Intuition in Mathematics

When I told a philosophy professor that (I think that) ZFC is probably inconsistent he looked at me like I was crazy. But he shouldn't have.

I don't actually think ZFC is inconsistent. But, I take *belief* in the consistency of ZFC to be something like belief in God. (In other contexts I like to say that belief in God is like belief in the consistency of ZFC.[1]) Obviously that's not to say that the belief is false (in fact, I don't think that it is). It is merely to say the belief is non-mathematical. (And therefore, not to say much at all.)

Let $\mathcal{L}$ be the language of sets, consider the set $t = \{\varphi \in \mathit{Sent}(\mathcal{L}) : \text{ZFC} \vdash \varphi\}$ of theorems in ZFC, and partition t into countably many disjoint sets indexed by ω, $(t_i)_{i<\aleph_0}$, where $t_i \subset t$ is the set of theorems of ZFC whose shortest proof is i-lines long. Ignoring the evident fact that we haven't even exhausted t_i for small i (and therefore also ignoring its implications), we certainly haven't exhausted t. Now the question I want to ask is whether there is a point in our partition of the theorems of ZFC after which it stabilizes, i.e. whether or not there is some $n < \aleph_0$ for which $t_i = \emptyset$ for all $i > n$ (so that $t = \bigcup_i^n t_i$). It's an irrelevant question, so I move on to the next one: given some φ, we (in doing mathematics) want to know whether or not it is in t. Once we've got that figured out, we are naturally drawn to the philosophical question: what changes as a result of it?

What I mean is the following. Our knowledge and understanding of t is merely a finite slice of it, call it t_f. Discovering some new φ to be in t often expands our understanding of t, i.e. generally $t_f \cup \{\varphi\}$ implies more things than t_f does alone (this is a sociological point).[2] Hence adding theorems into t_f is not without consequence: not only does it fill things in in the way we expect (by plopping another theorem in); φ usually brings a whole slew of friends along with him. Let's try not to underplay the significance of this phenomenon: I learn nothing new about 'reality' when upon e.g. dropping my pen for the gazillionth time it falls (as it always has). Discovering and proving theorems in ZFC is not like dropping things in a world with gravity.

The reason for making this point is to call into question some assumptions in (the philosophy of) mathematics which ground our intuitions. Our interactions with the physical world are finite, but we nevertheless make predictions and general claims about how the world is on account of them. I am not criticizing that. And I am not criticizing it in math either. I am only saying that the two situations are not the same; we generalize in the world (physics) because there is something (that we all take (agree) to be) there for us to generalize, but it is not so in mathematics. We may know the totality of $t_1 \cup \ldots \cup t_{2^{10^5}}$, but that by itself says nothing apriori[3] about what's in $t_{10^{10^{10}}}$, much less t itself. It is commonly believed that *if* ZFC were inconsistent, then that would have already been uncovered by now in the course of our working inside of it (in the last 100 years). But if $\varphi \wedge \neg\varphi \in t_{10^{10^{10}}}$, then it's not particularly surprising that we haven't found its proof yet, for if that proof were transcribed at one line per millisecond starting from the beginning of the universe, it would still not be done.

I should maybe be clear about exactly which intuitions I am challenging. I do not mean to suggest *that* there's a contradiction to be found in some t_n and that we should be quiet (say, following early Wittgenstein) about what we cannot say regarding some thing or class of things which we can't see (because we can't see). Instead I mean to say: talking about t independent of (separate from) math-as-an-activity is a temptation we should try to avoid. We say that $\text{ZFC} \vdash \varphi$ *every time* only when we've got a proof of φ in our hands. That's not an argument but an observation about how *mathematicians* act. Thus $t = t_f$, even if tomorrow t doesn't look like it looked like today. The reason we say ZFC is consistent is because it is. 'But... if $\varphi \wedge \neg\varphi \in t_n$ (some n)?' It *is* in there once you show me where.

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[1] Sometimes trying to be cute in philosophy doesn't pay off: I should say 'belief in the consistency of ZFC is like belief in fairies', because belief in God can fundamentally be—prescinding from questions of truth—based on our deepest human [felt] needs (e.g. our sense that there is *some* significance to this life after all etc. etc.). But it is important to remember that analogy is analogy-in-a-respect, and we don't always need to spell out what that respect is in order to use the analogy.

[2] Of course, $\bigcup_{\mathfrak{s}\subset t}\{\varphi : \text{ZFC} \cup \mathfrak{s} \vdash \varphi\} = t$; in other words, a theorem of ZFC can prove no more than ZFC "itself" can. (One direction is obvious; for the other, if $\mathfrak{s} \cup \text{ZFC} \vdash \varphi$, then let $\langle \varphi_0, \ldots, \varphi_k \rangle$ be a proof (in $\mathfrak{s} \cup \text{ZFC}$) of φ. Then wlog we can take $\varphi_i^0, \ldots, \varphi_i^{n_i}$ to be a proof (in ZFC) of φ_i, so that $\langle \varphi_0^0, \ldots, \varphi_0^{n_0}, \varphi_1^0, \ldots, \varphi_k^{n_k} \rangle \vdash_{\text{ZFC}} \varphi$.)

[3] In other words, without first telling us about $t_{2^{10^5}+1}$ and so on.