Math 231E Midterm 3 Concept Review

Hello everyone! My name is Alberto, and on behalf of CARE I've prepared this document with all the major concepts needed for this midterm, based off of questions from previous midterms from this class. This is not an exhaustive list, and I cannot guarantee this will be all the material on the exam (expect a few questions from last midterm), but if you understand this material well you should probably be good for 90% of the questions. Without further ado, let's get started!

Sequences and Series

Before we start talking about convergence tests, we need to have an understanding of the basics. A **sequence** is an ordered set of numbers, usually defined by some function. For example,

$$a_n = \left(\frac{1}{2}\right)^n = \left\{1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8}, \frac{1}{16},\ ...\right\}$$

is a sequence of numbers. Sequences almost always start at a_0 or a_1. A **series**, on the other hand, is the sum of all the terms in a given sequence. Using the above sequence, the series would be

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$$

A sequence converges if $\lim_{n\to\infty} a_n$ exists, and a series converges if $\lim_{k\to\infty} \sum_{n=0}^{k} a_n$ exists.

In the above examples, $a_n \to 0$ and $\sum a_n \to 2$.

A sequence can converge without the series converging (Take the sequence of all 1's). However, if a series converges, then the sequence MUST converge to zero. Equivalently, if a sequence does NOT converge, or if it converges to something other than 0, then the series does NOT converge as well. This is known as the Limit Test for convergence.

A series converges **absolutely** if $\sum_{n=0}^{\infty} |a_n|$ converges. Absolute convergence implies regular convergence; that is, if $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges as well. A series converges **conditionally** if it converges, but not absolutely.

For example, the series $\sum_{n=1}^{\infty}(-1)^n/n^2$ converges absolutely, while the sum $\sum_{n=1}^{\infty}(-1)^n/n$ only converges conditionally.

The **interval of convergence** is the interval on which a series convergence. For example, take the geometric series, $\sum_{n=0}^{\infty} x^n$. We know this series converges only when $|x| < 1$. Thus, the interval of convergence is $(-1, 1)$. The **radius of convergence** is literally just the radius of the interval of convergence (Intervals on the number line can be thought of as

1-dimensional circles / spheres). To simplify, if the interval of convergence looks like (a, b), then the radius of convergence is $(b - a)/2$. In the case of the geometric series, the radius of convergence is 1. In the case that a series only converges at a single point, then the radius of convergence is 0.

Note! You will always need to check the endpoints of an interval for convergence. i.e, the sum $\sum_{n=0}^{\infty}(x/2)^n$ only converges on $(-2, 2)$, but the series $\sum_{n=0}^{\infty}(-x/2)^n$ converges on the interval (-2,2]. There's no shortcut for this; you just have to plug in the endpoint and use your favorite test to see if it converges or not.

Solids of Revolutions

There are two equations used for finding volumes: The **disk method** and the **shell method** (Their names are important!). Both calculate the volume of a solid of revolution. A **solid of revolution** is a solid formed by rotating some profile (a function) around an axis, and considering the resulting solid. This can be difficult to visualize, so as practice, consider the area under the curve $y = x^2$ from 0 to 1.

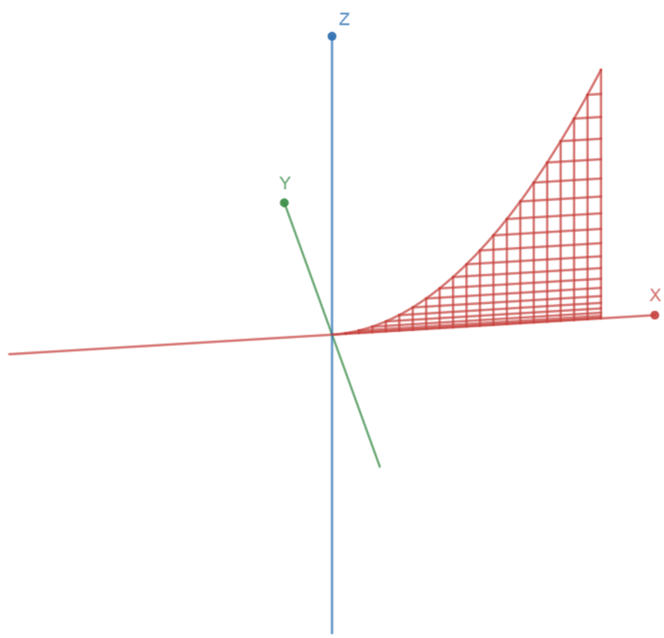

Figure 1: The curve $y = x^2$ from 0 to 1 (pretend the vertical axis is the y-axis). Axis y goes *into* the page.

The **disk method** is used when we want to rotate a function of X around the X axis. The equation for this is

$$V = \int_a^b \pi f(x)^2 \, dx$$

Graphically, it would look like a horn:

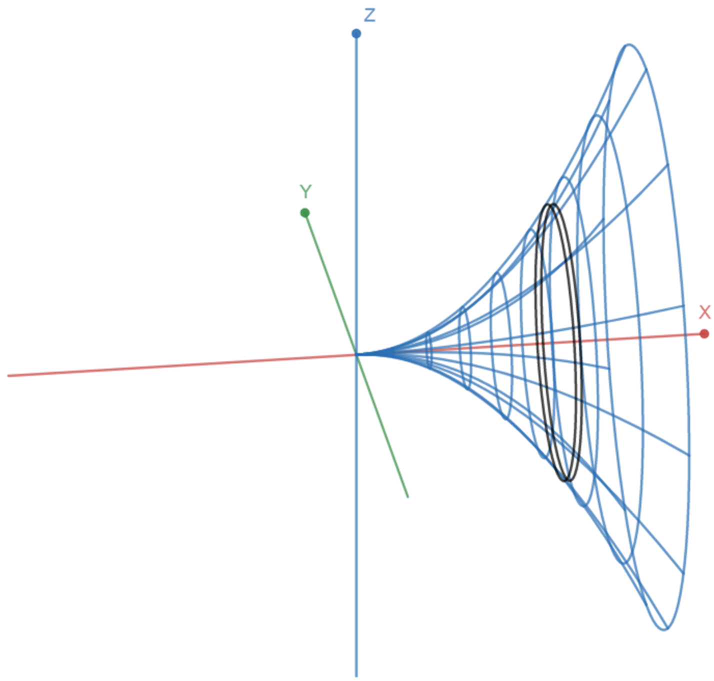

Figure 2: The curve $y = x^2$ from 0 to 1, revolved around the x-axis.

The equation comes from integrating the volumes of tiny disks. One such disk is shown in black - its volume is given by $\pi r^2 h = \pi f(x)^2 dx$. In the case that $y = x^2$, the volume would be $\int_0^1 \pi(x^2)^2 dx = \pi \int_0^1 x^4 dx = \pi/5$.

The **shell method** is used for revolving a function of X across the Y axis. The equation that gives this volume is

$$V = \int_a^b 2\pi x f(x)\, dx$$

The derivation for this equation is a bit more involved. The integral represents the volume of a hollow cylinder (the shell).

Graphically, revolving $y = x^2$ around the y-axis would look like a hollow bowl, and the volume of the resultant shape would be $V = \int_0^1 2\pi x f(x) dx = 2\pi \int_0^1 x^3 dx = \pi/2$

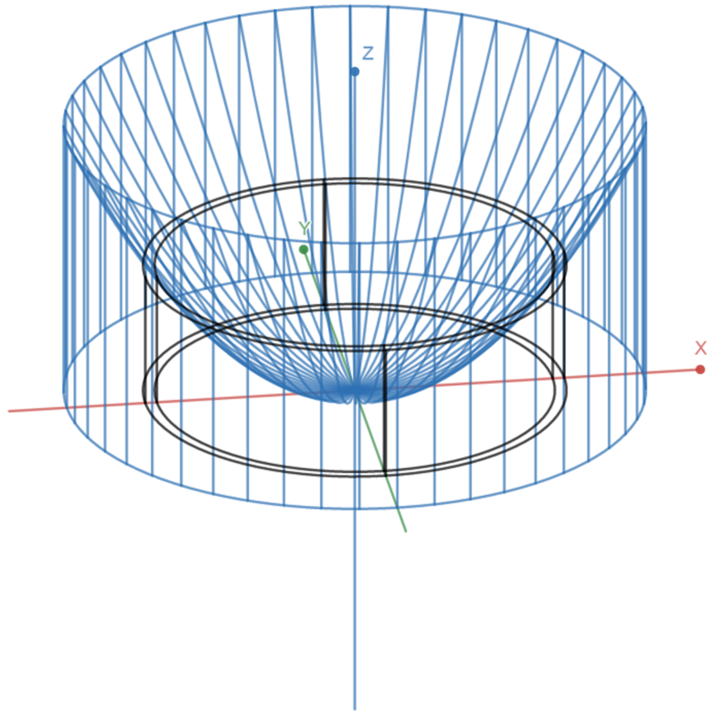

Figure 3: The curve $y = x^2$ from 0 to 1, revolved around the y-axis this time. Note the cylindrical shell!

It should be mentioned that the disk method equation can also be used to revolve a function of y around the y axis, and the shell method can also be used to revolve a function of y around the x axis.

The **arclength** of a curve is simply how long the curve would be if it were stretched out into a straight line. The equation for this is

$$A = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

In the case of $y = x^2$ from 0 to 1, we have that

$$A = \int_0^1 \sqrt{1 + \left(\frac{d}{dx}x^2\right)^2} = \int_0^1 \sqrt{1+4x^2}\, dx \approx 1.4789$$

The last equation in this section is the **surface area** formula. It represents the surface area of a function of x rotated about the x-axis (like the disk method). It is given by

$$SA = \int_0^1 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Note the similarity between this equation and the arclength equation. Due to the difficulty of evaluating this integral, you will rarely be asked to solve it (though you may be asked to write the formula down!). In the case that $y = x^2$ is the profile in question, the surface area from 0 to 1 is given by

$$SA = \int_0^1 2\pi x^2 \sqrt{1 + \left(\frac{d}{dx}x^2\right)^2}\, dx = 2\pi \int_0^1 x^2\sqrt{1+4x^2}\, dx \approx 3.8097$$

Integrals, Series, and P-Tests

Integral Test

Suppose $f(x)$ is a positive, decreasing function, and let a_n be a positive, decreasing sequence, such that $f(n) = a_n$. Then the series $\sum_{n=0}^{\infty} a_n$ converges if and only if the integral $\int_0^\infty f(x)\, dx$ converges. That is, either they both converge or they both diverge.

For example, does the series $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ converge? Well,

$$\int_1^\infty \frac{1}{x \ln x}\, dx = \int_0^\infty \frac{1}{u}\, du = \infty.$$

So the sum $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges, since the integral diverges.

P-tests

Consider the series $\sum_{n=1}^{\infty} 1/n^p$ and the integral $\int_1^\infty 1/x^p\, dx$. These two converge if and only if $p > 1$. Both diverge if $p \leq 1$.

An additional formula exists for integrals: Given an integral in the form $\int_0^1 1/x^p\, dx$, the opposite condition holds: The integral converges if and only if $p < 1$. It diverges if $p \geq 1$. This is one of the most important tests. P-test questions may seem simple, but they appear very often!

They also pair well with the limit comparison test. For example:

$$\int_1^\infty \frac{\sqrt{x^3+3x^2}}{\sqrt[3]{x^5+6x^4}}\, dx \sim \int_1^\infty \frac{\sqrt{x^3}}{\sqrt[3]{x^5}}\, dx = \int_1^\infty \frac{1}{x^{1/6}}\, dx, \text{ which diverges by the p test since } 1/6 < 1.$$

Estimating Series

An effective formula for placing bounds on the value of a series is the following inequality:

$$\int_1^\infty f(x)\, dx \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_1^\infty f(x)\, dx$$

For example, let $a_n = 1/n^2$. Then

$$\int_1^\infty \frac{1}{x^2}\, dx \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\, dx \qquad \text{so} \qquad 1 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2.$$

In fact, $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$! And indeed, $\pi^2/6 \approx 1.4646$.

Other useful facts

Euler's Number

One of the most important limits in all of mathematics is the limit definition for e:

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

This pops up more frequently than you may think. For example, when doing the ratio test on $n^n/n!$, the limit of the ratio of adjacent terms is precisely e. A more precise limit is

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^{bn} = e^{ab}$$

Average Value of a Function

The average value of a function f(x) on an interval [a,b] is given by

$$A = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Since $\int_a^b f(x)\,dx$ is the area under the curve, and (b-a) is the length of the base, the average height is given by area/base.

Improper Integrals

There are two types of improper integrals. **Type I** improper integrals are of the form

$$\int_0^{\infty} f(x)\,dx \qquad \text{or} \qquad \int_{-\infty}^{0} f(x)\,dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x)\,dx$$

These can be evaluated directly! If F(x) is the antiderivative of f(x), then

$$\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx = \lim_{n\to\infty} F(n) - F(1)$$

This may converge to a finite value, or it may diverge. The second type of improper integral is the **Type II** improper integral. These are all integrals that have an undefined point in their domain. For example, the integral $\int_2^5 \frac{1}{(x-2))}\,dx$ is Type II improper because the integrand is undefined at $x = 2$. These can also be evaluated directly: If $f(x)$ is discontinuous at $x = b$, and nowhere else on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to b} \int_a^n f(x)\,dx = \lim_{n\to b} F(n) - F(a).$$

This may also converge to a finite value, or it may diverge. You may need to split an integral into sections to evaluate these - The discontinuities MUST be on the endpoints.

Convergence Tests

As a general tip, the ratio test is the most applicable of the tests. Remember that $(n+1)! = (n+1)\cdot n!$ and $a^{n+1} = a \cdot a^n$. Also remember that $\left(\frac{n+1}{n}\right)^n \to e$. In the case of the ratio test and the root test, the test is inconclusive if the limit is 1. Remember that these two tests, if successful, tell you that the series is absolutely convergent!

TEST	SERIES	CONVERGES IF...	DIVERGES IF...	COMMENTS
nth Term Test for Divergence	$\sum_{n=1}^{\infty} a_n$	n/a	$\lim_{n\to\infty} \neq 0$	should be first test used. Inconclusive if limit = 0.
Geometric Series Test	$\sum_{n=1}^{\infty} a_n r^{n-1}$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	use if there is a "common ratio" $S_n = \frac{a}{1-r}$
P-Series Test	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	harmonic series when p=1. Useful for comparison tests.
Integral Test	$\sum_{n=1}^{\infty} a_n$ $a_n = f(x)$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	$f(x)$ must be continuous, positive, and decreasing
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$, $\sum_{n=1}^{\infty} b_n$ diverges	to show convergence, find a larger series. to show divergence, find a smaller series.
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ diverges	apply l'hospital's rule if necessary; inconclusive if limit equals 0 or ∞
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$a_{n+1} \leq a_n$, $\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	must prove that the limit equals 0 and that terms are decreasing
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	test fails if : $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	test fails if: $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$