Final Review

Exercise 0.1. Determine the convergence of the following series

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$ converges for every $\theta \in \mathbb{R}$
2. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ converges.
3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log(n)}{n}$ converges.
4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$ diverges.
5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 + 1}}$ diverges.
6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$ converges.
7. $\displaystyle\sum_{n=1}^{\infty} \frac{\log(n)}{n}$ diverges.
8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log(n)}$ diverges.
9. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log(n))^k}$ diverges.
10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log(n))^n}$ converges.
11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log(n^n)}$ diverges.
12. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^3 + 1}$ diverges.
13. $\displaystyle\sum_{n=2}^{\infty} \sin(1/n)$ diverges.
14. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p(\log(n))^q}$ converges or diverges.

Proof.
1. Comparison using the fact that $|\sin(x)| \leq 1$
2. Alternating series test. You should be able to recognize what this series is (since you are supposed to know everything about arctan) and therefore what it converges to.
3. Alternating series test, together with L'Hôpital, $\lim_{x \xrightarrow{\infty}} \frac{\log(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$.
4. Comparison (with what series?)
5. Limit comparison (hint: compare with the same series as the one above)
6. $n! > e^n > n^4$ for n large enough.
7. Integral test (and u substitution)
8. Comparison
9. Integral (can you solve the integral $\int(\log(x))^n dx$? You should! Hint: start with $\int \log(x)$ by IBP and try a couple more till you see a pattern.)
10. Comparison (with what? Hint: think geometric)
11. Integral test with $\log(n^n) = n\log(n)$.
12. Limit comparison
13. Correction: $\lim_{x\to 0} |\sin(x) - x| = 0$.
14. See worksheet solutions posted on site.

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