\right)+\epsilon E_{0,d}E_{0,d}^T+\frac{1}{\epsilon}X_d(E_{A,d}+E_{B,d}K_oC_{y,d})^T(E_{A,d}+E_{B,d}K_oC_{y,d})X_d < 0. \tag{4.13}
$$

By the congruence transformation with the transformation matrix diag$\{\epsilon I, \epsilon I\}$ and replacing X_d/ϵ with X_d, the unknown positive scalar variable ϵ is eliminated. Further, the equivalent inequality to (4.8) and (4.13) follows from the Schur complement lemma:

$$\begin{bmatrix} (A_d + B_{u,d}K_oC_{y,d})X_d + X_d(A_d + B_{u,d}K_oC_{y,d})^T & X_d(E_{A,d} + E_{B,d}K_oC_{y,d})^T & E_{0,d} \\ (E_{A,d} + E_{B,d}K_oC_{y,d})X_d & -\gamma_\theta I & 0 \\ E_{0,d}^T & 0 & -\gamma_\theta I \end{bmatrix} < 0. \tag{4.14}$$

Moreover, the matrix inequality (4.14) can be rewritten as (4.9). The congruence transformation with the transformation matrix $\hat{Y}$ gives (4.10). □

The inequality (4.9) (or (4.10)) is not jointly convex in X (or Y) and K_o. As shown before in Section 3.4.1, this feasibility problem for X (or Y) and K_o can be reduced as two linear matrix inequalities where X and its inverse Y appear in different convex constraints so that this problem can be solved through some alternating or successive projection algorithms–see Section 3.5.

Corollary 6. *(Coupled linear matrix inequality formulation) There exists a stabilizing SOF control gain matrix K_o such that $u(k) = K_oy(k)$ staiblizes the system (4.6) if and only if X and Y satisfies the following two matrix inequalities:*

$$\begin{bmatrix} B_{u,d}^\perp & 0 & 0 \\ 0 & E_{B,d}^\perp & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_dX_d + X_dA_d^T & X_dE_{A,d}^T & E_{0,d} \\ E_{A,d}X_d & -\gamma_\theta I & 0 \\ E_{0,d}^T & 0 & -\gamma_\theta I \end{bmatrix} \begin{bmatrix} (B_{u,d}^\perp)^T & 0 & 0 \\ 0 & (E_{B,d}^\perp)^T & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \tag{4.15}$$

and

$$\begin{bmatrix} (C_{y,d}^T)^\perp & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_d^TY_d + Y_dA_d & E_{A,d}^T & Y_dE_{0,d} \\ E_{A,d} & -\gamma_\theta I & 0 \\ E_{0,d}^TY_d & 0 & -\gamma_\theta I \end{bmatrix} \begin{bmatrix} ((C_{y,d}^T)^\perp)^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0, \tag{4.16}$$

where $XY = YX = I$ and $(\cdot)^\perp$ indicates a full-rank matrices which is orthogonal to $(\cdot)$.

Proof: The feasiblity of the matrix inequality (4.9) (or (4.10)) for some X (or Y) and K_o is equivalent to the feasiblity of the matrix inequality

$$\hat{B}^\perp(\hat{A}\hat{X} + \hat{X}\hat{A}^T)(\hat{B}^\perp)^T < 0, \quad \left(\text{or } (\hat{C}_y^T)^\perp(\hat{A}^T\hat{Y} + \hat{Y}\hat{A})((\hat{C}_y^T)^\perp)^T < 0\right). \tag{4.17}$$

The above two matrix inequalities can be rewritten as (4.15) and (4.16), respectively. □

In the Presence of Feedback-Connected Nonlinearities

Consider the following discrete-time system where the nominal linear time-invariant system is interconnected with a certain class of nonlinear operators:

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_p\phi(k, x(k)) + B_u u(k) \\
y(k) &= C_y x(k),
\end{aligned}
\tag{4.18}
$$

where the nonlinear operator ϕ is in the class $\bar{\Phi}_{sb}^{\alpha}$. As shown before, the stability condition for the above system can be written as a feasibility condition for a Lyapunov matrix Q^{-1} which is an eigenvalue problem:

$$
Q = Q^T > 0, \quad \begin{bmatrix} -Q & 0 & QA^T & QC_q^T \\ 0 & -I & B_p^T & 0 \\ AQ & B_p & -Q & 0 \\ C_qQ & 0 & 0 & -\gamma I \end{bmatrix} < 0,
\tag{4.19}
$$

where $\gamma \triangleq 1/\alpha^2$.

Replacing A by $A + B_u K_o C_y$ in the LMI constraint (4.19), if there exists a SOF control gain matrix K_o satisfying (4.19) for some $Q = Q^T > 0$ then the closed-loop system is stabilized by the control law $u(k) = K_o y(k)$. The corresponding LMI constraint can be rewritten in the same form as (4.8):

$$
(\bar{A} + \bar{B}_u K_o \bar{C}_y)\bar{Q} + \bar{Q}(\bar{A} + \bar{B}_u K_o \bar{C}_y)^T < 0,
\tag{4.20}
$$

where

$$
\bar{A} \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 & 0 \\ A & B_p & -\frac{1}{2}I & 0 \\ C_q & 0 & 0 & -\frac{1}{2}\gamma I \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} 0 \\ 0 \\ B_u \\ 0 \end{bmatrix}, \quad \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 & 0 & 0 \end{bmatrix},
\tag{4.21}
$$

$$
\bar{Q} \triangleq \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.
\tag{4.22}
$$

In order to apply the stabilizing SOF to the perturbed system with uncertainties, and to solve SOF controller synthesis problems using LMIs, consider the system with a control affine term

$$\begin{aligned} x(k+1) &= (A + \Delta A(\theta(k)))x(k) + B_p\phi(k, x(k)) + (B_u + \Delta B_u(\theta(k)))u(k) \\ y(k) &= C_y x(k) \end{aligned} \tag{4.23}$$

where $x(k) \in \mathbb{R}^n$ is the state variable and $u(k) \in \mathbb{R}^m$ is the control input at time $k \in \mathbb{Z}_+$, and $A \in \mathbb{R}^{n\times n}$ and $B_u \in \mathbb{R}^{n\times m}$ are known constant matrices. In addition, $\theta(k) \in \Theta$ represents the parametric uncertainty of the system and the subset $\Theta \subset \mathbb{R}^{n_\theta}$ is assumed to be closed and compact. Assume that the triplet nominal realization of the system (A, B_u, C_y) is stabilizable and detectable, and both the uncertainty mappings $\Delta A : \Theta \to \mathbb{R}^{n\times n}$ and $\Delta B_u : \Theta \to \mathbb{R}^{n\times m}$ are continuous in $\theta \in \Theta$ which is Lebesgue measurable.

Corollary 7. *The necessary and sufficient stability conditions for the closed-loop system with a SOF control law $u(k) = K_o y(k)$ is written as a feasibility problem for $Q = Q^T > 0$:*

$$(\bar{A} + \bar{B}_u K_o \bar{C}_y)\bar{Q} + \bar{Q}(\bar{A} + \bar{B}_u K_o \bar{C}_y)^T + \bar{E}_0 F(\theta)\bar{E}_{AB}\bar{Q} + \bar{Q}\bar{E}_{AB}^T F^T(\theta)\bar{E}_0^T < 0, \quad \theta \in \Theta, \tag{4.24}$$

where

$$\bar{A} \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 & 0 \\ A & B_p & -\frac{1}{2}I & 0 \\ C_q & 0 & 0 & -\frac{1}{2}\gamma I \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} 0 \\ 0 \\ B_u \\ 0 \end{bmatrix}, \quad \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 & 0 & 0 \end{bmatrix}, \tag{4.25}$$

and

$$\bar{E}_0 \triangleq \begin{bmatrix} 0 \\ 0 \\ E_0 \\ 0 \end{bmatrix}, \quad \bar{E}_{AB} \triangleq \begin{bmatrix} E_A + E_B K_o C_y & 0 & 0 & 0 \end{bmatrix}, \quad \bar{Q} \triangleq \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{4.26}$$

Further, the elimination of structured uncertain matrix lemma implies the equivalent matrix inequality for K_o, $Q = Q^T > 0$ and $\epsilon > 0$:

$$\begin{bmatrix} \gamma_\theta\left((\bar{A} + \bar{B}_u K_o \bar{C}_y)\bar{Q} + \bar{Q}(\bar{A} + \bar{B}_u K_o \bar{C}_y)^T\right) + \epsilon \bar{E}_o \bar{E}_o^T & \bar{Q}\bar{E}_{AB}^T \\ \bar{E}_{AB}\bar{Q} & -\epsilon I \end{bmatrix} < 0, \tag{4.27}$$

provided $\|F(\theta)\| \leq 1/\gamma_\theta$ for all $\theta \in \Theta$.

Proof: After small algebraic computation and by following the same procedure as the LTI system without feedback-connected nonlinearities, it is easy to derive the matrix inequalities. □

Lemma 18. *The system (4.23) is quadratically stabilized with a SOF control law* $u(k) = K_o y(k)$ *if and only if there exists a gain matrix* K_o *such that the following matrix inequality holds for some* $Q = Q^T > 0$ *(or* $P = P^T > 0$*) and* $\epsilon > 0$ *:*

$$(\bar{\bar{A}} + \bar{\bar{B}}K_o\bar{\bar{C}})\bar{\bar{Q}} + \bar{\bar{Q}}(\bar{\bar{A}} + \bar{\bar{B}}K_o\bar{\bar{C}})^T < 0, \tag{4.28}$$

or

$$\bar{\bar{P}}(\bar{\bar{A}} + \bar{\bar{B}}K_o\bar{\bar{C}}) + (\bar{\bar{A}} + \bar{\bar{B}}K_o\bar{\bar{C}})^T\bar{\bar{P}} < 0, \tag{4.29}$$

where

$$\bar{\bar{A}} \triangleq \begin{bmatrix} \bar{A} & 0 & \bar{E}_o \\ \bar{E}_A & -\frac{1}{2}\gamma_\theta I & 0 \\ 0 & 0 & -\frac{1}{2}\gamma_\theta I \end{bmatrix}, \quad \bar{\bar{B}} \triangleq \begin{bmatrix} \bar{B}_u \\ E_B \\ 0 \end{bmatrix}, \quad \bar{\bar{C}} \triangleq \begin{bmatrix} \bar{C}_y & 0 & 0 \end{bmatrix}, \quad \bar{\bar{Q}} \triangleq \begin{bmatrix} \bar{Q} & 0 & 0 \\ 0 & \epsilon I & 0 \\ 0 & 0 & \epsilon I \end{bmatrix}, \quad \bar{\bar{P}} \triangleq \begin{bmatrix} \bar{P} & 0 & 0 \\ 0 & \frac{1}{\epsilon}I & 0 \\ 0 & 0 & \frac{1}{\epsilon}I \end{bmatrix},$$

$$\bar{E}_A \triangleq \begin{bmatrix} E_A & 0 & 0 & 0 \end{bmatrix},$$

with $QP = I$ *such that* $\bar{Q}\bar{P} = I$ *and* $\bar{\bar{Q}}\bar{\bar{P}} = I$

Proof: Replace (A, B_u) in (4.20) by $(A + \Delta A(\theta(k)), B_u + \Delta B_u(\theta(k)))$ and assuming the matched uncertainty (4.7) for $(\Delta A, \Delta B_u)$. Then, after small algebraic computation and by following the same procedure as the LTI system without feedback-connected nonlinearities, it is easy to derive the matrix inequalities. □

Theorem 31. *(Coupled linear matrix inequality formulation) There exists a stabilizing SOF control gain matrix* K_o *such that* $u(k) = K_o y(k)$ *stabilizes the system (4.23) if and only if* Q *and* P *satisfies the following two matrix inequalities:*

$$\bar{\bar{B}}^\perp(\bar{\bar{A}}\bar{\bar{Q}} + \bar{\bar{Q}}\bar{\bar{A}})(\bar{\bar{B}}^\perp)^T < 0, \tag{4.30}$$

$$(\bar{\bar{C}}^T)^\perp(\bar{\bar{P}}\bar{\bar{A}} + \bar{\bar{A}}\bar{\bar{P}})((\bar{\bar{C}}^T)^\perp)^T < 0, \tag{4.31}$$

where $QP = PQ = I$ *such that* $\bar{\bar{Q}}\bar{\bar{P}} = \bar{\bar{P}}\bar{\bar{Q}} = I$ *and* $(\cdot)^\perp$ *indicates a full-rank matrix that is orthogonal to* $(\cdot)$.

Proof: The proof follows directly from the previous lemma. □

4.2.2 SOF Control via LMI Optimization for a Certain Class of Lur'e Systems under Matching Uncertainties–Case II

In the Absence of Feedback-Connected Nonlinearities

In order to apply the stabilizing SOF to the perturbed system with uncertainties, and to solve SOF controller synthesis problems using LMIs, consider the system with a control affine term

$$\begin{aligned} x(k+1) &= (A + \Delta A(\theta_a(k)))x(k) + B_u u(k) \\ y(k) &= (C_y + \Delta C_y(\theta_c(k)))x(k) \end{aligned} \tag{4.32}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^m$ is the control input at time $k \in \mathbb{Z}_+$, and $A \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times m}$, and $C_y \in \mathbb{R}^{n_y\times n}$ are known constant matrices. In addition, $\theta_r(k) \in \Theta_r$ represents the parametric uncertainty of the system and the subset $\Theta_r \subset \mathbb{R}^{n_{\theta_r}}$ is assumed to be compact for each $r = a, c$. Assume that the triplet nominal realization of the system (A, B_u, C_y) is stabilizable and detectable, and both the uncertainty mappings $\Delta A : \Theta_a \to \mathbb{R}^{n\times n}$ and $\Delta C_y : \Theta_c \to \mathbb{R}^{n_y\times n}$ are continuous in $\theta_r \in \Theta_r$ and θ_r is Lebesgue measurable for each $r = a, c$.

Assumption 2. *(Matching condition) Assume that the parametric uncertainty can be represented as*

$$\begin{bmatrix} \Delta A \\ \Delta C_y \end{bmatrix} = \begin{bmatrix} E_A \\ E_C \end{bmatrix} F(\theta)E_0, \tag{4.33}$$

where $||F(\theta)||_2 \leq 1/\gamma_\theta$ *for all* $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$.

Note that in a number of papers in the literature, matching conditions have different representations with which matching conditions are weakened in some degree and/or combined with additional assumptions.

Lemma 19. *The system (4.32) is GUAS if and only if there exists a gain matrix K_o such that the matrix inequality holds for some $X = X^T > 0$:*

$$(A_d{+}B_{u,d}K_oC_{y,d})X_d{+}X_d(A_d{+}B_{u,d}K_oC_{y,d})^T{+}(E_{A,d}{+}B_{u,d}K_oE_{C,d})F(\theta)E_{0,d}X_d{+}XE_{0,d}^TF^T(\theta)(E_{A,d}{+}B_{u,d}K_oE_{C,d})^T < 0 \quad \forall\theta \tag{4.34}$$

where $||F(\theta)|| \leq 1/\gamma_\theta$ *for all* $\theta \in \Theta$*. The definitions for* A_d*,* $B_{u,d}$*,* $C_{y,d}$*,* X_d *are given in (4.4) and*

$$E_{0,d} \triangleq [E_0 \;\; 0]\,, E_{A,d} \triangleq \begin{bmatrix} 0 \\ E_A \end{bmatrix}, E_{C,d} \triangleq E_C.$$

Further, (4.34) is feasible for some $X > 0$ and K_o if and only if there exists $X > 0$ (or its inverse Y) and K_o that satisfy the matrix inequality

$$(\hat{A} + \hat{B}_u K_o \hat{C})\hat{X} + \hat{X}(\hat{A} + \hat{B}_u K_o \hat{C})^T < 0, \tag{4.35}$$

or

$$\hat{Y}(\hat{A} + \hat{B}_u K_o \hat{C}) + (\hat{A} + \hat{B}_u K_o \hat{C})^T \hat{Y} < 0, \tag{4.36}$$

where

$$\hat{A} \triangleq \begin{bmatrix} A_d & E_{A,d} & 0 \\ 0 & -\frac{1}{2}\gamma_\theta I & 0 \\ E_{0,d} & 0 & -\frac{1}{2}\gamma_\theta I \end{bmatrix}, \quad \hat{B}_u \triangleq \begin{bmatrix} B_{u,d} \\ 0 \\ 0 \end{bmatrix}, \quad \hat{C} \triangleq \begin{bmatrix} C_{y,d} & E_{C,d} & 0 \end{bmatrix}, \quad \hat{X} \triangleq \begin{bmatrix} X_d & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \hat{Y} \triangleq \begin{bmatrix} Y_d & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{4.37}$$

Proof: Replace (A, C_y) in (4.4) by $(A + \Delta A(\theta(k)), C_y + \Delta C_y(\theta(k)))$ and assuming the matched uncertainty (4.33) for $(\Delta A, \Delta C_y)$. Then one needs small changes with the results given in the previous section. See the previous section for details. □

The inequality (4.35) (or (4.36)) is not jointly convex in X (or Y) and K_o. However, as discussed before, this feasiblity problem for X (or Y) and K_o can be reduced as two linear matrix inequalities where X and its inverse Y appear in different convex constraints so that this problem can be solved through some alternating or successive projection algorithms–see Section 3.5.

Corollary 8. *(Coupled linear matrix inequality formulation) There exists a stabilizing SOF control gain matrix K_o such that $u(k) = K_o y(k)$ staiblizes the system (4.32) if and only if X and Y satisfies the two matrix inequalities:*

$$\begin{bmatrix} B_{u,d}^\perp & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_d X_d + X_d A_d^T & E_{A,d} & X_d E_{0,d}^T \\ E_{A,d}^T & -\gamma_\theta I & 0 \\ E_{0,d} X & 0 & -\gamma_\theta I \end{bmatrix} \begin{bmatrix} (B_{u,d}^\perp)^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0; \tag{4.38}$$

$$\begin{bmatrix} (C_{y,d}^T)^\perp & 0 & 0 \\ 0 & (E_{C,d}^T)^\perp & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_d^T Y_d + Y_d A_d & Y_d E_{A,d} & E_{0,d}^T \\ E_{A,d}^T Y_d & -\gamma_\theta I & 0 \\ E_{0,d} & 0 & -\gamma_\theta I \end{bmatrix} \begin{bmatrix} ((C_{y,d}^T)^\perp)^T & 0 & 0 \\ 0 & ((E_{C,d}^T)^\perp)^T & 0 \\ 0 & 0 & I \end{bmatrix} < 0, \tag{4.39}$$

where $XY = YX = I$ and $(\cdot)^\perp$ indicates a full rank matrix orthogonal to $(\cdot)$.

Proof: The feasiblity of the matrix inequality (4.35) (or (4.36)) for some X (or Y) and K_o is equivalent to the feasibility of the matrix inequality

$$\hat{B}_u^{\perp}(\hat{A}\hat{X} + \hat{X}\hat{A}^T)(\hat{B}_u^{\perp})^T < 0, \quad \left(\text{or } (\hat{C}^T)^{\perp}(\hat{A}^T\hat{Y} + \hat{Y}\hat{A})((\hat{C}^T)^{\perp})^T < 0\right). \tag{4.40}$$

The above two matrix inequalities can be rewritten as (4.38) and (4.39), respectively. □

In the Presence of Feedback-Connected Nonlinearities

Similar to Case I, in order to apply the stabilizing SOF to the perturbed system with uncertainties, and to solve SOF controller synthesis problems using LMIs, consider the system with a control affine term

$$\begin{aligned} x(k+1) &= (A + \Delta A(\theta(k)))x(k) + B_p\phi(k, x(k)) + B_u u(k) \\ y(k) &= (C_y + \Delta C_y(\theta(k)))x(k) \end{aligned} \tag{4.41}$$

where $x(k) \in \mathbb{R}^n$ is the state variable and $u(k) \in \mathbb{R}^m$ is the control input at time $k \in \mathbb{Z}_+$, and $A \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times m}$, and $C_y \in \mathbb{R}^{n_y\times n}$ are known constant matrices. In addition, $\theta(k) \in \Theta$ represents the parametric uncertainty of the system and the subset $\Theta \subset \mathbb{R}^{n_{\theta_r}}$ is assumed to be closed and compact. Assume that the triplet nominal realization of the system (A, B_u, C_y) is stabilizable and detectable, and both the uncertainty mappings $\Delta A : \Theta \to \mathbb{R}^{n\times n}$ and $\Delta C_y : \Theta \to \mathbb{R}^{n_y\times n}$ are continuous in $\theta \in \Theta$, and θ is Lebesgue measurable.

Corollary 9. *The necessary and sufficient stability conditions for the closed-loop system with a SOF control law $u(k) = K_o y(k)$ is written as a feasibility problem for $Q = Q^T > 0$:*

$$(\bar{A}+\bar{B}_uK_o\bar{C}_y)\bar{Q}+\bar{Q}(\bar{A}+\bar{B}_uK_o\bar{C}_y)^T+(\bar{E}_A+\bar{B}_uK_oE_C)F(\theta)\bar{E}_0\bar{Q}+\bar{Q}\bar{E}_0^TF^T(\theta)(\bar{E}_A+\bar{B}_uK_oE_C)^T < 0, \quad \theta \in \Theta, \tag{4.42}$$

where

$$\bar{A} \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 & 0 \\ A & B_p & -\frac{1}{2}I & 0 \\ C_q & 0 & 0 & -\frac{1}{2}\gamma I \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} 0 \\ 0 \\ B_u \\ 0 \end{bmatrix}, \quad \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 & 0 & 0 \end{bmatrix}, \tag{4.43}$$

$$\bar{E}_A \triangleq \begin{bmatrix} 0 \\ 0 \\ E_A \\ 0 \end{bmatrix}, \quad \bar{E}_0 \triangleq \begin{bmatrix} E_0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{Q} \triangleq \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{4.44}$$

Further, using the elimination of structured uncertain matrix lemma we have the following equivalent matrix inequality for $Q = Q^T > 0$ and $\epsilon > 0$:

$$\begin{bmatrix} \gamma_\theta \left((\bar{A} + \bar{B}_u K_o \bar{C}_y)\bar{Q}\frac{1}{\epsilon} + \frac{1}{\epsilon}\bar{Q}(\bar{A} + \bar{B}_u K_o \bar{C}_y)^T \right) + \frac{1}{\epsilon^2}\bar{Q}\bar{E}_o\bar{E}_o^T\bar{Q} & \bar{E}_A + \bar{B}_u K_o E_C \\ (\bar{E}_A + \bar{B}_u K_o E_C)^T & -I \end{bmatrix} < 0 \quad \theta \in \Theta, \tag{4.45}$$

Proof: See the previous section for details. One needs small changes with the results given in the previous section. □

Lemma 20. *The system (4.41) is quadratically stabilized with a SOF control law $u(k) = K_o y(k)$ if and only if there exists a gain matrix K_o such that the following matrix inequality holds for some $Q = Q^T > 0$ (or $P = P^T > 0$) and $\epsilon > 0$:*

$$(\bar{\bar{A}} + \bar{\bar{B}} K_o \bar{\bar{C}})\bar{\bar{Q}} + \bar{\bar{Q}}(\bar{\bar{A}} + \bar{\bar{B}} K_o \bar{\bar{C}})^T < 0, \tag{4.46}$$

or

$$\bar{\bar{P}}(\bar{\bar{A}} + \bar{\bar{B}} K_o \bar{\bar{C}}) + (\bar{\bar{A}} + \bar{\bar{B}} K_o \bar{\bar{C}})^T \bar{\bar{P}} < 0, \tag{4.47}$$

where

$$\bar{\bar{A}} \triangleq \begin{bmatrix} \bar{A} & 0 & \bar{E}_A \\ 0 & -\frac{1}{2}\gamma_\theta I & 0 \\ \bar{E}_0 & 0 & -\frac{1}{2}\gamma_\theta I \end{bmatrix}, \quad \bar{\bar{B}} \triangleq \begin{bmatrix} \bar{B}_u \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\bar{C}} \triangleq \begin{bmatrix} \bar{C}_y & E_C & 0 \end{bmatrix}, \quad \bar{\bar{Q}} \triangleq \begin{bmatrix} \bar{Q} & 0 & 0 \\ 0 & \epsilon I & 0 \\ 0 & 0 & \epsilon I \end{bmatrix}, \quad \bar{\bar{P}} \triangleq \begin{bmatrix} \bar{P} & 0 & 0 \\ 0 & \frac{1}{\epsilon} I & 0 \\ 0 & 0 & \frac{1}{\epsilon} I \end{bmatrix},$$

with $QP = I$ such that $\bar{Q}\bar{P} = I$ and $\bar{\bar{Q}}\bar{\bar{P}} = I$

Proof: Replace (A, B_u) in (4.20) by $(A + \Delta A(\theta(k)), C_y + \Delta C_y(\theta(k)))$ and assuming the matched uncertainty

(4.33) for $(\Delta A, \Delta C_y)$. Then one needs small changes with the results given in the previous section. See the previous section for details. □

Theorem 32. *(Coupled linear matrix inequality formulation) There exists a stabilizing SOF control gain matrix K_o such that $u(k) = K_o y(k)$ stabilizes the system (4.41) if and only if Q and P satisfies the two matrix inequalities:*

$$\bar{\bar{B}}^{\perp}(\bar{\bar{A}}\bar{\bar{Q}} + \bar{\bar{Q}}\bar{\bar{A}})(\bar{\bar{B}}^{\perp})^T < 0, \tag{4.48}$$

$$(\bar{\bar{C}}^T)^{\perp}(\bar{\bar{P}}\bar{\bar{A}} + \bar{\bar{A}}\bar{\bar{P}})((\bar{\bar{C}}^T)^{\perp})^T < 0, \tag{4.49}$$

where $QP = PQ = I$ such that $\bar{\bar{Q}}\bar{\bar{P}} = \bar{\bar{P}}\bar{\bar{Q}} = I$ and $(\cdot)^{\perp}$ indicates a full rank matrix orthogonal to $(\cdot)$.

Proof: The proof directly follows from the previous lemma. □

4.3 Static Output-Feedback Control via LMI Optimization for a Certain Class of Lur'e Systems in the Presence of Polytopic Uncertainties

This section considers the polytopic parameter-dependent uncertain model. The LMI formulations derived in this section are based on the assumption where the parametric uncertainties θ_i (or their auxiliary variables $\rho_i(\theta)$) are all normalized such that $\theta_i \in [0, 1]$ (or $\rho_i(\theta) \in [0, 1]$, $\forall\theta$) for each index i. This normalization of the size of the parametric uncertainties can be achieved by introducing the constant weight blocks in the LFT representation for the system and those are augmented into the nominal LTI plant whose transfer function is $G(s)$ in Figures 4.1 and 4.2. The SOF controller synthesis problems are formulated with the consideration of (i) a single (or common) and (ii) parameter-dependent Lyapunov functions. The second type of Lyapunov function is known as its anonym PLDF. The degree of conservatism introduced in each method is compared with different SOF control schemes.

4.3.1 Parameter-Dependent Lyapunov Function (PLDF) and Its Applications

For analysis and controller synthesis problems for polytopic uncertain models, the simplest approach consists of looking for a common quadratic Lyapunov function that proves stability of the polytope of matrices and provides a sufficient condition that amounts to solving matrix inequalities of a Lyapunov matrix

X (or $Y = X^{-1}$) and the output-feedback control gain K_o at each vertex of PLDI realization of uncertain models. The main disadvantage in searching for a single quadratic Lyapunov function for a certain class of uncertain system models is the conservatism of the analysis and synthesis problems. As pointed out in many papers in the literature [28, 115, 14, 118, 74, 104, 56, 103], the Lyapunov function

$$V(x, x_\delta, x_\phi) \triangleq x^T P x + x_\delta^T P_\Delta x_\delta + x_\phi^T P_\Phi x_\phi \tag{4.50}$$

is less conservative for analysis and controller synthesis problems for the system in Figures 4.1 and 4.2 where the set-valued uncertain mapping Δ and nonlinear mapping Φ are dynamic in general. If the set-valued uncertain mapping Δ is in the set of LTI operators such that the FDI

$$\Delta(e^{j\omega}) + \Delta^*(e^{j\omega}) \geq 0$$

holds for all $\omega \in [0, 2\pi]$, that is,

$$\Delta = \Delta(z) \triangleq \left[\begin{array}{c|c} A_\Delta & B_\Delta \\ \hline C_\Delta & D_\Delta \end{array}\right]$$

is positive real, then the feasibility of its equivalent LMI condition

$$\begin{bmatrix} A_\Delta^T P_\Delta A_\Delta - P_\Delta & A_\Delta^T P_\Delta B_\Delta - C_\Delta^T \\ B_\Delta^T P_\Delta A_\Delta - C_\Delta & -D_\Delta - D_\Delta^T \end{bmatrix} \leq 0$$

for $P_\Delta = P_\Delta^T > 0$ guarantees the stability of the closed-loop system in Figure 4.2 and 4.1, provided $-G(z)$ is positive real and $\phi \equiv 0$. As introduced in Section 1.1.2 of Chapter 1, the FDI and LMI conditions for the positive realness of $-G(z)$ are formulated in terms of the Lyapuov matrix P for the nominal LTI plant. In particular, noting that the positive realness implies the passivity of the transfer function, such P_Δ and P are said to prove passivity or positive realness of the LTI perturbation $\Delta(z)$ and the nominal LTI plant $-G(z)$, respectively. Note that the closed-loop system that consists of the negative-feedback interconnection of two passive operators is stable. Furthermore, if the set-valued nonlinear mapping ϕ or the set-valued uncertain mapping Δ is static, then the last two terms in (4.50) are replaced by their integrals

$$\int_0^t x_\delta^T P_\Delta x_\delta d\tau \quad \text{and} \quad \int_0^t x_\phi^T P_\Phi x_\phi d\tau,$$

respectively. To consider such Lyapunov functions corresponds to the so-called IQC [96, 74, 104]. Note that the Lyapunov function (4.50) depends on the uncertainties characterized Δ and Φ as well as the nominal

LTI plant. Thus, the analysis and controller synthesis problems based on the Lyapunov function (4.50) are less conservative.

Now, consider the uncertain system

$$x(k+1) = A(\theta(k))x(k), \tag{4.51}$$

and the Lyapunov matrix that are convex in the parametric uncertainty vector $\theta = [\theta_i, \cdots, \theta_{n_\theta}]^T \in \mathbb{R}^{n_\theta}$, i.e.,

$$X(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k))P_j, \tag{4.52}$$

where $\sum_{j=1}^{J} \rho_j(\theta(k)) = 1$, $\rho_j(\theta(k)) \in [0, 1]$ for all $\theta(k) \in \Theta \subset \mathbb{R}^{n_\theta}$, and $P_j = P_j^T$ is real for each index $j = 1, \ldots, J$. In addition, suppose that $\theta \in \Theta$ and Θ is a convex hull of a finite set of vertices Θ_v, that is, $\Theta = \mathbf{Co}(\Theta_v)$. It is well known that if the matrix inequality

$$W(\theta(k), \theta(k+1)) \triangleq A^T(\theta(k))X(\theta(k+1))A(\theta(k)) - X(\theta(k)) < 0 \tag{4.53}$$

holds for all $\theta(k), \theta(k+1) \in \Theta \subset \mathbb{R}^{n_\theta}$ at each sampling time $k \in \mathbb{Z}_+$, then the origin of the uncertain system (4.51) is GUAS. Since the parameter-dependent matrix $W(\theta(k), \theta(k+1))$ is not jointly convex in $\theta(k)$ and $\theta(k+1)$, it is not sufficient to check the feasibility of (4.53) in the set of finite vertices Θ_v. The purpose is to find an equivalent LMI condition to (4.53) such that it is jointly convex in $\theta(k)$ and $\theta(k+1)$. The next lemma shows that the importance of the convexity of a function f in δ where δ is assumed to in a convex hull Δ.

Lemma 21. *Let a scalar-valued function $f : \Delta \to \mathbb{R}$ be convex and Δ is a convex hull such that Δ_0 is a set of its vertices, i.e., $\Delta = \boldsymbol{Co}(\Delta_0)$. Then $f(\delta) \leq \gamma$ for all $\delta \in \Delta$ if and only if $f(\delta) \leq \gamma$ for all $\delta \in \Delta_0$.*

Proof: The (only if) part is trivial, since $\Delta_0 \subset \Delta$. To prove the (if) part, from the definition of a convex hull, $\delta \in \Delta$ can be rewritten as a convex combination of vertices, i.e., $\delta = \sum_i \alpha_i \delta_i$ where $\sum_i \delta_i = 1$, $\delta_i \in [0, 1]$, and $\delta_i \in \Delta_0$ $\forall i$. Since the function f is convex on Δ,

$$f(\delta) = f(\sum_i \alpha_i \delta_i) \leq \sum_i \alpha_i f(\delta_i) \leq \gamma.$$

□

The next lemma shows how to handle the troublesome bilinear dependency of the matrix inequalities on the unknown parametric uncertainty by introducing new decision variables. The details are given [71] and here a brief introduction for the idea will be introduced and applied to our problems to find a SOF control gain matrix K_o.

Lemma 22. *(Equivalence conditions with Schur stability) The following statements are equivalent:*

(i) the system matrix A is Schur stable;

(ii) there exists a Lyapunov matrix $P = P^T > 0$ such that

$$A^T PA - P < 0; \tag{4.54}$$

(iii) there exist matrices $P = P^T > 0$ and G of compatible dimensions such that

$$\begin{bmatrix} P & A^T G^T \\ GA & G + G^T - P \end{bmatrix} > 0. \tag{4.55}$$

(iv) there exists a positive definite matrix $Q = Q^T > 0$ such that

$$Q^{1/2} A^T Q^{-1} A Q^{1/2} - Q < 0, \tag{4.56}$$

or equivalently

$$\begin{bmatrix} -Q & QA^T \\ AQ & -Q \end{bmatrix} < 0, \tag{4.57}$$

(v) there exist matrices $Q = Q^T$ and H of compatible dimensions such that

$$\begin{bmatrix} H + H^T - Q & H^T A \\ AH & Q \end{bmatrix} > 0. \tag{4.58}$$

Proof: The equivalence between the first two statements is well known. To prove $(iii) \Rightarrow (ii)$, let suppose that the inequality (4.55) holds for some $P = P^T > 0$ and G. Then, since $P > 0$ the matrix inequality $(G - P)P^{-1}(G - P)^T \geq 0$ holds for such matrices P and G. We can easily see that

$$0 < (4.55) \leq \begin{bmatrix} P & A^T G^T \\ GA & GP^{-1}G^T \end{bmatrix} \Rightarrow \left(\begin{bmatrix} P & A^T \\ A & P^{-1} \end{bmatrix} > 0 \Leftrightarrow P - A^T PA > 0 \right). \tag{4.59}$$

To prove $(ii) \Rightarrow (iii)$, let suppose that the inequality (4.54) holds for some $P = P^T > 0$. Then setting $G = G^T = P$ for (4.54) one has the inequality (4.55). The equivalence of the inequalities (4.54), (4.56), and (iv) follows from congruence transformations. In addition, by following the similar procedure in (4.59), the equivalence between the feasibilities of the inequalities (iv) and (v) is shown. □

Lemma 23. *(Polytopic parameter dependent systems) The origin of the uncertain system (4.51) is GUAS for any time-varying uncertain vector $\theta(k) \in \Theta \subset \mathbb{R}^{n_\theta}$ if either of the following inequalities holds for the specified decision variables:*

(i) there exists a Lyapunov matrix $P(\theta(k)) = P^T(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k))P_j > 0$ such that

$$A^T(\theta(k))P(\theta(k+1))A(\theta(k)) - P(\theta(k)) < 0, \quad \forall \theta \in \Theta; \tag{4.60}$$

(ii) there exist matrices $P(\theta(k)) = P^T(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k))P_j > 0$ and G of compatible dimensions such that

$$\begin{bmatrix} P(\theta(k)) & A^T(\theta(k))G^T \\ GA(\theta(k)) & G + G^T - P(\theta(k+1)) \end{bmatrix} > 0, \quad \forall \theta \in \Theta \tag{4.61}$$

or equivalently,

$$\begin{bmatrix} P_j & A_j^T G^T \\ GA_j & G + G^T - P_i \end{bmatrix} > 0, \quad \forall i, j = 1, \ldots, J; \tag{4.62}$$

Note that the inequality (4.61) is convex in the uncertain parameter vector $\theta(k)$, so that it is enough to check its feasibility in the vertices of the PLDI (4.62).

(iii) there exists the inverse of a Lyapunov matrix, say $Q(\theta(k)) = Q^T(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k))Q_j > 0$, such that

$$\begin{bmatrix} Q(\theta(k)) & Q(\theta(k+1))A^T(\theta(k)) \\ A(\theta(k))Q(\theta(k+1)) & Q(\theta(k+1)) \end{bmatrix} > 0, \quad \forall \theta \in \Theta; \tag{4.63}$$

(iv) there exist matrices $Q(\theta(k)) = Q^T(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k))Q_j > 0$ and H of compatible dimensions such that

$$\begin{bmatrix} H + H^T - Q(\theta(k)) & H^T A^T(\theta(k)) \\ A(\theta(k))H & Q(\theta(k+1)) \end{bmatrix} > 0, \quad \forall \theta \in \Theta \tag{4.64}$$

or equivalently,

$$\begin{bmatrix} H + H^T - Q_j & H^T A_j^T \\ A_j H & Q_i \end{bmatrix} > 0, \quad \forall i, j = 1, \ldots, J. \tag{4.65}$$

Note that the inequality (4.64) is convex in the uncertain parameter vector $\theta(k)$, so that it is enough to check its feasibility in the vertices of the PLDI (4.65).

Proof: Using the Schur complement lemma, the matrix inequality

$$\bar{W}(\theta(k), \theta(k+1)) \triangleq \begin{bmatrix} -X(\theta(k)) & A^T(\theta(k))X(\theta(k+1)) \\ X(\theta(k+1))A(\theta(k)) & -X(\theta(k+1)) \end{bmatrix} < 0 \tag{4.66}$$

is equivalent to (4.53). From Lemma 22, the matrix inequality

$$\bar{W}_G(\theta(k), \theta(k+1)) \triangleq \begin{bmatrix} X(\theta(k)) & A^T(\theta(k))X(\theta(k+1)) \\ X(\theta(k+1))A(\theta(k)) & G + G^T - X(\theta(k+1)) \end{bmatrix} > 0 \tag{4.67}$$

is equivalent to (4.66) and (4.53). Now, replacing $X(\cdot)$ by $P(\cdot)$, the other parts are nothing but an extension of Lemma 22. □

Comment 5. *Note that there is no product terms of P_j and A_j and this property plays a crucial role in controller synthesis problems, which is addressed in the remainder of this section.*

4.3.2 SOF Control via LMI Optimization for a Certain Class of Lur'e Systems under Polytopic Uncertainties–Case I

In the Absence of Feedback-Connected Nonlinearities

In order to apply the stabilizing output-feedback to the perturbed system with uncertainties, and to solve SOF controller synthesis problems using LMIs, consider the system with a control affine term

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + B_u u(k) \\ y(k) &= C_y(\theta(k))x(k), \end{aligned} \tag{4.68}$$

where $x(k) \in \mathbb{R}^n$ is the state variable and $u(k) \in \mathbb{R}^{n_u}$ is the control input at time $k \in \mathbb{Z}_+$. In addition, $\theta(k) \in \Theta$ represents the parametric uncertainty of the system and the subset $\Theta \subset \mathbb{R}^{n_\theta}$ is assumed to be closed and compact. Assume that the mappings $A : \Theta \to \mathbb{R}^{n\times n}$ and $C_y : \Theta \to \mathbb{R}^{n_y\times n}$ are continuous in $\theta(k) \in \Theta$ which is Lebesgue measurable for all $k \in \mathbb{Z}_+$.

Definition 12. *(Polytopic LDIs [100]) Of particular interest will be the systems in which the realization* *matrices of the system are affinely dependent on the parametric uncertainty function $\theta : \mathbb{Z}_+ \to \Theta$:*

$$\begin{bmatrix} A(\theta) \\ C_y(\theta) \end{bmatrix} \in \Omega_{AC} \triangleq \boldsymbol{Co}\left\{ \begin{bmatrix} A_1 \\ C_{y,1} \end{bmatrix}, \cdots, \begin{bmatrix} A_{n_J} \\ C_{y,J} \end{bmatrix} \right\} = \boldsymbol{Co}(\Omega_{AC}^v), \quad \forall \theta \in \Theta, \tag{4.69}$$

where $J = 2^{n_\theta}$ and the uncertain parameter θ is assumed to be time-varying in general and the time-dependence notation is dropped for convenience. In other words,

$$\Pi(\theta) = \sum_{i=1}^{J} \rho_i(\theta)\Pi_i \text{ for each } \Pi \in \{A, C_y\}, \tag{4.70}$$

where $\sum_{i=1}^{J} \rho_i(\theta) = 1$ and $\rho_i(\theta) \in [0, 1]$ for all $\theta \in \Theta$ and each index $i = 1, \ldots, J$. Then, the system (4.68) is referred to as a convex parameter-dependent model.

Now, consider the SOF controller synthesis problem for the system (4.68) with (i) a single (or common) and (ii) parameter-dependent Lyapunov functions.

A. With a common (simultaneous) Lyapunov function

For a certain class of state-space representation used for describing the system, a sufficient LMI condition for the original non-convex feasibility problem for SOF control, which has been derived in [24], was introduced in Section 3.4.1.

Proposition 20. *(Robust stabilizing SOF control) Consider a particular state-space representation for the system where the uncertain system realization $(A(\theta), C_y(\theta))$, which is represented as a PLDI (4.69), and B_u is assumed to be full column rank. If there exist the matrices Y, M, N such that the following two matrix inequalities and one matrix equality constraint*

$$\begin{aligned} YA_i + A_i^T Y + B_u N C_{y,i} + C_{y,i}^T N^T B_u^T &< 0 \\ Y &> 0 \\ B_u M &= Y B_u \end{aligned} \tag{4.71}$$

are feasible for all $i = 1, \ldots, J$, then a stabilizing SOF control gain K_o is $M^{-1}N$, i.e., the feedback control signal

$$u(k) = M^{-1}Ny(k)$$

stabilizes the system (4.68) whose uncertain model is represented by the PLDI in (4.69). In other words, the existence of the matrices satisfying (4.71) is sufficient for the GUAS of the system (4.68) with a common

Lyapunov matrix Y, where the quadratic Lyapunov function is $V(\xi) = \xi^T Y \xi$, and $K_o = M^{-1}N$.

Proof: The proofs for the previous two propositions are given in [24], which requires simple linear algebra. □

Proposition 21. *(Robust stabilizing SOF control–Coupled LMI formulation) There exists a stabilizing SOF control gain matrix K_o for $X = X^T > 0$ and $Y = Y^T > 0$ if and only if X and Y satisfy the following two matrix inequalities for all $i = 1, \ldots, J$:*

$$B_u^{\perp}(A_i X + X A_i^T)(B_u^{\perp})^T < 0, \tag{4.72}$$

$$(C_{y,i}^T)^{\perp}(A_i^T Y + Y A_i)((C_{y,i}^T)^{\perp})^T < 0, \tag{4.73}$$

where $XY = YX = I$ and $(\cdot)^{\perp}$ indicates a full rank matrix orthogonal to $(\cdot)$.

The importance of this result lies in the fact that robust quadratic stability can be concluded from a finite test of matrix inequalities whenever the uncertainty polytope consists of a finite number of vertices. The next lemma gives a more general, but complex, tool to design a SOF controller while lemma itself is about the existence of a stabilizing static feedback control law. The next proposition is also a variation of the SOF scheme, which was derived in [41], introduced in Section 3.4.1.

B. With parameter-dependent Lyapunov function (PDLF)

Consider a PLDF (4.52) for SOF controller synthesis problems of the system (4.68). From the results in Section 4.3.1, similarly to the case with a common Lyapunov function, two SOF control schemes can be derived.

Lemma 24. *(Robust stabilizing SOF control) Consider a particular state-space representation for the system where the uncertain system realization $(A(\theta(k)), C_y(\theta(k)))$, which is represented as a PLDI (4.69), and B_u is assumed to be full column rank. If there exist the matrices $P(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k))P_j$, G, M, and N such that the following matrix inequalities and a matrix equality*

$$\begin{aligned} \begin{bmatrix} P_j & A_j^T G^T + C_{y,j}^T N_g^T B_u^T \\ G A_j + B_u N_g C_{y,j} & G + G^T - P_i \end{bmatrix} &> 0 \\ P_j, P_i &> 0 \\ B_u M_g &= G B_u \end{aligned} \tag{4.74}$$

are feasible for all $i, j = 1, \ldots, J$, then a stabilizing SOF control gain K_o is $M_g^{-1}N_g$, i.e., the feedback control *signal*

$$u(k) = M_g^{-1} N_g y(k)$$

stabilizes the system (4.68) whose uncertain model is represented by the PLDI (4.69). In other words, the existence of the decision matrix variables satisfying (4.74) is sufficient for the GUAS of the system (4.68) with time-varying parameter dependent Lyapunov matrix $P(\theta(k)) = \sum_{j=1}^{T} \rho_j(\theta(k))P_j$*, where the quadratic Lyapunov function is* $V(\xi(k)) = \xi(k)^T P(\theta(k))\xi(k)$*, and the output-feedback gain matrix factorized as* $K_o = M_g^{-1} N_g$*.*

Proof: Replacing $A(\theta(k))$ by $A(\theta(k)) + B_u K_o C_y(\theta(k))$ in (4.61) and (4.62), one can conclude that if there exist $P_j = P_j^T > 0$ for $j = 1, \dots, J$, G, and K_o such that

$$\begin{bmatrix} P_j & A_j^T G^T + C_{y,j}^T K_o^T B_u^T G^T \\ G A_j + G B_u K_o C_{y,j} & G + G^T - P_i \end{bmatrix} > 0 \tag{4.75}$$

holds for all $i, j = 1, \dots, J$, then $u(k) = K_o y(k)$ is a stabilizing control law. When B_u is full column rank, the feasibility of (4.75) reduces as (4.74). $\square$

The next lemma gives a more general, but complex, tool to design a SOF controller while lemma itself is about the existence of a stabilizing static feedback control law.

Lemma 25. *(Robust stabilizing SOF control–Coupled LMI formulation) There exists a stabilizing SOF control gain matrix* K_o *such that the closed loop system (4.68) is stabilized with the control law* $u(k) = K_o y(k)$ *if and only if there exist the matrices* G, G^{-1}, $P = P^T = \sum_{j=1}^{J} \rho_j(\theta(k))P_j > 0$*, and* H *such that the following matrix inequalities hold for all* $i, j = 1, \dots, J$*:*

$$\begin{bmatrix} I & 0 & 0 \\ 0 & B_u^\perp & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} H + H^T - P_j^{-1} & H^T A_j^T & 0 \\ A_j H & G^{-1} + G^{-T} & G^{-1} \\ 0 & G^{-T} & P_i^{-1} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & (B_u^\perp)^T & 0 \\ 0 & 0 & I \end{bmatrix} > 0, \tag{4.76}$$

$$\begin{bmatrix} (C_{y,j}^T)^\perp & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_j & A_j^T G^T \\ G A_j & G + G^T - P_i \end{bmatrix} \begin{bmatrix} ((C_{y,j}^T)^\perp)^T & 0 \\ 0 & I \end{bmatrix} > 0, \tag{4.77}$$

where $(\cdot)^\perp$ *indicates a full rank matrix orthogonal to* $(\cdot)$*.*

Proof: The matrix inequality (4.75) can be rewritten as the following two equivalent

$$\bar{G}\left(\mathcal{A}_j(P_j,P_i,G)+\bar{B}_uK_o\bar{C}_{y,j}\right)+\left(\mathcal{A}_j^T(P_j,P_i,G)+\bar{C}_{y,j}^TK_o^T\bar{B}_u^T\right)\bar{G}^T>0 \quad \forall i,j; \tag{4.78}$$

$$\left(\mathcal{A}_j(P_j,P_i,G)+\bar{B}_uK_o\bar{C}_{y,j}\right)\bar{G}^{-T}+\bar{G}^{-1}\left(\mathcal{A}_j^T(P_j,P_i,G)+\bar{C}_{y,j}^TK_o^T\bar{B}_u^T\right)>0 \quad \forall i,j, \tag{4.79}$$

where

$$\mathcal{A}_j(P_j,P_i,G)\triangleq\begin{bmatrix}\frac{1}{2}P_j & 0\\ A_j & I-\frac{1}{2}G^{-1}P_i\end{bmatrix},\ \bar{G}\triangleq\begin{bmatrix}I & 0\\ 0 & G\end{bmatrix},\ \bar{B}_u\triangleq\begin{bmatrix}0\\ B_u\end{bmatrix},\ \bar{C}_{y,j}\triangleq\begin{bmatrix}C_{y,j} & 0\end{bmatrix}. \tag{4.80}$$

Using the coupled LMI formulation for stabilizing SOF control, there exists a SOF control gain K_o satisfying the inequality (4.75) with some P_j, P_i, and G if and only if P_j, P_i, and G satisfy the following two inequalities for each index pair (i,j):

$$\bar{B}_u^{\perp}\left(\mathcal{A}_j(P_j,P_i,G)\bar{G}^{-T}+\bar{G}^{-1}\mathcal{A}_j^T((P_j,P_i,G)\right)(\bar{B}_u^{\perp})^T>0, \tag{4.81}$$

$$(\bar{C}_{y,j}^T)^{\perp}\left(\bar{G}\mathcal{A}_j(P_j,P_i,G)+\mathcal{A}_j^T(P_j,P_i,G)\bar{G}^T\right)((\bar{C}_{y,j}^T)^{\perp})^T>0. \tag{4.82}$$

The second matrix inequality (4.82) can be rewritten as (4.77). The first matrix inequality (4.81) can be rewritten as

$$\begin{bmatrix}I & 0\\ 0 & B_u^{\perp}\end{bmatrix}\begin{bmatrix}P_j & A_j^T\\ A_j & G^{-1}+G^{-T}-G^{-1}P_iG^{-T}\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & (B_u^{\perp})^T\end{bmatrix}>0. \tag{4.83}$$

Applying the Schur complement lemma and taking a congruence transformation results in the equivalent matrix inequality:

$$\begin{bmatrix}I & 0 & 0\\ 0 & B_u^{\perp} & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}P_j & A_j^T & 0\\ A_j & G^{-1}+G^{-T} & G^{-1}\\ 0 & G^{-T} & P_i^{-1}\end{bmatrix}\begin{bmatrix}I & 0 & 0\\ 0 & (B_u^{\perp})^T & 0\\ 0 & 0 & I\end{bmatrix}>0. \tag{4.84}$$

With simple algebraic manipulation, the inequality (4.84) can be rewritten as

$$\begin{bmatrix}I & 0 & 0\\ 0 & B_u^{\perp} & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}P_j & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}P_j^{-1} & P_j^{-1}A_j^T & 0\\ A_jP_j^{-1} & G^{-1}+G^{-T} & G^{-1}\\ 0 & G^{-T} & P_i^{-1}\end{bmatrix}\begin{bmatrix}P_j & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}I & 0 & 0\\ 0 & (B_u^{\perp})^T & 0\\ 0 & 0 & I\end{bmatrix}>0, \tag{4.85}$$

which is equivalent to the inequality

$$\begin{bmatrix} P_j & 0 & 0 \\ 0 & B_u^{\perp} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_j^{-1} & P_j^{-1}A_j^T & 0 \\ A_jP_j^{-1} & G^{-1}+G^{-T} & G^{-1} \\ 0 & G^{-T} & P_i^{-1} \end{bmatrix} \begin{bmatrix} P_j & 0 & 0 \\ 0 & (B_u^{\perp})^T & 0 \\ 0 & 0 & I \end{bmatrix} > 0. \tag{4.86}$$

Now, taking a congruence transformation with the transformation matrix diag$\{P_j^{-1}, I, I\}$, an equivalent LMI condition for P_j^{-1}, P_i^{-1}, and G^{-1} is derived:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & B_u^{\perp} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_j^{-1} & P_j^{-1}A_j^T & 0 \\ A_jP_j^{-1} & G^{-1}+G^{-T} & G^{-1} \\ 0 & G^{-T} & P_i^{-1} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & (B_u^{\perp})^T & 0 \\ 0 & 0 & I \end{bmatrix} > 0. \tag{4.87}$$

To remove the product terms of A_j and P_j, the results of Lemma 22 are adapted. □

Robust quadratic stability can be concluded from a finite test of matrix inequalities whenever the uncertainty polytope consists of a finite number of vertices and further, if the uncertain parameter θ is time-invariant then the subscript i is replaced by j.

In the Presence of Feedback-Connected Nonlinearities

In order to apply the stabilizing output-feedback to the perturbed system with uncertainties, and to solve SOF controller synthesis problems using LMIs, consider the system with a control affine term and feedback connected nonlinearity $\phi \in \bar{\Phi}_{ab}^{\alpha}$

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + B_u u(k) + B_p \phi(k, x(k)) \\ y(k) &= C_y(\theta(k))x(k), \end{aligned} \tag{4.88}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^{n_u}$ is the control input at time $k \in \mathbb{Z}_+$. In addition, $\theta(k) \in \Theta$ represents the parametric uncertainty of the system and the subset $\Theta \subset \mathbb{R}^{n_\theta}$ is assumed to be closed and compact. It is assumed that the mappings $A : \Theta \to \mathbb{R}^{n\times n}$ and $C_y : \Theta \to \mathbb{R}^{n_y\times n}$ satisfy the PLDI (4.69).

A. With a common (simultaneous) Lyapunov function

Recalling the stability condition for a Lur'e system with the maximum sector bound $\alpha = \frac{1}{\sqrt{\gamma}}$ in (4.19), we have the following matrix inequality for the system (4.88) provided no feedback control, i.e., $u(k) \equiv 0$:

$$\begin{bmatrix} P & 0 & A^T(\theta(k))P & C_q^T \\ 0 & I & B_p^T P & 0 \\ PA(\theta(k)) & PB_p & P & 0 \\ C_q & 0 & 0 & \gamma I \end{bmatrix} > 0. \tag{4.89}$$

Now, replacing $A(\theta(k))$ by $A(\theta(k)) + B_u K_o C_y(\theta(k))$ in the LMI constraint (4.89) and assuming the PLDI relation (4.69), if there exists a SOF control gain matrix K_o satisfying (4.89) for some $P = P^T > 0$ then the closed-loop system is stabilized by that control law $u(k) = K_o y(k)$. The LMI constraint (4.89) can be rewritten in the standard form of SOF control problem as (4.2):

$$\bar{P}(\bar{A}_j + \bar{B}_u K_o \bar{C}_{y,j}) + (\bar{A}_j + \bar{B}_{u,j} K_o \bar{C}_y)^T \bar{P} < 0, \tag{4.90}$$

where

$$\bar{A}_j \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 & 0 \\ A_j & B_p & -\frac{1}{2}I & 0 \\ C_q & 0 & 0 & -\frac{1}{2}\gamma I \end{bmatrix}, \quad \bar{B}_u \triangleq \begin{bmatrix} 0 \\ 0 \\ B_u \\ 0 \end{bmatrix}, \quad \bar{C}_{y,j} \triangleq \begin{bmatrix} C_{y,j} & 0 & 0 & 0 \end{bmatrix}, \tag{4.91}$$

$$\bar{P} \triangleq \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{4.92}$$

Similarly to the case without the feedback connected nonlinear mapping, for a certain class of state-space representation used for describing the system, a sufficient LMI condition can be derived and a more general, but complex, tool to design a SOF controller with the condition for existence of such a stabilizing static feedback control law is also formulated.

Lemma 26. *(Robust stabilizing SOF control) Consider a particular state-space representation for the system where the uncertain system realization $(A(\theta), C_y(\theta))$, which is represented as a PLDI (4.69), and B_u is assumed to be full column rank. If there exist the matrices P, M, and N such that the following two matrix* *inequalities and one matrix equality constraint*

$$\begin{aligned} \bar{P}\bar{A}_j + \bar{A}_j^T\bar{P} + \bar{B}_u N\bar{C}_{y,j} + \bar{C}_{y,j}^T N^T \bar{B}_u^T &< 0 \\ P &> 0 \\ B_u M &= P B_u \end{aligned} \tag{4.93}$$

are feasible, then a stabilizing SOF control gain K_o is $M^{-1}N$, i.e., the feedback control signal

$$u(k) = M^{-1}Ny(k)$$

stabilizes the system (4.88) whose uncertain model is represented by the PLDI (4.69). In other words, the existence of the matrices satisfying (4.93) is sufficient for the GUAS of the system (4.88) with the same Lyapunov matrix P, where the quadratic Lyapunov function is $V(\xi) = \xi^T P\xi$, and $K_o = M^{-1}N$.

Lemma 27. *(Robust stabilizing SOF control–Coupled LMI formulation) There exists a stabilizing SOF control gain matrix K_o for $P = P^T > 0$ if and only if P satisfies the following two matrix inequalities for all $j = 1, \ldots, J$:*

$$\bar{B}_u^{\perp}(\bar{A}_j\bar{P}^{-1} + \bar{P}^{-1}\bar{A}_j^T)(\bar{B}_u^{\perp})^T < 0, \tag{4.94}$$

$$(\bar{C}_{y,j}^T)^{\perp}(\bar{A}_j^T\bar{P} + \bar{P}\bar{A}_j)((\bar{C}_{y,j}^T)^{\perp})^T < 0, \tag{4.95}$$

where $(\cdot)^{\perp}$ indicates a full rank matrix orthogonal to $(\cdot)$.

Robust quadratic stability can be concluded from a finite test of matrix inequalities whenever the uncertainty polytope consists of a finite number of vertices.

B. With parameter-dependent Lyapunov function (PDLF)

Consider a PLDF (4.52) for SOF controller synthesis problems of the system (4.88). Two different SOF controller design schemes previously introduced are also considered.

Theorem 33. *(Robust stabilizing SOF control) Consider a particular state-space representation for the system (4.88) where the uncertain system realization $(A(\theta(k)), C_y(\theta(k)))$, which is represented as a PLDI (4.69), and B_u is assumed to be full column rank. Then if there exist the matrices $P(\theta(k)) = \sum_{j=1}^{J}\theta_j(k)P_j$,* *G, M, and N such that the following matrix inequalities and matrix equality*

$$\begin{bmatrix} P_j & 0 & A_j^T G^T + C_{y,j}^T N_g^T B_u^T & C_q^T \\ 0 & I & B_p^T G^T & 0 \\ GA_j + B_u N_g C_{y,j} & GB_p & G + G^T - P_i & 0 \\ C_q & 0 & 0 & \gamma I \end{bmatrix} > 0$$

$$P_j > 0 \tag{4.96}$$

$$B_u M_g = GB_u$$

are feasible for all $i, j = 1, \ldots, J$, then a stabilizing SOF control gain K_o is $M_g^{-1} N_g$, i.e., the feedback control signal

$$u(k) = M_g^{-1} N_g y(k)$$

stabilizes the system (4.88) whose uncertain model is represented by the PLDI (4.69). In other words, the existence of the decision matrix variables satisfying (4.96) is sufficient for the GUAS of the system (4.88) with parameter dependent Lyapunov matrix $P(\theta(k))$, where the quadratic Lyapunov function is $V(\xi(k)) = \xi(k)^T P(\theta(k))\xi(k)$, and the output-feedback gain matrix which is factorized as $K_o = M_g^{-1} N_g$.

Proof: Replacing $A(\theta(k))$ by $A(\theta(k)) + B_u K_o C_y(\theta(k))$ in (4.89), and H by G, one can conclude that if there exist $P_j = P_j > 0$ for each $j = 1, \ldots, J$, G, and K_o such that the matrix inequality

$$\begin{bmatrix} P_j & 0 & A_j^T G^T + C_{y,j}^T K_o^T B_u^T G^T + & C_q^T \\ 0 & I & B_p^T G^T & 0 \\ GA_j + GB_u K_o C_{y,j} & GB_p & G + G^T - P_i & 0 \\ C_q & 0 & 0 & \gamma I \end{bmatrix} > 0 \tag{4.97}$$

holds for all $i, j = 1, \ldots, J$, then $u(k) = K_o y(k)$ is a stabilizing control law. The inequality (4.97) reduces as the constraints in (4.96), which consist of linear matrix inequalities and equality, provided that B_u has full column rank. □

Theorem 34. *(Robust stabilizing SOF control–Coupled LMI formulation) There exists a stabilizing SOF control gain matrix K_o if and only if there exist the matrices G, G^{-1}, $Q(\theta(k)) = Q^T(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k)) Q_j >$*

0, and H such that the following matrix inequalities hold for all $i, j = 1, \ldots, J$:

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & B_{u,j}^{\perp} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} H + H^T - P_j^{-1} & 0 & H^T A_j^T & H C_q^T & 0 \\ 0 & I & B_p^T & 0 & 0 \\ A_j H & B_p & G^{-1} + G^{-T} & 0 & G^{-1} \\ C_q H & 0 & 0 & \gamma I & 0 \\ 0 & 0 & G^{-T} & 0 & P_i^{-1} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & (B_{u,j}^{\perp})^T & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} > 0, \tag{4.98}$$

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_j & 0 & A_j^T G^T & C_q^T \\ 0 & I & B_p^T G^T & 0 \\ G A_j & G B_p & G + G^T - P_i & 0 \\ C_q & 0 & 0 & \gamma I \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} > 0, \tag{4.99}$$

where $(\cdot)^{\perp}$ indicates a full rank matrix orthogonal to $(\cdot)$.

Proof: For the proof of this theorem, simple algebra is only required. Detailed computations are left for the reader to derive. □

4.3.3 SOF Control via LMI Optimization for a Certain Class of Lur'e Systems under Polytopic Uncertainties–Case II

In the Absence of Feedback-Connected Nonlinearities

In order to apply the stabilizing output-feedback to the perturbed system with parameteric uncertainties, and to solve SOF controller synthesis problems using LMIs, consider the system with a control affine term

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + B_u(\theta(k))u(k) \\ y(k) &= C_y x(k), \end{aligned} \tag{4.100}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^{n_u}$ is the control input at time $k \in \mathbb{Z}_+$. In addition, $\theta(k) \in \Theta$ represents the parametric uncertainty of the system and the subset $\Theta \subset \mathbb{R}^{n_\theta}$ is assumed to be closed and compact. It is assumed that the mappings $A : \Theta \to \mathbb{R}^{n \times n}$ and $B_u : \Theta \to \mathbb{R}^{n \times n_u}$ are continuous in $\theta \in \Theta$ which is Lebesgue measurable.

Definition 13. *(Polytopic LDIs [100]) Of particular interest will be the systems in which the realization matrices of the system are affinely dependent on the parametric uncertainty function* $\theta : \mathbb{Z}_+ \to \Theta$.

$$\begin{bmatrix} A(\theta) & B_u(\theta) \end{bmatrix} \in \Omega_{AB} \triangleq \boldsymbol{Co}\left\{ \begin{bmatrix} A_1 & B_{u,1} \end{bmatrix}, \cdots, \begin{bmatrix} A_J & B_J \end{bmatrix} \right\} = \boldsymbol{Co}(\Omega_{AB}^v) \quad, \forall \theta \in \Theta, \tag{4.101}$$

where $J = 2^{n_\theta}$ *and the uncertain parameter* θ *is assumed to be time-varying in general and the time-dependence notation is dropped for convenience. In other words,*

$$\Pi(\theta) = \sum_{i=1}^{J} \rho_j(\theta)\Pi_i \text{ for each } \Pi \in \{A, B_u\}, \tag{4.102}$$

where $\sum_{i=1}^{J} \rho_i(\theta) = 1$, $\rho_i(\theta) \in [0,1]$ *for all* $\theta \in \Theta$ *and each* $i = 1, \ldots, J$. *Then, the system (4.100) is referred to as a* convex parameter-dependent model.

Now, consider the SOF controller synthesis problem for the system (4.100) with (i) a single (or common) and (ii) parameter-dependent Lyapunov functions. All of the LMI formulations for SOF control are similar to the results in Section 4.3.2.

A. With a common (simultaneous) Lyapunov function

For a certain class of state-space representation used for describing the system, a sufficient LMI condition for the original non-convex feasibility problem for SOF control, which has been derived in [24], was introduced in Section 3.4.1.

Proposition 22. *(Robust stabilizing SOF control [24]) Consider a particular state-space representation for the system (4.100) where the uncertain system realization* $(A(\theta(k)), B_u(\theta(k)))$, *which is represented as a PLDI (4.101), and* C_y *is assumed to be full row rank. If there exist the matrices* X, M, N *such that the following two matrix inequalities and one matrix equality constraint*

$$\begin{aligned} A_i X + X A_i^T + B_{u,i} N C_y + C_y^T N^T B_{u,i}^T &< 0 \\ X &> 0 \\ M C_y &= C_y X \end{aligned} \tag{4.103}$$

are feasible for all $i = 1, \ldots, J$, *then a stabilizing SOF control gain* K_o *is* NM^{-1}, *i.e., the feedback control signal*

$$u(k) = NM^{-1} y(k)$$

stabilizes the system (4.100) whose uncertain model is represented by the PLDI (4.101). In other words, the existence of the matrices satisfying (4.103) is sufficient for the GUAS of the system (4.100) with the same Lyapunov matrix X, where the quadratic Lyapunov function is $V(\xi) = \xi^T X \xi$, and $K_o = M^{-1}N$.

The next lemma gives a more general, but complex, tool to design a SOF controller while lemma itself is about the existence of a stabilizing static feedback control law.

Proposition 23. *(Robust stabilizing SOF control–Coupled LMI formulation) There exists a stabilizing SOF control gain matrix K_o for $X = X^T > 0$ and $Y = Y^T > 0$ if and only if X and Y satisfy the following two matrix inequalities for all $i = 1, \ldots, J$:*

$$B_{u,i}^{\perp}(A_i X + X A_i^T)(B_{u,i}^{\perp})^T < 0, \tag{4.104}$$

$$(C_y^T)^{\perp}(A_i^T Y + Y A_i)((C_y^T)^{\perp})^T < 0, \tag{4.105}$$

where $XY = YX = I$ and $(\cdot)^{\perp}$ indicates a full rank matrix orthogonal to $(\cdot)$.

B. With parameter-dependent Lyapunov function (PDLF)

Consider a PLDF (4.52) for SOF controller synthesis problems of the system (4.100). From the results in Section 4.3.1, similarly to the case with a common Lyapunov function, two SOF control schemes can be derived.

Theorem 35. *(Robust stabilizing SOF control) Consider a particular state-space representation for the system (4.100) where the uncertain system realization $(A(\theta(k)), B_u(\theta(k)))$, which is represented as a PLDI (4.101), and C_y is assumed to be full row-rank. If there exist the matrices $Q(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k)) Q_j$, G, M_g, and N_g such that the following matrix inequalities and a matrix equality*

$$\begin{aligned} \begin{bmatrix} G + G^T - Q_j & G^T A_j^T + C_y^T N_g^T B_{u,j}^T \\ A_j G + B_{u,j} N_g C_y & Q_i \end{bmatrix} &> 0 \\ Q_j &> 0 \\ M_g C_y &= C_y G \end{aligned} \tag{4.106}$$

are feasible for all $i, j = 1, \ldots, J$, then a stabilizing SOF control gain K_o is $N_g M_g^{-1}$, i.e., the feedback control signal

$$u(k) = N_g M_g^{-1} y(k)$$

stabilizes the system (4.100) whose uncertain model is represented by the PLDI (4.101). In other words, *the existence of the decision matrix variables satisfying (4.106) is sufficient for the GUAS of the system (4.100) with parameter dependent Lyapunov matrix $Q^{-1}(\theta(k))$, where the quadratic Lyapunov function is $V(\xi(k)) = \xi(k)^T Q^{-1}(\theta(k))\xi(k)$, and the output-feedback gain matrix which is factorized as $K_o = N_g M_g^{-1}$.*

Proof: Replacing $A(\theta(k))$ by $A(\theta(k)) + B_u(\theta(k))K_o C_y$ in (4.63) and (4.64), and H by G, one can conclude that if there exist $Q_j = Q_j > 0$, G, and K_o such that

$$\begin{bmatrix} G + G^T - Q_j & G^T A_j^T + G^T C_y^T K_o^T B_{u,j}^T \\ A_j G + B_{u,j} K_o C_y G & Q_i \end{bmatrix} > 0 \tag{4.107}$$

hold then $u(k) = K_o y(k)$ is a stabilizing control law. The inequality (4.107) reduces as the constraints in (4.106), which consist of linear matrix inequalities and equality, provided that C_y has full row rank. □

The next theorem gives a more general, but complex, tool to design a SOF controller while lemma itself is about the existence of a stabilizing static feedback control law.

Theorem 36. *(Robust stabilizing SOF control–Coupled LMI formulation) There exists a stabilizing SOF control gain matrix K_o if and only if there exist the matrices G, G^{-1}, $Q(\theta(k)) = Q^T(\theta(k)) = \sum_{j=1}^J \rho_j(\theta(k))Q_j > 0$, and H such that the following matrix inequalities hold for all $i, j = 1, \ldots, J$:*

$$\begin{bmatrix} I & 0 \\ 0 & B_{u,j}^{\perp} \end{bmatrix} \begin{bmatrix} G + G^T - Q_j & G^T A_j^T \\ A_j G & Q_i \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (B_{u,j}^{\perp})^T \end{bmatrix} > 0, \tag{4.108}$$

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} G^{-1} + G^{-T} & A_j^T H^T & G^{-T} \\ H A_j & H + H^T - Q_i^{-1} & 0 \\ G^{-1} & 0 & Q_j^{-1} \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} > 0, \tag{4.109}$$

where $(\cdot)^{\perp}$ indicates a full rank matrix orthogonal to $(\cdot)$.

Proof: The matrix inequality (4.107) can be rewritten as the two equivalent inequalities which are equivalent each other:

$$\left(\mathcal{A}_j(Q_j, Q_i, G^{-1}) + \bar{B}_{u,j} K_o \bar{C}_y\right) \bar{G} + \bar{G}^T \left(\mathcal{A}_j^T(Q_j, Q_i, G^{-1}) + \bar{C}_y^T K_o^T \bar{B}_{u,j}^T\right) > 0 \quad \forall i, j; \tag{4.110}$$

$$\bar{G}^{-T} \left(\mathcal{A}_j(Q_j, Q_i, G^{-1}) + \bar{B}_{u,j} K_o \bar{C}_y\right) + \left(\mathcal{A}_j^T(Q_j, Q_i, G^{-1}) + \bar{C}_y^T K_o^T \bar{B}_{u,j}^T\right) \bar{G}^{-1} > 0 \quad \forall i, j, \tag{4.111}$$

where

$$\mathcal{A}_j(Q_j, G^{-1}) \triangleq \begin{bmatrix} I - \frac{1}{2}Q_jG^{-1} & 0 \\ A_j & \frac{1}{2}Q_i \end{bmatrix}, \; \bar{G} \triangleq \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix}, \; \bar{B}_{u,j} \triangleq \begin{bmatrix} 0 \\ B_{u,j} \end{bmatrix}, \; \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 \end{bmatrix}. \tag{4.112}$$

The coupled LMI formulation for stabilizing SOF control implies that there exists a SOF control gain K_o satisfying the inequality (4.107) with some Q_j, Q_i, and G if and only if Q_j, Q_i, and G satisfy the following two inequalities for each index pair (i, j):

$$\bar{B}_{u,j}^{\perp} \left(\mathcal{A}_j(Q_j, G^{-1})\bar{G} + \bar{G}^T \mathcal{A}_j(Q_j, G^{-1}) \right) (\bar{B}_{u,j}^{\perp})^T < 0, \tag{4.113}$$

$$(\bar{C}_y^T)^{\perp} \left(\bar{G}^{-T} \mathcal{A}_j(Q_j, G^{-1}) + \mathcal{A}_j(Q_j, G^{-1})\bar{G}^{-1} \right) ((\bar{C}_y^T)^{\perp})^T < 0. \tag{4.114}$$

The first matrix inequality (4.113) can be rewritten as (4.130). The second matrix inequality (4.114) can be rewritten as

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G^{-1} + G^{-T} - G^{-T}Q_jG^{-1} & A_j^T \\ A_j & Q_i \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 \\ 0 & I \end{bmatrix} > 0. \tag{4.115}$$

The Schur complement lemma results in the equivalent matrix inequality

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} G^{-1} + G^{-T} & A_j^T & G^{-T} \\ A_j & Q_i & 0 \\ G^{-1} & 0 & Q_j^{-1} \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} > 0. \tag{4.116}$$

With simple algebraic manipulation, the inequality (4.116) can be rewritten as

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & Q_i & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} G^{-1} + G^{-T} & A_j^T Q_i^{-1} & G^{-T} \\ Q_i^{-1} A_j & Q_i^{-1} & 0 \\ G^{-1} & 0 & Q_j^{-1} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & Q_i & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} > 0, \tag{4.117}$$

which is equivalent to the inequality

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 & 0 \\ 0 & Q_i & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} G^{-1} + G^{-T} & A_j^T Q_i^{-1} & G^{-T} \\ Q_i^{-1} A_j & Q_i^{-1} & 0 \\ G^{-1} & 0 & Q_j^{-1} \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 & 0 \\ 0 & Q_i & 0 \\ 0 & 0 & I \end{bmatrix} > 0. \tag{4.118}$$

Taking a congruence transformation with the transformation matrix $\mathrm{diag}\{I, Q_i^{-1}, I\}$, an equivalent LMI condition for Q_j^{-1}, Q_i^{-1}, and G^{-1} is derived:

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} G^{-1}+G^{-T} & A_j^T Q_i^{-1} & G^{-T} \\ Q_i^{-1}A_j & Q_i^{-1} & 0 \\ G^{-1} & 0 & Q_j^{-1} \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} > 0. \tag{4.119}$$

Lemma 22 can be used to derive another equivalent linear matrix inequality (4.131) with the introduction of a new matrix decision variable H. □

In the Presence of Feedback-Connected Nonlinearities

In order to apply the stabilizing output-feedback to the perturbed system with uncertainties, and to solve SOF controller synthesis problems using LMIs, consider the system with a control affine term

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + B_u(\theta(k))u(k) + B_p\phi(k,x(k)) \\ y(k) &= C_y x(k), \end{aligned} \tag{4.120}$$

where $x(k) \in \mathbb{R}^n$ is the state variable and $u(k) \in \mathbb{R}^{n_u}$ is the control input at time $k \in \mathbb{Z}_+$. In addition, $\theta(k) \in \Theta$ represents the parametric uncertainty of the system and the subset $\Theta \subset \mathbb{R}^{n_\theta}$ is assumed to be closed and compact. It is assumed that the mappings $A : \Theta \to \mathbb{R}^{n\times n}$ and $B_u : \Theta \to \mathbb{R}^{n\times n_u}$ are continuous in $\theta(k) \in \Theta$ which is Lebesgue measurable for all $k \in \mathbb{Z}_+$ and satisfy the PLDI (4.101).

A. With a common (simultaneous) Lyapunov function

Recalling the stability condition for a Lur'e system with the maximum sector value $\alpha = \frac{1}{\sqrt{\gamma}}$ in (4.19), we have the following matrix inequality for the system the system (4.120), provided no feedback control, i.e., $u(k) \equiv 0$:

$$\begin{bmatrix} -Q & 0 & QA^T(\theta(k)) & QC_q^T \\ 0 & -I & B_p^T & 0 \\ A(\theta(k))Q & B_p & -Q & 0 \\ C_qQ & 0 & 0 & -\gamma I \end{bmatrix} < 0. \tag{4.121}$$

Replacing $A(\theta(k))$ by $A(\theta(k)) + B_u(\theta(k))K_oC_y$ in the LMI constraint (4.121) and assuming that the PLDI relation (4.101), if there exists a SOF control gain matrix K_o satisfying (4.121) for some $Q = Q^T > 0$ then the closed-loop system is stabilized by that control law $u(k) = K_oy(k)$. The LMI constraint (4.121) can be rewritten in the same form as (3.176):

$$(\bar{A}_j + \bar{B}_{u,j} K_o \bar{C}_y)\bar{Q} + \bar{Q}(\bar{A}_j + \bar{B}_{u,j} K_o \bar{C}_y)^T < 0, \tag{4.122}$$

where

$$\bar{A}_j \triangleq \begin{bmatrix} -\frac{1}{2}I & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 & 0 \\ A_j & B_p & -\frac{1}{2}I & 0 \\ C_q & 0 & 0 & -\frac{1}{2}\gamma I \end{bmatrix}, \quad \bar{B}_{u,j} \triangleq \begin{bmatrix} 0 \\ 0 \\ B_{u,j} \\ 0 \end{bmatrix}, \quad \bar{C}_y \triangleq \begin{bmatrix} C_y & 0 & 0 & 0 \end{bmatrix}, \tag{4.123}$$

$$\bar{Q} \triangleq \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{4.124}$$

Similarly to the case without feedback-connected nonlinear mapping, for a certain class of state-space representation used for describing the system, a sufficient LMI condition can be derived and a more general, but complex, tool to design a SOF controller with the condition for existence of such control law is also formulated.

Theorem 37. *(Robust stabilizing SOF control) Consider a particular state-space representation for the system (4.120) where the uncertain system realization $(A(\theta(k)), B_u(\theta(k)))$, which is represented as a PLDI (4.101), and C_y is assumed to be full row rank. If there exist the matrices X, M, N such that the following two matrix inequalities and one matrix equality constraint*

$$\begin{aligned} \bar{A}_j\bar{Q} + \bar{Q}A_j^T + \bar{B}_{u,j}N\bar{C}_y + \bar{C}_y^T N^T \bar{B}_{u,i}^T &< 0 \\ \bar{Q} &> 0 \\ MC_y = C_y Q \end{aligned} \tag{4.125}$$

are feasible, then a stabilizing SOF control gain K_o is NM^{-1}, i.e., the feedback control signal

$$u(k) = NM^{-1}y(k)$$

stabilizes the system (4.120) whose uncertain model is represented by the PLDI (4.101). In other words, the *existence of the matrices satisfying (4.103) is sufficient for the GUAS of the system (4.120) with the same Lyapunov matrix $\bar{Q}$, where the quadratic Lyapunov function is $V(\xi) = \xi^T \bar{Q}^{-1} \xi$, and $K_o = M^{-1}N$.*

Theorem 38. *(Robust stabilizing SOF control–Coupled LMI formulation) There exists a stabilizing SOF control gain matrix K_o for $Q = Q^T > 0$ if and only if Q satisfies the following two matrix inequalities for all $j = 1, \ldots, J$:*

$$\bar{B}_{u,j}^{\perp}(\bar{A}_j \bar{Q} + \bar{Q} \bar{A}_j^T)(\bar{B}_{u,j}^{\perp})^T < 0, \tag{4.126}$$

$$(\bar{C}_y^T)^{\perp}(\bar{A}_j^T \bar{Q}^{-1} + \bar{Q}^{-1} \bar{A}_j)((\bar{C}_y^T)^{\perp})^T < 0, \tag{4.127}$$

where $(\cdot)^{\perp}$ indicates a full rank matrix orthogonal to $(\cdot)$.

B. With parameter-dependent Lyapunov function (PLDF)

Now consider a PLDF (4.52) for SOF controller synthesis problems of the system (4.120). Two different previously introduced SOF controller design schemes are also considered.

Theorem 39. *(Robust stabilizing SOF control) Consider a particular state-space representation for the system (4.120) where the uncertain system realization $(A(\theta(k)), B_u(\theta(k)))$ which is represented as a PLDI (4.101) and C_y is assumed to be full row rank. If there exist the matrices $Q(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k)) Q_j$, G, M_g, and N_g such that the following matrix inequalities and a matrix equality*

$$\begin{bmatrix} G + G^T - Q_j & 0 & G^T A_j^T + C_y^T N_g^T B_{u,j}^T & G C_q^T \\ 0 & I & B_p^T & 0 \\ A_j G + B_{u,j} N_g C_y & B_p & Q_i & 0 \\ C_q G & 0 & 0 & \gamma I \end{bmatrix} > 0$$

$$Q_j > 0 \tag{4.128}$$

$$M_g C_y = C_y G$$

are feasible for all $i, j = 1, \ldots, J$, then a stabilizing SOF control gain K_o is $N_g^{-1} M_g$, i.e., the feedback control signal

$$u(k) = N_g^{-1} M_g y(k)$$

stabilizes the system (4.120) whose uncertain model is represented by the PLDI (4.101). In other words, the existence of the decision matrix variables satisfying (4.128) is sufficient for the GUAS of the system (4.120) with parameter dependent Lyapunov matrix $Q^{-1}(\theta(k))$, where the quadratic Lyapunov function is

$V(\xi(k)) = \xi(k)^T Q^{-1}(\theta(k))\xi(k)$*, and the output-feedback gain matrix which is factorized as* $K_o = N_g^{-1} M_g$.

Proof: Replacing $A(\theta(k))$ by $A(\theta(k)) + B_u(\theta(k))K_oC_y$ in (4.121) and from the results in (iv) of Lemma 23 with replacing H by G, one can conclude that if there exist $Q_j = Q_j^T > 0$, G, and K_o such that the matrix inequality

$$\begin{bmatrix} G + G^T - Q_j & 0 & G^T A_j^T + GC_y^T K_o^T B_{u,j}^T + & GC_q^T \\ 0 & I & B_p^T & 0 \\ A_jG + B_{u,j}K_oC_yG & B_p & Q_j & 0 \\ C_qG & 0 & 0 & \gamma I \end{bmatrix} > 0 \tag{4.129}$$

holds for all $i, j = 1, \ldots, J$, then $u(k) = K_o y(k)$ is a stabilizing control law. The inequality (4.129) reduces as the constraints in (4.128) which consist of linear matrix inequalities and equality, provided that C_y has full row rank. □

Theorem 40. *(Robust stabilizing SOF control–Coupled LMI formulation) There exists a stabilizing SOF control gain matrix* K_o *if and only if there exist the matrices* G, G^{-1}, $Q(\theta(k)) = Q^T(\theta(k)) = \sum_{j=1}^{J} \rho_j(\theta(k))Q_j > 0$*, and* H *such that the following matrix inequalities hold for all* $i, j = 1, \ldots, J$*:*

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & B_{u,j}^{\perp} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} G + G^T - Q_j & 0 & G^T A_j^T & G^T C_q^T \\ 0 & I & B_p^T & 0 \\ A_jG & B_p & Q_i & 0 \\ C_qG & 0 & 0 & \gamma I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & (B_{u,j}^{\perp})^T & 0 \\ 0 & 0 & 0 & I \end{bmatrix} > 0, \tag{4.130}$$

$$\begin{bmatrix} (C_y^T)^{\perp} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} G^{-1} + G^{-T} & 0 & A_j^T H^T & C_q^T & G^{-T} \\ 0 & I & B_p^T H^T & 0 & 0 \\ HA_j & HB_p & H + H^T - Q_i^{-1} & 0 & 0 \\ C_q & 0 & 0 & \gamma I & 0 \\ G^{-1} & 0 & 0 & 0 & Q_j^{-1} \end{bmatrix} \begin{bmatrix} ((C_y^T)^{\perp})^T & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} > 0, \tag{4.131}$$

where $(\cdot)^{\perp}$ *indicates a full rank matrix orthogonal to* $(\cdot)$.

Proof: The proof is very similar to the one for Theorem 36.

SOF Control Schemes	α^*
Theorem 37	0.1412
Theorem 38	0.1641
Theorem 39	0.1433
Theorem 40	0.1641

Table 4.1: The maximal upper sector-bound achieved by the output-feedback control schemes

4.3.4 Illustrative Examples

We consider an illustrative example to show the applicability of the previously defined LMI synthesis problems for a certain case of polytopic uncertain Lur'e systems where the feedback connected nonlinear mapping is classified as $\phi \in \bar{\Phi}_{sb}^{\alpha}$. Consider the system (4.120) whose uncertain system realization is defined by the polytopic LDI:

$$A_1 = \begin{bmatrix} 1.0996 & -0.1103 & 0.0000 \\ 0.1103 & 1.0996 & 0.0000 \\ 0.0000 & 0.0000 & 0.9512 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.8886 & -0.7485 & 0.0000 \\ 0.7485 & 0.8886 & 0.0000 \\ 0.0000 & 0.0000 & 1.1052 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.9087 & -0.2811 & 0.0000 \\ 0.2811 & 0.9087 & 0.0000 \\ 0.0000 & 0.0000 & 1.2214 \end{bmatrix},$$

$$B_{u,1} = B_{u,2} = B_{u,3} = \begin{bmatrix} 0.0238 & -0.0216 \\ 0.0441 & 0.0843 \\ 0.0632 & 0.0398 \end{bmatrix},\ B_p = \begin{bmatrix} -0.1600 & -1.2900 \\ 0.8100 & 0.9600 \\ 0.4100 & 0.6500 \end{bmatrix},$$

and

$$C_{y,1} = C_{y,2} = C_{y,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ C_q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Suppose that the control objective is to maximize the upper bound α on the sector condition for the feedback-connected nonlinear mapping $\phi \in \bar{\Phi}_{sb}^{\alpha}$ such that the closed-loop system (4.120) is stabilized by the output-feedback control law $u(k) = K_o y(k)$. Then the optimal solutions K_o^* for the optimization problems in Theorem 37, 38, 39, and 40 are obtained as

$$\text{Theorem 37} : K_o^* = \begin{bmatrix} -71.9574 & -21.0591 & 19.3357 \\ 55.9335 & 3.1531 & -27.0185 \end{bmatrix},\ \text{Theorem 38} : K_o^* = \begin{bmatrix} -30.5649 & 0.9110 & -0.6719 \\ 23.5970 & -9.3111 & -6.5752 \end{bmatrix},$$

$$\text{Theorem 39} : K_o^* = \begin{bmatrix} -67.2531 & -24.0286 & 15.7645 \\ 53.2382 & 4.3839 & -25.4228 \end{bmatrix},\ \text{Theorem 40} : K_o^* = \begin{bmatrix} -9.4265 & 5.8384 & -12.0600 \\ 6.6509 & -12.2241 & 3.3370 \end{bmatrix}.$$

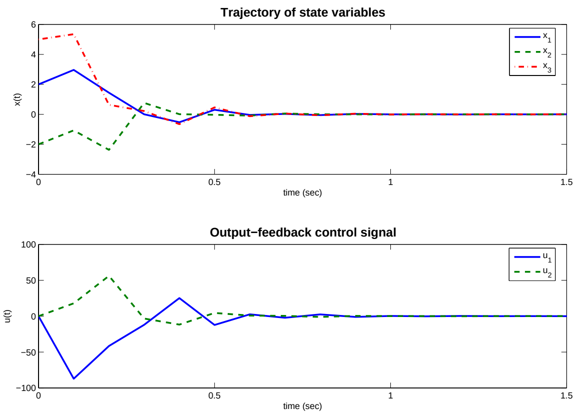


Figure 4.3: Theorem 37

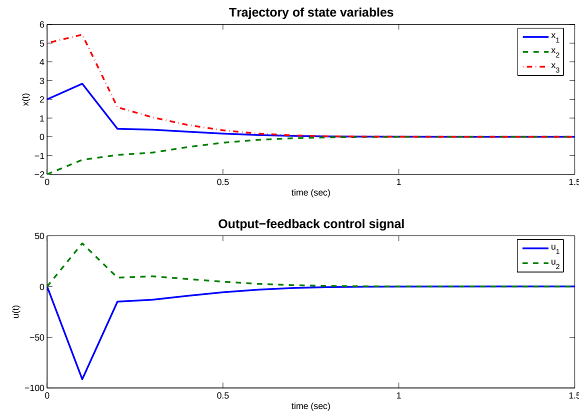


Figure 4.4: Theorem 38

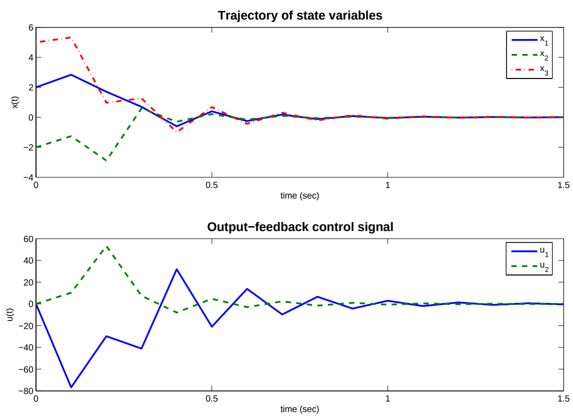


Figure 4.5: Theorem 39

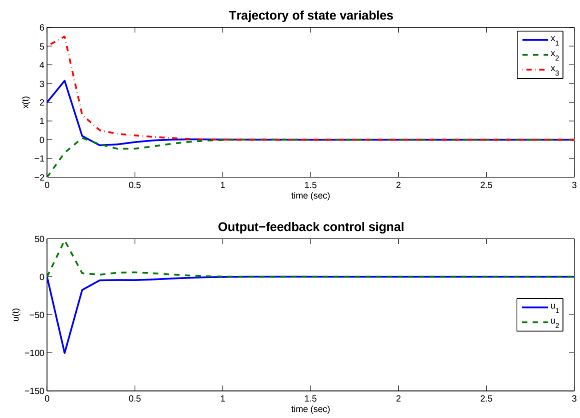


Figure 4.6: Theorem 40

Figures 4.3, 4.4, 4.5, and 4.6 show the stability of the closed-loop systems with the output-feedback control laws $u(k) = K_o^* y(k)$ that are obtained from four different optimization problems in Theorems 37, 38, 39, and 40. In the simulation, it is assumed that the feedback connected nonlinear functions satisfy the sector condition $\phi \in \bar{\Phi}_{sb}^{\alpha^*}$ where the different upper bounds α^* are given in Table 4.1.

4.4 Parameter Dependent State-Feedback Control via LMI Optimization for Lur'e Systems with Uncertainties

The main idea in this section is to combine the parameter-dependent control Lyapunov functions, which give a number of control gain matrices, with the well-known adaptive scheme to tune the absolute weights of each control gain matrices. That is, the overall control law is a linear combination of a number of state

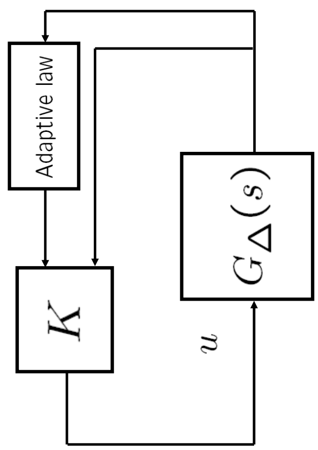


Figure 4.7: Parameter-Dependent State-Feedback Control for LTI systems

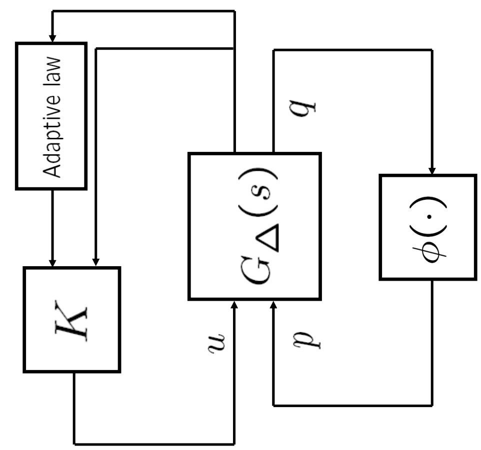


Figure 4.8: Parameter-Dependent State-Feedback Control for Lur'e systems

feedbacks weighted by the time-varying parameters which are tuned by following the pre-determined adaptive laws.

4.4.1 Continuous-Time LTI Systems with Time-Invariant Parametric Uncertainty

Consider the continuous-time linear parameter-varying (LPV) systems given by

$$\dot{x}(t) = A(\theta)x(t) + B_u u(t) \tag{4.132}$$

where the system matrix A depends on a time-invariant parameter $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ and Θ is assumed to be compact. For example, if each element of the time-varying parameter vector $\theta = (\theta_1, \cdots, \theta_{n_\theta})^T$ lies in a bounded interval, $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$, then a compact set Θ is a cube whose vertices can be defined by a finite set

$$\Theta_v \triangleq \left\{\theta_v \in \Theta | \theta_v = (\theta_{v,1}, \cdots, \theta_{v,n_\theta})^T, \text{ where } \theta_{v,i} \in \{\underline{\theta}_i, \overline{\theta}_i\} \text{ for } i = 1, \ldots, n_\theta \right\}. \tag{4.133}$$

In other words, $\Theta = \mathbf{Co}(\Theta_v)$. Further, the LPV system (4.132) is described by a polytopic parameter dependent uncertain system such that there exists an unknown time-invariant vector-valued function $\rho : \Theta \to \mathbb{R}_+^{2^{n_\theta}}$ satisfying

$$A(\theta) = \sum_{i=1}^{2^{n_\theta}} \rho_i(\theta) A_i, \quad \sum_{i=1}^{2^{n_\theta}} \rho_i(\theta) = 1. \tag{4.134}$$

That is, $A(\theta) \in \mathbf{Co}\{A_1, \cdots, A_{2^{n_\theta}}\}$ for all $\theta(t) \in \Theta$. The control objective is to design a state-feedback controller $u(t) = K(\theta)x(t)$ such that the closed-loop system

$$\dot{x}(t) = (A(\theta) + B_u K(\theta))x(t) \tag{4.135}$$

is stable for all $\theta \in \Theta$. $K(\theta)$ indicates a parameter dependent control gain matrix to be determined. Since θ is assumed to be unknown and nonmeasurable, (4.135) can be considered as a closed-loop system with an ideal linear parameter dependent control law $u(t) = K(\theta)x(t)$. To estimate the unknown time-invariant parameter-dependent control gain matrix $K(\theta)$, consider the following closed-loop system dynamics with the state-feedback control law where the control gain matrices are weighted by the parameter adaptation:

$$\dot{x}(t) = \left(A(\theta) + B_u K(\hat{\theta}(t))\right) x(t) \tag{4.136}$$

$$= \left(A(\theta) + B_u \sum_{i=1}^{2^{n_\theta}} \hat{\rho}_i(t) K_i\right) x(t) \tag{4.137}$$

$$= \left(\sum_{i=1}^{2^{n_\theta}} \rho_i (A_i + B_u K_i) + \sum_{i=1}^{2^{n_\theta}} \tilde{\rho}_i(t)(B_u K_i)\right) x(t), \tag{4.138}$$

where

$$\hat{\rho}(\hat{\theta}(t)) \triangleq \left[\hat{\rho}_i(\hat{\theta}(t)), \cdots, \hat{\rho}_{2^{n_\theta}}(\hat{\theta}(t))\right]^T \tag{4.139}$$

is a parameter adaptation vector to be determined and its estimation error is $\tilde{\rho}(\tilde{\theta}(t)) \triangleq \hat{\rho}(\hat{\theta}(t)) - \rho(\theta)$ and shortly $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$.

Theorem 41. *(Adaptive laws with the predetermined state feedback control gains) Suppose that the parametric uncertainty ρ (or θ) is unknown and nonmeasurable, but is a constant ρ_0 (or θ_0). If an adaptation parameter-dependent control gain matrix given in (4.137) satisfies the following matrix inequality for all $i = 1, \ldots, 2^{n_\theta}$:*

$$(A_i + B_u K_i)Q + Q(A_i + B_u K_i)^T < 0 \tag{4.140}$$

for some $Q = Q^T > 0$, with the following adaptive law

$$\dot{\hat{\rho}}_i(t) = -\gamma_i x^T(t)\left(Q^{-1}B_u K_i + K_i^T B_u^T Q^{-1}\right)x(t) \quad \textit{for } i = 1, \dots, 2^{n_\theta}, \tag{4.141}$$

where γ_i indicates the adaptation gain for each index i, then the origin of the closed-loop system is GUAS.

Proof: To analyze the stability of the closed-loop system with a parameter estimation based state-feedback control law $u(t) = K(\hat{\theta}(t))x(t)$, consider the Lyapunov function:

$$V(x(t), \tilde{\rho}(t)) = x^T(t)Q^{-1}x(t) + \frac{1}{2}\tilde{\rho}^T(t)\Gamma^{-1}\tilde{\rho}(t), \tag{4.142}$$

where $Q = Q^T > 0$ and $\Gamma = \text{diag}\{\gamma_i, \cdots, \gamma_{2^{n_\theta}}\} > 0$. Its time-derivative becomes

$$\frac{d}{dt}V(x(t), \tilde{\rho}(t)) = x^T(t)Q^{-1}\left(\sum_{i=1}^{2^{n_\theta}}\rho_i(A_i + B_u K_i) + \sum_{i=1}^{2^{n_\theta}}\tilde{\rho}_i(t)(B_u K_i)\right)x(t) \tag{4.143}$$

$$+ x^T(t)\left(\sum_{i=1}^{2^{n_\theta}}\rho_i(A_i + B_u K_i) + \sum_{i=1}^{2^{n_\theta}}\tilde{\rho}_i(t)(B_u K_i)\right)^T Q^{-1}x(t) + \tilde{\rho}^T(t)\Gamma^{-1}\dot{\tilde{\rho}}(t). \tag{4.144}$$

Therefore, if K_i and Q satisfy the inequality (4.140), and the adaptive law for $\hat{\rho}(t)$ is given by (4.141), then there exists $\Pi = \Pi^T > 0$ such that

$$\frac{d}{dt}V(x(t), \tilde{\rho}(t)) = -x^T(t)\Pi x(t) \le 0 \quad \forall x(t) \in \mathbb{R}^n. \tag{4.145}$$

This implies that, since the Lyapunov function $V(x(t), \tilde{\rho}(t))$ is positive semi-definite and non-increasing along the vector field, $x(t)$ and $\tilde{\rho}(t)$ are bounded in the following sense:

$$V(x(t), \tilde{\rho}(t)) = \left\| \begin{bmatrix} x(t) & \tilde{\rho}(t) \end{bmatrix}^T \right\|^2_{(Q,\Gamma)} \triangleq \|x(t)\|_Q^2 + \|\tilde{\rho}(t)\|_\Gamma^2 \le V((x(0), \tilde{\rho}(0))). \tag{4.146}$$

Further, since $\ddot{V}$ is bounded such that $\dot{V}$ is uniformly continuous it follows from Barbalat's lemma [115, 56] that $x \to 0$ as $t \to \infty$. Note that nothing can be concluded about the convergence of $\tilde{\rho}(t)$, except that it is bounded. □

The basic assumption in Theorem 41 is that the LTI plant is free of noise disturbances, unmodeled dynamics and unknown nonlinearities. Because these assumptions will be easily violated in real applications, it is also of interest to investigate the stability of the closed-loop system with the adaptive scheme for ρ (or θ) in the presence of disturbances and unmodeled dynamics. Moreover, in the presence of input saturation constraints the global stabilizing property of an unlimited control input does not hold anymore in general. One can only conclude the local stability for a specific domain of attraction. There are some modifications for the adaptive law which that help us to derive such a region of attraction, with which an instability pheonomenon due to the so-called parameter drift can be prevented.

Definition 14. *(Projection operator [88]) Let consider the scalar-valued smooth convex function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(\theta) = \frac{\|\theta\|^2 - \theta_0^2}{\epsilon_\theta \theta_0^2},$$

which is a special form of ellipsoidal functions, where θ_0 denotes the norm bound imposed on the unknown parameter vector θ and $\epsilon_\theta > 0$ is the user-defined convergence tolerance. Define a convex compact set with a smooth boundary given by

$$\Omega_c = \{\theta \in \mathbb{R}^n_\theta \mid f(\theta) \le c\}, \quad 0 \le c \le 1,$$

whose convexity is guaranteed by the convexity of the function f since sublevel sets of a convex function are convex for any value of c–but, the converse is not true, i.e., a function can have all its sublevel sets convex, but not be a convex function. Provided that the true value of the parameter $\theta^* \in \Omega_0$, the projection operator given by

$$Proj(\theta, y) = \begin{cases} y - f(\theta)\langle \frac{\nabla f}{\|\nabla f\|}, y\rangle \frac{\nabla f}{\|\nabla f\|} & \text{if } f(\theta) \ge 0 \text{ and } \nabla f^T y > 0 \\ y & \text{else} \end{cases} \tag{4.147}$$

prevents the potential parameter drift. Furthermore, the projection operator has the following property:

$$\langle \theta - \theta^*, Proj(\theta, y) - y\rangle \le 0. \tag{4.148}$$

Remark 12. *(Projection-type adaptive law) From the property of the projection operator defined earlier, if the parameter estimation θ starts inside the compact convex set Ω_1 then it never leaves Ω_1 when the adaptive law is given by $\dot{\theta}(t) = Proj(\theta(t), y(t))$. More specifically, the boundaries of outer set of Ω_1 is defined by the inequality $\|\theta\|^2 \le (1 + \epsilon_\theta)\theta_0^2$ so that ϵ_θ specifies the maximum tolerance one allows the parameter estimation to exceed in terms of the maximum conservative value preselected by the users.*

Example 11. *The example is taken from [1]. Suppose that*

$$A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, A_2 = \begin{bmatrix} 1.5 & -7 & 0 \\ 7 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} -0.5 & -3 & 0 \\ 3 & -0.5 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

and

$$B_1 = B_2 = B_3 = \begin{bmatrix} 0.2477 & -0.1645 \\ 0.4070 & 0.8115 \\ 0.6481 & 0.4083 \end{bmatrix},$$

such that $A(\theta) = \rho_1(\theta)A_1 + \rho_2(\theta)A_2 + \rho_3(\theta)A_3$ *with the true parameters* $\rho_1 = 0.1$, $\rho_2 = 0.7$, *and* $\rho_3 = 0.2$. *Then, the solutions of the LMI condition (4.140) are given by*

$$Q = \begin{bmatrix} 168.2410 & 30.0532 & 117.7142 \\ 30.0532 & 387.2462 & 60.7599 \\ 117.7142 & 60.7599 & 444.5127 \end{bmatrix},$$

and

$$K_1 = \begin{bmatrix} 7.5209 & -10.4670 & -4.2129 \\ 4.8160 & 2.7440 & 5.3709 \end{bmatrix}, K_2 = \begin{bmatrix} -10.1890 & 9.7062 & 4.2108 \\ 6.7711 & -7.2755 & -7.8200 \end{bmatrix}, K_3 = \begin{bmatrix} 5.9821 & 1.2671 & -4.8637 \\ 1.6403 & -0.5531 & -0.3588 \end{bmatrix}.$$

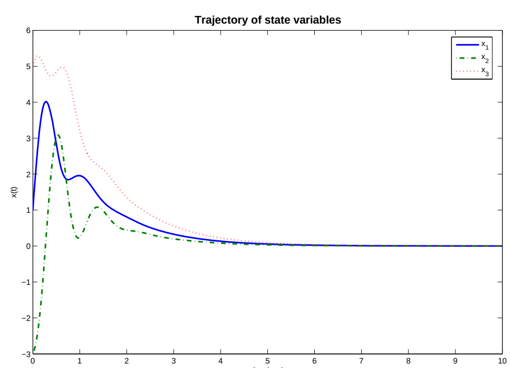


Figure 4.9: Trajectory of solution for the closed-loop system

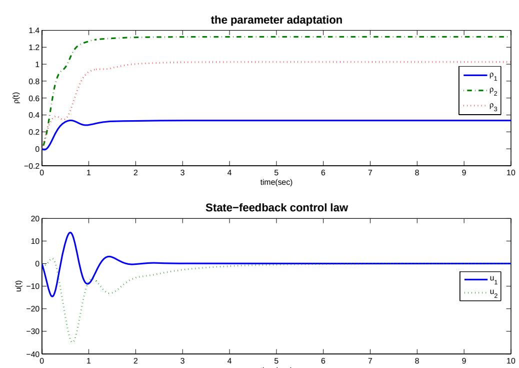


Figure 4.10: State-feedback control law $u(k) = K(\hat{\theta})x(k)$ and the adaptatation parameters

4.4.2 Continuous Lur'e Systems with Time-Invariant Parametric Uncertainty

Consider the continuous Lur'e-LPV system given by

$$\dot{x}(t) = A(\theta)x(t) + B_p\phi(t, x(t)) + B_u u(t) \tag{4.149}$$

where the system matrix A depends on a time-invariant parameter $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ and Θ is assumed to be compact. In addition, $\phi \in \Phi_{sb}^{\alpha}$ is a set of nonlinear functions defined previously. For example, if each element of the time-invariant parameter vector $\theta = (\theta_1, \cdots, \theta_{n_\theta})^T$ lies in a bounded interval, $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$, then a compact set Θ is a cube whose vertices can be defined by a finite set

$$\Theta_v \triangleq \left\{\theta_v \in \Theta | \theta_v = (\theta_{v,1}, \cdots, \theta_{v,n_\theta})^T, \text{ where } \theta_{v,i} \in \{\underline{\theta}_i, \overline{\theta}_i\} \text{ for } i = 1, \ldots, n_\theta \right\}. \tag{4.150}$$

In other words, $\Theta = \mathbf{Co}(\Theta_v)$. Further, the LPV system (4.149) is described by a polytopic parameter dependent uncertain system such that there exists an unknown time-invariant vector-valued function $\rho : \Theta \to \mathbb{R}_+^{2^{n_\theta}}$ satisfying

$$A(\theta) = \sum_{i=1}^{2^{n_\theta}} \rho_i(\theta) A_i, \quad \sum_{i=1}^{2^{n_\theta}} \rho_i(\theta) = 1. \tag{4.151}$$

That is, $A(\theta) \in \mathbf{Co}\{A_1, \cdots, A_{2^{n_\theta}}\}$ for all $\theta \in \Theta$. The control objective is to design a state-feedback controller $u(t) = K(\theta)x(t)$ such that the closed-loop system

$$\dot{x}(t) = (A(\theta) + B_u K(\theta))x(t) + B_p \phi(t, x(t)) \tag{4.152}$$

is stable for all $\theta \in \Theta$. $K(\theta)$ indicates a parameter dependent control gain matrix to be determined. Since θ is unknown and nonmeasurable, (4.152) can be considered as a closed-loop system with an ideal linear parameter dependent control law $u(t) = K(\theta)x(t)$. Now, to estimate the unknown time-invariant parametric uncertain matrix $K(\theta)$, consider the following closed-loop system dynamics with an estimator-based control law:

$$\begin{aligned}
\dot{x}(t) &= \left(A(\theta) + B_u K(\hat{\theta}(t))\right) x(t) + B_p \phi(t, x(t)) && (4.153)\\
&= \left(A(\theta) + B_u \sum_{i=1}^{2^{n_\theta}} \hat{\rho}_i(t) K_i\right) x(t) + B_p \phi(t, x(t)) && (4.154)\\
&= \left(\sum_{i=1}^{2^{n_\theta}} \rho_i (A_i + B_u K_i) + \sum_{i=1}^{2^{n_\theta}} \tilde{\rho}_i(t)(B_u K_i)\right) x(t) + B_p \phi(t, x(t)), && (4.155)
\end{aligned}$$

where

$$\hat{\rho}(\hat{\theta}(t)) \triangleq \left[\hat{\rho}_i(\hat{\theta}(t)), \cdots, \hat{\rho}_{2^{n_\theta}}(\hat{\theta}(t))\right]^T \tag{4.156}$$

is a parameter adaptation vector to be determined and its estimation error is $\tilde{\rho}(\tilde{\theta}(t)) \triangleq \hat{\rho}(\hat{\theta}(t)) - \rho(\theta)$ and shortly $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$.

Theorem 42. *Suppose that the parametric uncertainty $\rho(t)$ is unknown, but a constant ρ_0. If an adaptation* *parameter-dependent control gain matrix given in (4.154) satisfies the following matrix inequality for all* $i = 1, \ldots, 2^{n_\theta}$*:*

$$\begin{bmatrix} (A_i + B_u K_i)Q + Q(A_i + B_u K_i)^T & B_p & QC_q^T \\ B_p^T & -I & 0 \\ C_q Q & 0 & -\gamma I \end{bmatrix} < 0 \tag{4.157}$$

for some $Q = Q^T > 0$*, with the following adaptive law*

$$\dot{\tilde{\rho}}_i(t) = -\gamma_i x^T(t) \left(Q^{-1} B_u K_i + K_i^T B_u^T Q^{-1} \right) x(t) \quad \text{for } i = 1, \ldots, 2^{n_\theta}, \tag{4.158}$$

where γ_i *indicates the adaptation gain for each index* i*, then the origin of the closed-loop Lur'e system is GUAS.*

Proof: To analyze the stability of the closed-loop system with a parameter estimation based state-feedback control law $u(t) = K(\hat{\theta}(t))x(t)$, consider the Lyapunov function:

$$V(x(t), \tilde{\rho}(t)) = x^T(t) Q^{-1} x(t) + \frac{1}{2} \tilde{\rho}^T(t) \Gamma^{-1} \tilde{\rho}(t), \tag{4.159}$$

where $Q = Q^T > 0$ and $\Gamma = \text{diag}\{\gamma_i, \cdots, \gamma_{2^{n_\theta}}\} > 0$. Its time-derivative is

$$\frac{d}{dt} V(x(t), \tilde{\rho}(t)) = x^T(t) Q^{-1} \left(\sum_{i=1}^{2^{n_\theta}} \rho_i (A_i + B_u K_i) + \sum_{i=1}^{2^{n_\theta}} \tilde{\rho}_i(t)(B_u K_i) + B_p \phi(t, x(t)) \right) x(t) \tag{4.160}$$

$$+ x^T(t) \left(\sum_{i=1}^{2^{n_\theta}} \rho_i (A_i + B_u K_i) + \sum_{i=1}^{2^{n_\theta}} \tilde{\rho}_i(t)(B_u K_i) + B_p \phi(t, x(t)) \right)^T Q^{-1} x(t) + \tilde{\rho}^T(t) \Gamma^{-1} \dot{\tilde{\rho}}(t). \tag{4.161}$$

If K_i and Q satisfy the inequality (4.157), and the adaptive law for $\hat{\rho}(t)$ is given by (4.158), then there exists $\Pi = \Pi^T > 0$ such that

$$\frac{d}{dt} V(x(t), \tilde{\rho}(t)) = \begin{bmatrix} x(t) \\ \phi(t, x(t)) \end{bmatrix}^T \Pi \begin{bmatrix} x(t) \\ \phi(t, x(t)) \end{bmatrix} \leq 0 \quad \forall \begin{bmatrix} x(t) \\ \phi(t, x(t)) \end{bmatrix} \in \mathbb{R}^{2n}. \tag{4.162}$$

This implies that since the Lyapunov function $V(x(t), \tilde{\rho}(t))$ is positive semi-definite and non-increasing along the vector field, $x(t)$ and $\tilde{\rho}(t)$ are bounded in the following sense:

$$V(x(t), \tilde{\rho}(t)) = \left\| \begin{bmatrix} x(t) & \tilde{\rho}(t) \end{bmatrix}^T \right\|_{(Q,\Gamma)}^2 \triangleq \|x(t)\|_Q^2 + \|\tilde{\rho}(t)\|_\Gamma^2 \leq V((x(0), \tilde{\rho}(0))). \tag{4.163}$$

Further, from the property of the feedback-connected nonlinear mapping $\phi \in \Phi_\alpha$ one can see that $\phi(t, x(t))$ is also bounded for all time history. Therefore, since $\ddot{V}$ is bounded such that $\dot{V}$ is uniformly continuous it follows from Barbalat's lemma that $x \to 0$ as $t \to \infty$. Note that nothing can be concluded about the convergence of $\tilde{\rho}(t)$, except that it is bounded. □

Example 12. *(Maximum upper sector bounds on the feedback connected unknown nonlinear function) The control objective is to design a parameter dependent state-feedback controller that stabilizes the closed-loop system in the presence of the feedback connected unknown nonlinear function whose input-output relation is characterized by the sector bound. In this example, construct an optimal control law that maximizes the degree of robustness* α. *Consider the same system introduced in Example 11. The numerical computation shows that* $\alpha^* = 1/\sqrt{\gamma^*} = 2.7959 \times 10^4$ *where* $\gamma^* = \arg\min \gamma$ *subject to* (4.157). *Consider the system that consists of the polytopic uncertain plant given in Example 11 and the nonlinear function* $\phi(\sigma) = \tanh(\sigma)$ *where the mapping is component-wise. Then the solutions are given by*

$$Q = \begin{bmatrix} 152.8018 & -70.1085 & 90.0603 \\ -70.1085 & 400.4534 & -97.0002 \\ 90.0603 & -97.0002 & 446.7448 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -7.2529 & -0.3559 & 1.3848 \\ 5.8120 & -1.2764 & -1.3064 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -19.8037 & 6.0846 & 5.3134 \\ 17.8497 & -4.3926 & -8.7829 \end{bmatrix}, \quad K_3 = \begin{bmatrix} -6.3754 & 10.7042 & 3.6094 \\ 0.6269 & -6.6817 & -7.6672 \end{bmatrix}.$$

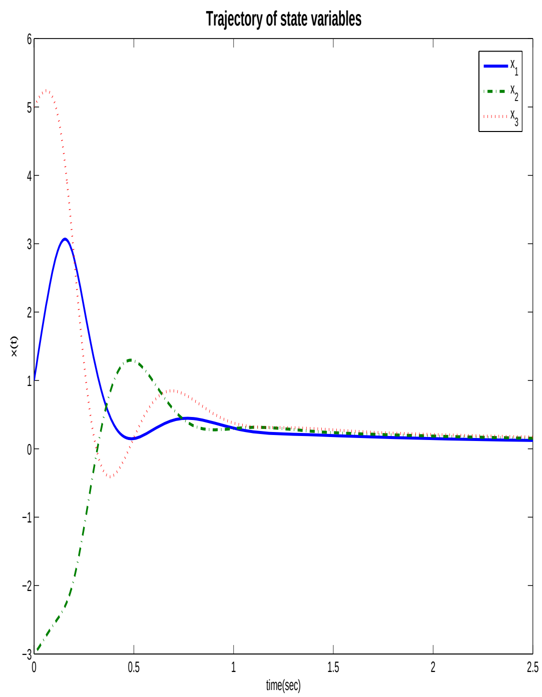


Figure 4.11: Trajectory of solution for the closed-loop system

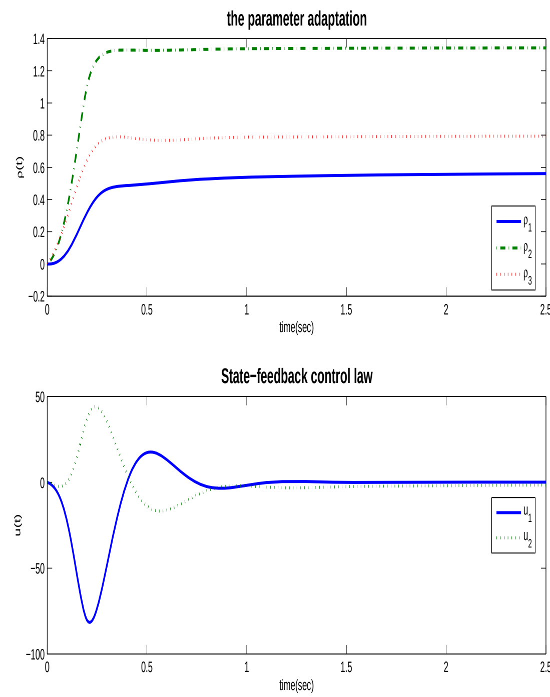


Figure 4.12: State-feedback control law $u(k) = K(\hat{\theta})x(k)$ and the adaptation parameters

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