Extra Problems on Continuity

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *lipschitz* on open interval $(a, b) \subset \mathbb{R}$ if there is a positive constant $C > 0$ such that $|f(x) - f(y)| < C|x - y|$ whenever $a < x < y < b$.

Further recall the definition of continuity:

Definition 2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous at p* if for every $\varepsilon > 0$ there is an associated $\delta_\varepsilon > 0$ (depending on ε) such that $|f(x) - f(p)| < \varepsilon$ whenever $|x - p| < \delta_\varepsilon$. A function is *continuous* (with no qualification) if it is continuous at p for every p in the domain.

Here is an illustration of how these proofs go.

Exercise 0.1 (Example)**.** For any $a \in \mathbb{R}$, function ax is continuous on $\mathbb{R}$.

Proof. Let $a \in \mathbb{R}$ be given, and suppose (one case) that $a > 0$ (what do you do when $a = 0$ and when $a < 0$?). Next, fix $p \in \mathbb{R}$. It doesn't matter what p is, but we need it to be a fixed value to continue with the proof.[1] Next, suppose $\varepsilon > 0$ is given (not chosen). We need to find a suitable δ so that *if* $|x - p| < \delta$ then we can be assured that $|f(x) - f(p)| < \varepsilon$. Well, let's start with what we want to achieve: we want that $|ax - ap| < \varepsilon$. This holds if and only if (after factoring out a), $|a(x - p)| < \varepsilon$ which holds if and only if $|x - p| < \varepsilon/a$. In other words, *if* $|x - p| < \varepsilon/a$, *then* $|ax - ap|$ will surely be less than ε. So let $\delta = \varepsilon/a$.

If we want to be really complete, we can reverse the steps to verify that this δ works (but it isn't necessary):

$$
\begin{aligned}
& |x - p| && < \delta \\
\Leftrightarrow\ & |x - p| && < \varepsilon/a \\
\Leftrightarrow\ & a|x - p| && < \varepsilon \\
\Leftrightarrow\ & |a(x - p)| && < \varepsilon \\
\Leftrightarrow\ & |ax - ap| && < \varepsilon \\
\Leftrightarrow\ & |f(x) - f(p) && < \varepsilon
\end{aligned}
$$

In particular since $A \Leftrightarrow B$ implies that $A \Rightarrow B$, we have that $|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$. □

Notice that we fixed p in the beginning, so technically we only showed continuity at p. But then again, I didn't really tell you where p was, so p could have been anywhere. This establishes that the argument works for *any* p and hence that ax is continuous everywhere. (This is actually sortof what we do with ε, since technically ε can be anything, but we need it to be fixed in order to find δ.)

Ok, I'm gonna give another example which is a little bit more involved.

Exercise 0.2 (Example)**.** Show that $\log(x)$ is continuous on $(0, \infty)$.

[1] This is a technical point about the definition of continuity, which is that continuity is continuity *at* a point. So even though ε can be whatever, it is whatever after you have selected which point you want to show continuity around.

Proof. Let $p \in (0, \infty)$, and suppose that $\varepsilon > 0$ is given. Start with what we need: $|f(x) - f(p)| < \varepsilon$, and from this let's try to derive what would be required of $|x - p|$. Well, $|\log(x) - \log(p)| < \varepsilon$ if and only if $-\varepsilon < \log(x) - \log(p) < \varepsilon$ if and only if $\log(p) - \varepsilon < \log(x) < \log(p) + \varepsilon$ if and only if $e^{\log(p)-\varepsilon} < x < e^{\log(p)+\varepsilon}$ (think about why the inequality direction is preserved when we apply the exponential).

Therefore *if* $x \in (e^{\log(p)-\varepsilon}, e^{\log(p)+\varepsilon})$, *then* we are guaranteed that $|f(x) - f(p)| < \varepsilon$. Great!, but we don't yet have $|x - p| < \delta$. That's ok: we know that $e^{\log(p)-\varepsilon} < p < e^{\log(p)+\varepsilon}$ (how?: recall that $p = e^{\log(p)}$!), or in other words that p is "somewhere" in the middle. Chances are it's closer to one side than the other. So let

$$\delta = \min\left\{|p - e^{\log(p)-\varepsilon}|, |p - e^{\log(p)+\varepsilon}|\right\}.$$

Beautiful, so given this δ, if $|x - p| < \delta$, then for sure $x \in (e^{\log(p)-\varepsilon}, e^{\log(p)+\varepsilon})$, and from our calculation above that implies that $|f(x) - f(p)| < \varepsilon$. (Again, this can be verified by repeating an analogous computation as in the preceding problem, but usually this is not necessary; just become comfortable with why we can stop here.) □

A general comment on the following solutions: these are not easy problems, and so even reading and understanding the solutions won't necessarily be a walk in the park. Don't be discouraged if you don't understand the argument on a first couple passes. Just try to concentrate on what is required for continuity to hold, and why each step is taken to achieve that.

I strongly recommend trying to follow the argument for (0.7), continuity of polynomials, because it uses things in calculus you should know, and gives one concrete way of understanding the derivative as measure of change which is not *only* local. You need to understand the definition of Lipschitz, but you know everything else. There is a logical subtlety at the end, which technically is important, but if you at least understand the calculus that is good.

Exercise 0.3. Suppose that f is lipschitz on $(a, b) \subset \mathbb{R}$. Then prove that f is continuous on (a, b).

Key idea: linear relation between $\delta - \varepsilon$ given from linear relation between $|x - p|$ and $|f(x) - f(p)|$

Proof. Let $p \in (a, b)$ and C as in the definition of Lipschitz, so that $|f(x) - f(p)| < C|x - p|$ for every $x \in (a, b)$. Suppose that $\varepsilon > 0$ is given, and we want to find $\delta_\varepsilon > 0$ so that $|f(x) - f(p)| < \varepsilon$ as long as $|x - p| < \delta_\varepsilon$. We *define* $\delta_\varepsilon = \varepsilon/C$ and verify that it works (following the argument backwards will give a hint as to how we came up with this δ_ε!).

Suppose that $|x - p| < \delta_\varepsilon$. Then for sure, $C|x - p| < C\delta_\varepsilon$. But the left hand side bounds by above $|f(x) - f(p)|$ since by assumption f is Lipschitz on (a, b). So $|f(x) - f(p)| < C\delta_\varepsilon$ but $\delta_\varepsilon = \varepsilon/C$ implies that

$$|f(x) - f(p)| < C\varepsilon/C = \varepsilon$$

which is exactly what we wanted. □

Exercise 0.4. Suppose that f is differentiable at p. Then prove it is also continuous at p.

Hint. There will be two sets of (δ, ε) floating around, the first from the definition of derivative as

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p} = f'(p)$$

which (recall, by definition of limit) *means* that for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that $|x - p| < \delta$ implies that $\left|\frac{f(x)-f(p)}{x-p} - f'(p)\right| < \varepsilon$. But you need to show that given any $\varepsilon' > 0$, there is (for *this* epsilon) an associated δ' such that $|x - p| < \delta'$ (not δ) implies that $|f(x) - f(p)| < \varepsilon'$. Also, remember the triangle inequality which says that $|x + y| \leq |x| + |y|$. And one more thing, add zero, like this: $f(x) - f(p) = f(x) - f(p) - a + a$. Hint again (because this is a challenging problem): let $a = f'(p)(x - p)$. $\square$

Warning, this one is pretty tough (in hindsight I apologize for giving it), but it *is* digestible and if you understand this, then you've really got a good handle on limits!

Proof. Let $\varepsilon' > 0$ be given; we are tasked with finding a $\delta' > 0$ such that $|x - p| < \delta'$ implies that $|f(x) - f(p)| < \varepsilon'$. As in the hint, differentiability implies that for any $\varepsilon > 0$ (this time not ε'!), there is a $\delta > 0$ so that $\left|\frac{f(x)-f(p)}{x-p} - f'(p)\right| < \varepsilon$ as long as $|x - p| < \delta$. Now, since we have to find δ' not δ (that is assumed to exist for any given ε), we are actually free to choose ε (but not ε'), and could happily do so if needed. We will not need.

As in the hint $f(x) - f(p) = f(x) - f(p) + 0 = f(x) - f(p) - f'(p)(x - p) + f'(p)(x - p)$ so applying absolute values and the triangle inequality, we conclude that

$$|f(x) - f(p) - f'(p)(x - p) + f'(p)(x - p)| \leq |f(x) - f(p) - f'(p)(x - p)| + |f'(p)(x - p)|.$$

The first term on the right is bounded above by $\varepsilon|x - p|$ *when* $|x - p| < \delta$ (where *this* delta depends on the epsilon from the definition of limit in derivative, not continuity). Hence, *as long as* $|x - p| < \delta$,

$$|f(x) - f(p) - f'(p)(x - p)| + |f'(p)(x - p)| < \varepsilon|x - p| + |f'(p)||x - p| = (\varepsilon + |f'(p)|)|x - p|$$

but since $|x - p|$ is assumed to be less than δ, we actually have

$$|f(x) - f(p) - f'(p)(x - p)| + |f'(p)(x - p)| < (\varepsilon + |f'(p)|)\delta$$

but again we are only guaranteed this as long as $|x - p| < \delta$.

Ok, we are almost home free, because if we chain together all the inequalities that we've cooked up, we have that $|f(x) - f(p)| < (\varepsilon + |f'(p)|)\delta$ when $|x - p| < \delta$. But, we are required to have $|f(x) - f(p)| < \varepsilon'$, not all this junk on the right. That's ok, we can actually make δ smaller if we want to. For if $|x - p| < \delta$ implies that $|f(x) - f(p)| < \varepsilon$, then for sure if $\delta' < \delta$, then $|x - p| < \delta'$ will definitely imply that $|f(x) - f(p)| < \varepsilon$. (Draw a picture if this isn't clear: the interval of radius δ' centered at p is smaller than the interval of radius δ centered at p.)

Therefore, given a $\delta - \varepsilon$ pair *from the definition of the derivative existing at* p, we *want*

$$(\varepsilon + |f'(p)|)\delta < \varepsilon'.$$

Either that inequality is satisfied already, in which case we have our desired implication (trace through the steps to see why we're done here!) and we can simply let $\delta' = \delta$, or it's not in which case we need to make $(\varepsilon + |f'(p)|)\delta$ smaller. We do that by shrinking δ: since $\varepsilon > 0$ was fixed, we can satisfy our desired inequality by taking $\delta' < \frac{\varepsilon'}{\varepsilon + |f'(p)|}$. This $\delta' < \delta$ and for such $|x - p| < \delta'$, we will have that $|f(x) - f(p)| < \varepsilon'$, as desired. □

Exercise 0.5. Let $f : \mathbb{Q} \to \mathbb{R}$ be defined by $f(x) = x$ (defined only for rational numbers). Prove that f is continuous.

Note that this function is not defined for irrational reals. That is not a problem in terms of the definition of continuity (can you explain in words why not?)!

There's no point to this problem other than to say that continuity does not require being able to draw the function "continuously". If this doesn't make sense at all, totally ignore it, because probably function definitions as involving domain, codomain, etc. may not be something you've seen in depth.

Proof. Once getting over the intimidation that the function is not defined for all $\mathbb{R}$, the $\delta - \varepsilon$ part is almost free. Since $f(x)$ doesn't make sense when x is irrational, we only need to worry about the implication $|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$ for x rational. But by definition $f(x) = x$ and $f(p) = p$ so $|f(x) - f(p)| = |x - p|$ which is ensured to be less than ε as long as we force $|x - p| < \varepsilon$, i.e. we can simply take $\delta = \varepsilon$. □

Exercise 0.6. Let $n \in \mathbb{N}$ be a natural number. Then show the n-th power function x^n is continuous on $\mathbb{R}$.

Hint. Use the a similar strategy to the problems in Worksheet 2B. Another strategy would be to show that this function is Lipschitz on any bounded interval. □

Ok, this problem is in form and argument identical to what you should have already done on the worksheet, except that we are replacing the concrete number 2 with the abstract one n.

Proof. We are going to basically copy the argument from example 0.2. Fix a real number $p \in \mathbb{R}$ and we will show that x^n is continuous at p. Let $\varepsilon > 0$ be given, and suppose without loss of generality that $\varepsilon < p^n$ (why 'without loss of generality?: because $\min\{\varepsilon, p^n\} < \varepsilon$). We want to find $\delta > 0$ so that $|x^n - p^n| < \varepsilon$ whenever $|x - p| < \delta$.

Well

$$\begin{aligned} & |x^n - p^n| < \varepsilon \\ \Leftrightarrow \quad & -\varepsilon < x^n - p^n < \varepsilon \\ \Leftrightarrow \quad & p^n - \varepsilon < x^n < p^n + \varepsilon \\ \Leftrightarrow \quad & \sqrt[n]{p^n - \varepsilon} < x < \sqrt[n]{p^n + \varepsilon} \end{aligned}$$

and so if $x \in (\sqrt[n]{p^n - \varepsilon}, \sqrt[n]{p^n + \varepsilon})$ then we are guaranteed that $|f(x) - f(p)| < \varepsilon$. Therefore, let $\delta := \min\{|p - \sqrt[n]{p^n - \varepsilon}|, |p - \sqrt[n]{p^n + \varepsilon}|\}$, so that $(p - \delta, p + \delta) \subset (\sqrt[n]{p^n - \varepsilon}, \sqrt[n]{p^n + \varepsilon})$ and therefore satisfies the required $\delta - \varepsilon$ implication $|x - p| < \delta$ implies that $|f(x) - f(p)| < \varepsilon$. (Recall that

$|x-p| < \delta$ if and only if $-\delta < x-p < \delta$ if and only if $p-\delta < x < p+\delta$, i.e. if and only if $x \in (p-\delta, p+\delta)$. □

Exercise 0.7. Let $p(x) = \sum_{j=0}^{n} a_j x^j$ be a polynomial in $\mathbb{R}$ (with $a_j \in \mathbb{R}$). So for example $1 + x^2 - 3x^5$. Prove that p is continuous on $\mathbb{R}$.

Hint. This is a challenging problem. There are two strategies. One is to show lipschitzness on any bounded interval. The other is to show that addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $(a, b) \mapsto a + b$ is continuous. This is also difficult, so another hint is that for a function defined on two variables, we no longer have $|x - p|$ but $\|\mathbf{x} - \mathbf{p}\| = \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2}$. Then there is another inequality which you can use, which is that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. □

Key ideas: fundamental theorem of calculus, extreme value theorem, mathematical induction.

Proof. I am going to pretend like I didn't give the second hint, because I think the generalization to a function on more than one variables, in the context of continuity, is at first conceptually difficult. Let's use Lipschitzness instead because it highlights some nice calculus facts which we should already have a feel for.

We start with a fact, prove it, and then show how it is relevant in our case. **Fact**: Suppose that f is differentiable on an open interval (a, b) and that there is some M such that $|f'(x)| < M$ for every $x \in (a, b)$.[2] Then f is Lipschitz on (a,b) (with a constant of M). To see this, we use the fundamental theorem of calculus; let $x_1, x_2 \in (a, b)$. Then:

$$\begin{aligned} |f'(x_2) - f'(x_1)| &= \left| \int_{x_1}^{x_2} f'(x) \right| \\ &\leq \left| \int_{x_1}^{x_2} M dx \right| \\ &= M|x_2 - x_1| \end{aligned}$$

where the first equality follows by the fundamental theorem and the next line follows by the assumption on the boundedness of the derivative. Since x_1, x_2 were arbitrary, this shows the Lipschitz bound on (a,b), as claimed.

Ok, next, we want to show that polynomials are lipschitz, which will imply by the first problem (0.3) that they are continuous. Let $f(x) = \sum_{j=0}^{n} a_j x^j$ with $a_j \in \mathbb{R}$ be some random polynomial.

[2] One of the confusing things for student when first learning about continuity is the way quantifiers ('for all', 'there exists') work, especially when they are used together. There is a crucial difference between: 'there is..., such that for all...' and 'for all..., there is..., such that' and it is important to have an intuitive feeling for it. Consider the following two ordinary English examples: there is a pizza type at Giordanos that everyone likes (i.e. everyone likes the *same* pizza type) and everyone has *a* favorite pizza type at Giordanos. In the one case, the one pizza type is *independent* of everyone else (since everyone likes the same thing), and in the other case, it depends directly on the person. This is the difference between having 'there is...' appear first or second, respectively.

Let (a, b) be an open bounded interval; we will show that f is Lipschitz on (a, b). Using the preceding paragraph, it suffices to show that f' is bounded on (a, b). Well, differentiating, we have that

$$f'(x) = \sum_{j=1}^{n} j a_j x^{j-1}$$

which is also a polynomial, and in fact is also defined all all of $\mathbb{R}$, and hence on the *closed* interval $[a, b]$.

This next step may seem confusing, especially if you have never seen induction. We are going to make the assumption that every polynomial of degree strictly less than n is continuous and use *that* to prove that degree n polynomials are continuous. Why does this work? Think of dominoes lined up: when you knock the first one down, every one thereafter follows. With induction, it's the same: we know that constant functions (degree 0 polynomials) are continuous. Therefore using this strategy, we obtain that degree 1 polynomials are continuous. Then continuing, to show degree 2 polynomials are continuous, using that degree 1 polynomials are implies that degree $1 + 1$ polynomials are, and so on.

How do we use this principle?: well, when you differentiate a polynomial, you knock its degree down by one. So we get a degree $n - 1$ polynomial $f'(x)$, and we're *assuming* that it is continuous. What do we know about continuous functions on a closed bounded interval?: that they take their extreme value somewhere on it, i.e. there is some $q \in [a, b]$ for which $|f'(q)| \geq |f'(x)|$ for every $x \in [a, b]$, and in particular, also for every $x \in (a, b)$. This shows that $f'(x)$ is bounded above by $M = |f'(q)|$ on (a, b), and therefore that f is Lipschitz on (a, b), and therefore (what we really wanted) continuous, on all of (a, b). Since (a, b) was arbitrary, we can take any interval, and hence for any point $p \in \mathbb{R}$ any interval containing *it*; this shows that polynomials are continuous everywhere on $\mathbb{R}$. □

Exercise 0.8. Let $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ be defined by $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{else.} \end{cases}$ So for example $\chi_{\mathbb{Q}}(1/2) = 1$ and $\chi_{\mathbb{Q}}(\pi) = 0$. Prove that $\chi_{\mathbb{Q}}$ is nowhere continuous.

Hint. You need a fact in order to solve this problem which is that both the rationals *and* irrationals are "dense" in $\mathbb{R}$. What does that mean? Given *any* $a < b$ in $\mathbb{R}$, there are both a rational point $a < r < b$ and irrational point $a < t < b$ (and therefore, infinitely many of each). □

This is more a logical example to get some feel for what 'not continuous' means in terms of $\delta - \varepsilon$. Notice, for example, that we only need to use one *particular* epsilon, whereas in the 'yes continuous' case we need to consider all of them. There's nothing sacrosanct about $\varepsilon = 1/2$ (we could have taken any $\varepsilon < 1$), but it is imporant that we used *an* epsilon less than 1 (can you see why?).

Proof. Fix $p \in \mathbb{R}$ and suppose that it is rational (so that $f(p) = 1$). The argument for p irrational is identical; but we need p to be fixed so it has to be one or the other. Let $\varepsilon = 1/2$. We will show that no matter what $\delta > 0$ we choose, we cannot enforce the following implication $|x - p| < \delta \Rightarrow |f(x) - f(p)| < 1/2$. The reason is that for *any* $\delta > 0$, the interval $(p - \delta, p + \delta)$ contains infinitely many irrational numbers (according to the hint). In other words, there are some values

$x \in (p-\delta, p+\delta)$ for which $f(x) = 0$ and hence $|f(x) - f(p)| = |0 - 1| = 1 > 1/2$. Since $\delta > 0$ was arbitrary, this proves that no such $\delta > 0$ can be found for $\varepsilon = 1/2$. $\square$