At the end of this worksheet, think if you can generalize results for the following case. Let $\ln^n(x)^q := (\ln \circ \cdots \circ \ln(x))^q$ (alternatively, defined recursively by $(\ln^1(x))^q = (\ln(x))^q$ and $(\ln^n(x))^q := (\ln \circ \ln^{n-1}(x))^q$. Then can we determine for what values $p, q_1, \ldots, q_n$ the following integral converges:

$$\int_{e^{\cdot^{\cdot^{e}}}}^{\infty} \frac{dx}{x^p \prod_{j=1}^{n} (\ln^j(x))^{q_j}}?$$

Exercise 0.1. 1. When $p > 1$, $\displaystyle\int_e^\infty \frac{dx}{x^p(\ln(x))^q} < \infty$.

2. When $0 < p < 1$, the integral diverges.

3. When $p = 1$, it converges for only $q > 1$.

Proof. 1. Notice that $\ln(x) > 1$ for all $x > e$, and and so $(\ln(x))^q > 1$ for $x > e$, which implies that $x^p(\ln(x))^1 > x^p$ when $x > e$; hence the reciprocals are swapped, $\frac{1}{x^p} < \frac{1}{x^p(\ln(x))^q}$ for $x > e$, and we conclude that

$$\int_e^\infty \frac{dx}{x^p(\ln(x))^q} < \int_e^\infty \frac{dx}{x^p} < \infty.$$

2. For this part, we want to use the fact that $\displaystyle\int \frac{1}{x^p} = \infty$ when $p < 1$. We start by choosing some $p' \in (p, 1)$, and rewrite the integrand

$$\frac{1}{x^p(\ln(x))^q} = \frac{x^{p'}}{x^{p'}x^p(\ln(x))^q} = \frac{x^{p'-p}}{x^{p'}(\ln(x))^q} = \frac{1}{x^{p'}} \cdot \frac{x^{p'-p}}{(\ln(x))^q}. \tag{1}$$

What motivates this manipulation of the expression? Well, we are given that $\lim_{x\to\infty} \ln(x)/x^b = 0$ for *any* $b > 0$, and if we can make *that* pop up somewhere, maybe we will be able to use it (hint: we *will* make it show up, and we *will* use it).[1]

In fact, $\frac{x^{p'-p}}{(\ln(x))^q} = \left(\frac{x^{\frac{(p'-p)}{q}}}{\ln(x)}\right)^q$ and $p' > p$ implies that $\frac{p'-p}{q} > 0$, so by R3, the stuff inside $(\cdot)^q$ goes to infinity as $x \to \infty$. But that implies that the whole thing goes to infinity. Sweet, so now we need to dust of our trusty $M - N$ definition of infinite limits: for every $M > 0$ there is an N_M (N is a 'function' of M) such that

$$\left(\frac{x^{\frac{(p'-p)}{q}}}{\ln(x)}\right)^q > M \tag{2}$$

whenever $x > N_M$. So for kicks and giggles, if we let $M = 1$ then the right hand expression of Equation (1) bounds $\frac{1}{x^p}$ by above *when* $x > N_1$. That implies:

$$\int_{N_1}^\infty \frac{dx}{x^p(\ln(x))^q} > \int_{N_1}^\infty \frac{dx}{x^p} = \infty.$$

[1]Bt dubs, we are also given that $\displaystyle\int_e^\infty \frac{dx}{x^p \ln(x)}$ diverges when $p \leq 1$, but you don't know that this is true when you raise $\ln(x)$ to the power $q > 1$.

Who cares what happens to $\int_e^{N_1} \frac{dx}{x^p(\ln(x))^q}$; it will be finite, and won't effect what happens to the integral at the tail. (For a bounded function (which $\frac{dx}{x^p \ln(x)^q}$ is *on* $[e, \infty)$, divergence and convergence is a question of "tail behavior".)

3. When $p = 1$, we are given the hint use $u = \ln(x)$ so that $du = \frac{1}{x}dx$, $u(e) = \ln(e) = 1$ and $u(\infty)$" $=$" $\ln(\infty) = \infty$. Then $\int_e^{\infty} \frac{dx}{x(\ln(x))^q} = \int_1^{\infty} \frac{du}{u^q} = \frac{1}{1-q} u^{1-q}|_1^{\infty} = \begin{cases} \frac{1}{q-1} & \text{if } q > 1 \\ \infty & \text{if } > q < 1. \end{cases}$ In other words, convergence if $q > 1$ and divergence otherwise (case when $q = 1$ given by R2).

□

Exercise 0.2. 1. When $p > 1$, the integral $\int_{e^e}^{\infty} \frac{dx}{x^p(\ln(x))^q(\ln(\ln(x)))^r}$ converges.

2. The integral diverges when $0 < p < 1$ or if $p = 1$ and $0 < q < 1$.

3. What happens when $p = q = 1$

Proof. 1. Notice that $\ln(\ln(e^e)) = \ln(e) = 1$, and so $\ln(\ln(x)) > 1$ for all $x > 1$ (and in general, $\ln^n(\exp^n(1)) = 1$ for all n (recall from the beginning (and also translate for exp) that $\exp^n(x)$ noes *not* mean $(\exp(x))^n$. It means the n-fold composition of exp then taking x as its argument)). Therefore the integrand

$$\frac{1}{x^p(\ln(x))^q(\ln(\ln(x)))^r} < \frac{1}{x^p(\ln(x)^q)}$$

and we already saw that the integral of the right hand integrand converges for $p > 1$ (converges, integrating from $e \to \infty$, and *a fortiori* on $e^e \to \infty$).

The next part is like problem 1.b, so let's take a break to reexamine the technique applied there.

2. The hint asks us to show that $\lim_{x\to\infty} \frac{(\ln(\ln(x))^{\alpha}}{(\ln(x))^{\beta}} = 0$ for any $\alpha, \beta > 0$. When $\alpha = 1$, one application of L'Hôpital gives that this limit is $\lim_{x\to\infty} \frac{1}{\beta(\ln(x))^{\beta}} = 0$. Good, next we're gonna

apply the same trick as in Equation (2): for $\alpha \neq 1$,

$$\frac{(\ln(\ln(x))^{\alpha}}{(\ln(x))^{\beta}} = \left(\frac{(\ln(\ln(x))}{(\ln(x))^{\beta/\alpha}}\right)^{\alpha} \xrightarrow{x\to\infty} 0$$

because the inside does.

At this point, we want to make a grandiose assumption about how the universe operates, namely that the future resembles the past, (`https://plato.stanford.edu/entries/induction-problem/`), and hope for the best that we can basically copy what we did in problem 1.c

Indeed, let $q' \in (q, 1)$ and note that

$$\frac{1}{x^p(\ln(x))^q(\ln(\ln(x)))^r} = \frac{(\ln(x))^{q'}}{x^p(\ln(x))^q(\ln(x))^{q'}(\ln(\ln(x)))^r} = \frac{1}{x^p(\ln(x))^{q'}} \cdot \frac{(\ln(x))^{q'-q}}{(\ln(\ln(x)))^r}.$$

By the hint, the right hand multiple of the expression on the right hand side goes to infinity (note: $q' - q > 0$!), so for some N_1, $\frac{(\ln(x))^{q'-q}}{(\ln(\ln(x)))^r} > 1$ for $x > N_1$ (without loss of generality, we can [and need to] take $N_1 > e^e$). Then

$$\int_{N_1}^{\infty} \frac{dx}{x^p(\ln(x))^q(\ln(\ln(x)))^r} > \int_{N_1}^{\infty} \frac{dx}{x^p(\ln(x))^q} = \infty,$$

where now we're citing our results from part 1. Beautiful.

3. When $p = q = 1$, we make the substitution $u = \ln^2(x) = \ln(\ln(x))$, so $du = \frac{dx}{x\ln(x)}$ and $u(e^e) = 1$, etc., so

$$\int_{e^e}^{\infty} \frac{dx}{x\ln(x)(\ln(\ln(x)))^r} = \int_{1}^{\infty} \frac{du}{u^r}.$$

The final integral (i.e. after substitution) is the exact same as that of problem 1.c!

□