Extra Problems on Continuity

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *lipschitz* on open interval $(a, b) \subset \mathbb{R}$ if there is a positive constant $C > 0$ such that $|f(x) - f(y)| < C|x - y|$ whenever $a < x < y < b$.

Further recall the definition of continuity:

Definition 2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous at p* if for every $\varepsilon > 0$ there is an associated $\delta_\varepsilon > 0$ (depending on ε) such that $|f(x) - f(p)| < \varepsilon$ whenever $|x - p| < \delta_\varepsilon$. A function is *continuous* (with no qualification) if it is continuous at p for every p in the domain.

Here is an illustration of how these proofs go.

Exercise 0.1 (Example)**.** For any $a \in \mathbb{R}$, function ax is continuous on $\mathbb{R}$.

Proof. Let $a \in \mathbb{R}$ be given, and suppose (one case) that $a > 0$ (what do you do when $a = 0$ and when $a < 0$?). Next, fix $p \in \mathbb{R}$. It doesn't matter what p is, but we need it to be a fixed value to continue with the proof.[1] Next, suppose $\varepsilon > 0$ is given (not chosen). We need to find a suitable δ so that *if* $|x - p| < \delta$ then we can be assured that $|f(x) - f(p)| < \varepsilon$. Well, let's start with what we want to achieve: we want that $|ax - ap| < \varepsilon$. This holds if and only if (after factoring out a), $|a(x - p)| < \varepsilon$ which holds if and only if $|x - p| < \epsilon/a$. In other words, *if* $|x - p| < \epsilon/a$, *then* $|ax - ap|$ will surely be less than ϵ. So let $\delta = \varepsilon/a$.

If we want to be really complete, we can reverse the steps to verify that this δ works (but it isn't necessary):

$$\begin{aligned} & |x-p| && < \delta \\ \Leftrightarrow\ & |x-p| && < \varepsilon/a \\ \Leftrightarrow\ & a|x-p| && < \varepsilon \\ \Leftrightarrow\ & |a(x-p)| && < \varepsilon \\ \Leftrightarrow\ & |ax - ap| && < \varepsilon \\ \Leftrightarrow\ & |f(x) - f(p) && < \varepsilon \end{aligned}$$

In particular since $A \Leftrightarrow B$ implies that $A \Rightarrow B$, we have that $|x - p| < \delta \to |f(x) - f(p)| < \varepsilon$. □

Notice that we fixed p in the beginning, so technically we only showed continuity at p. But then again, I didn't really tell you where p was, so p could have been anywhere. This establishes that the argument works for *any* p and hence that ax is continuous everywhere. (This is actually sortof what we do with ε, since technically ε can be anything, but we need it to be fixed in order to find δ.)

Ok, I'm gonna give another example which is a little bit more involved.

Exercise 0.2 (Example)**.** Show that $\log(x)$ is continuous on $(0, \infty)$.

[1] This is a technical point about the definition of continuity, which is that continuity is continuity *at* a point. So even though ε can be whatever, it is whatever after you have selected which point you want to show continuity around.

Proof. Let $p \in (0, \infty)$, and suppose that $\varepsilon > 0$ is given. Start with what we need: $|f(x) - f(p)| < \varepsilon$, and from this let's try to derive what would be required of $|x - p|$. Well, $|\log(x) - \log(p)| < \varepsilon$ if and only if $-\varepsilon < \log(x) - \log(p) < \varepsilon$ if and only if $\log(p) - \varepsilon < \log(x) < \log(p) + \varepsilon$ if and only if $e^{\log(p)-\varepsilon} < x < e^{\log(p)+\varepsilon}$ (think about why the inequality direction is preserved when we apply the exponential).

Therefore *if* $x \in (e^{\log(p)-\varepsilon}, e^{\log(p)+\varepsilon})$, *then* we are guaranteed that $|f(x) - f(p)| < \varepsilon$. Great!, but we don't yet have $|x - p| < \delta$. That's ok: we know that $e^{\log(p)-\varepsilon} < p < e^{\log(p)+\varepsilon}$ (how?: recall that $p = e^{\log(p)}$!), or in other words that p is "somewhere" in the middle. Chances are it's closer to one side than the other. So let

$$\delta = \min\left\{|p - e^{\log(p)-\varepsilon}|, |p - e^{\log(p)+\varepsilon}|\right\}.$$

Beautiful, so given this δ, if $|x - p| < \delta$, then for sure $x \in (e^{\log(p)-\varepsilon}, e^{\log(p)+\varepsilon})$, and from our calculation above that implies that $|f(x) - f(p)| < \varepsilon$. (Again, this can be verified by repeating an analogous computation as in the preceding problem, but usually this is not necessary; just become comfortable with why we can stop here.)

□

Exercise 0.3. Suppose that f is lipschitz on $(a, b) \subset \mathbb{R}$. Then prove that f is continuous on (a, b).

Exercise 0.4. Suppose that f is differentiable at p. Then prove it is also continuous at p.

Hint. There will be two sets of (δ, ε) floating around, the first from the definition of derivative as

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p} = f'(p)$$

which (recall, by definition of limit) *means* that for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that $|x - p| < \delta$ implies that $\left|\frac{f(x)-f(p)}{x-p} - f'(p)\right| < \varepsilon$. But you need to show that given any $\varepsilon' > 0$, there is (for *this* epsilon) an associated δ' such that $|x - p| < \delta'$ (not δ) implies that $|f(x) - f(p)| < \varepsilon'$. Also, remember the triangle inequality which says that $|x + y| \leq |x| + |y|$. And one more thing, add zero, like this: $f(x) - f(p) = f(x) - f(p) - a + a$. Hint again (because this is a challenging problem): let $a = f'(p)(x - p)$. □

Exercise 0.5. Let $f : \mathbb{Q} \to \mathbb{R}$ be defined by $f(x) = x$ (defined only for rational numbers). Prove that f is continuous.

Note that this function is not defined for irrational reals. That is not a problem in terms of the definition of continuity (can you explain in words why not?)!

Exercise 0.6. Let $n \in \mathbb{N}$ be a natural number. Then show the n-th power function x^n is continuous on $\mathbb{R}$.

Hint. Use the a similar strategy to the problems in Worksheet 2B. Another strategy would be to show that this function is Lipschitz on any bounded interval. □

Exercise 0.7. Let $p(x) = \sum_{j=0}^{n} a_j x^j$ be a polynomial in $\mathbb{R}$ (with $a_j \in \mathbb{R}$). So for example $1 + x^2 - 3x^5$. Prove that p is continuous on $\mathbb{R}$.

Hint. This is a challenging problem. There are two strategies. One is to show lipshitzness on any bounded interval. The other is to show that addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $(a, b) \mapsto a + b$ is continuous. This is also difficult, so another hint is that for a function defined on two variables, we no longer have $|x - p|$ but $||\mathbf{x} - \mathbf{p}|| = \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2}$. Then there is another inequality which you can use, which is that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$. $\square$

Exercise 0.8. Let $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ be defined by $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{else.} \end{cases}$ So for example $\chi_{\mathbb{Q}}(1/2) = 1$ and $\chi_{\mathbb{Q}}(\pi) = 0$. Prove that $\chi_{\mathbb{Q}}$ is nowhere continuous.

Hint. You need a fact in order to solve this problem which is that both the rationals *and* irrationals are "dense" in $\mathbb{R}$. What does that mean? Given *any* $a < b$ in $\mathbb{R}$, there are both a rational point $a < r < b$ and irrational point $a < t < b$ (and therefore, infinitely many of each). $\square$