Group: ______________ ______________ ______________ Name: ______________

Math 231E. Worksheet 2B. September 6, 2018
A Brief History of π

People have been using series to compute π since at least the 17^{th} century. We will use some things that we know about series to explore these ideas in this worksheet.

Problem 1. One popular starting point for computations of π is the Taylor series for the arctangent

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

a) Use the formula above together with $\arctan(1) = \frac{\pi}{4}$ to find the series $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots)$. This formular is known as "Gregory's series" and dates from the late 17^{th} century. Use the first four terms to find an approximation to π. (It won't be very good!)

b) The series above converges very slowly. In fact we will see later in the course that this series is only conditionally convergent. We expect better convergence if x is smaller. Use the fact that $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$ to find the series $\pi = \frac{6}{\sqrt{3}}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \ldots\right)$. Use the first four terms to approximate π. How does your answer compare to the previous estimate?

c) A more rapidly convergent series comes from the fact that $\pi = 16\arctan(\frac{1}{5}) - 4\arctan(\frac{1}{239})$. Approximate π using four terms of the series for $\arctan(\frac{1}{5})$. How many terms of the series for $\arctan(\frac{1}{239})$ should you then keep? Explain your reasoning.

Problem 2. Many of you may have seen the number $\frac{22}{7}$ as a rational approximation to π. This was originally due to Archimedes in the Third century BCE, who approximated the circle by a 96-gon to show that $\frac{223}{71} < \pi < \frac{22}{7}$. This was later improved by the Chinese mathematician Liu Hui (260 CE) who used an 192-gon.

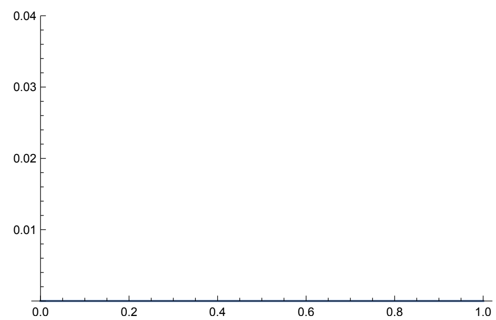
In fact one has the following identity, which can be proved by a tedious integration

$$\pi = 22/7 - \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$$

Throughout this exercise you may find the following identity useful: if n and m are integers then

$$\int_0^1 x^n(1-x)^m dx = \frac{n!m!}{(n+m+1)!}$$

a) Use your calculator to draw a graph of the function $\frac{x^4(1-x)^4}{1+x^2}$ on the interval $[0, 1]$. You should find that the function $\frac{x^4(1-x)^4}{1+x^2}$ achieves its maximum value at approximately $x \approx \frac{1}{2}$.[1]



b) Find the first two terms of the Taylor series for $\frac{1}{1+x^2}$ about $a = \frac{1}{2}$.

c) Take the integral $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ and replace the $\frac{1}{1+x^2}$ by the *first* term in its Taylor series about $a = \frac{1}{2}$ to approximate the integral $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$.

d) What does this give you as an approximation to π. How does it compare with $\frac{22}{7}$?

[1]Actually it is at $\frac{1}{9}\left(1 - \frac{7\,5^{2/3}}{\sqrt[3]{38+9\sqrt{39}}} + \sqrt[3]{5\left(38+9\sqrt{39}\right)}\right) \approx .475$