Senior Thesis

Forcing and Group Theory

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A. James Schmidt

Adviser: Cameron Donnay Hill

1 Introduction

Forcing was introduced as a technique by Paul Cohen who used it to prove the independence of the continuum hypothesis in 1963, i.e. that both $ZFC + CH$ and $ZFC + \neg CH$ are consistent. There is a close connection, given by the Soundness and Completeness Theorems, between consistency and the existence of models: to show that a theory is consistent, it is enough to construct a model for it. Forcing gave the necessary machinery for constructing such models. Its utility has been extended to other independence results in mathematics, but it is not limited only to independence results. Naturally independence results tell us something interesting about various axiomatic frameworks, but often times we just have particular ones already established and we want to know what's true in them. For this end, models of the theories we are looking at are exactly what we want, and forcing gives us the tools for getting our hands on them.

In the present thesis, we will provide a short background of model theoretic basics. We then present forcing, in terms of how we will be using it, namely through the playing of 'games'; intuitively in a game two players make moves on a certain kind of board and at the end there is some criterion which allows us to decide who won. In our case, games are infinite (usually countably so) and moves will correspond to picking objects to put down in an increasing sequence of elements from a particular set. We will want that sequence to have certain properties, among which will be the existence of a model which 'says' something about it and concomitantly brings other stuff with it. The model we want is called the 'canonical model' of the atomic sentences of the sequence constructed in the course of our game. We will present a proof of its existence and uniqueness.

The machinery of forcing by games will then be used to prove a basic theorem from model theory: Compactness. It will give us a good example of how the machinery is used. We will then present an unrelated example, the random graph. After these two examples, we will give further characterizations of forcing and also present a variant of normal forcing, called Robinson forcing, relative to particular theories.

In the second part of the thesis, we will discuss classes of structures and existential closure, the model theoretic analogue of algebraic closure, which is essentially the property of a model which guarantees that if an extension of it has elements which witness certain claims in the language, then it does too. Then we will consider existential closure in groups proper, and present a nice theorem about embeddings of existentially closed groups.

Before diving into the material proper, we present a few basic theorems of model theory which relate the logical notion of 'proof' and syntax with the mathematical notion of 'truth' and meaning. In theory, every theorem of mathematics can be proven directly from the axioms ZFC, but we tend to avoid doing so for the simple reason that it is overly formal and would needlessly prolong our arguments. Recall that the forcing of Paul Cohen proved the independence of CH from ZFC by constructing a model. This strategy relies on soundness, namely that the existence of a model guarantees the consistency of the theory it is a model for. The following theorem bridges the gap in the other direction; it tells us that we can, in theory, cook up a model for any consistent theory:

Theorem 1. *Let $\mathcal{T}$ be a consistent theory in a first order language $\mathcal{L}$. Then $\mathcal{T}$ has a model, i.e. there is some structure $\mathcal{A}$ such that $\mathcal{A} \vDash \mathcal{T}$.*

A formal connection between a syntactic proof system and truth in a model is required in order to prove the Completeness Theorem. Since the formal system isn't used when working with models (and therefore isn't needed), it won't be necessary to give a proof of Completeness. It is worth mentioning, however, that Completeness immediately implies the Compactness Theorem, which we will be using:

Theorem 2. *Suppose $\mathcal{T}$ is a theory in the first order language $\mathcal{L}$, and that every finite $S \subset \mathcal{T}$ has a model. Then $\mathcal{T}$ has a model.*

Proof. Since every finite $S \subset \mathcal{T}$ has a model, every finite $S \subset \mathcal{T}$ is consistent, by the Soundness Theorem. Suppose that $\mathcal{T}$ didn't have a model. By Completeness, that would mean that $\mathcal{T}$ isn't consistent. But then $\mathcal{T} \vdash \varphi \wedge \neg\varphi$ for some sentence $\varphi \in \mathcal{L}$. Consider some $S \subset_{\text{fin}} \mathcal{T}$ such that $S \vdash \varphi \wedge \neg\varphi$. That means that S doesn't have a model, but $S \subset \mathcal{T}$ is finite, contradiction. $\square$

Compactness is interesting for model theoretic purposes, but it is also interesting for our purposes because it provides a good example of forcing. We will return to the theorem later after laying out the necessary background.

1.1 Basic Definitions and Notation

We will fix the following notation: $\mathcal{L}$ will refer to a first order language, and unless otherwise specified, we will be working in such a language. For the most part, we will assume that $|\mathcal{L}| = \omega$. Recall that a language is a set of constant, function, and relation symbols. Each function and relation symbol takes on a particular number of inputs, called the arity of the function or relation. Those inputs are terms. Terms are defined inductively from the set of constant symbols in $\mathcal{L}$ and an arbitrary set of variables, as follows: every constant symbol and every variable is a term, and given a tuple $\bar{t} = (t_0, \ldots, t_{n_f})$ of terms, for any function symbol $f \in \mathcal{L}$, $f(\bar{t})$ is a term. A *closed* term is a term containing no variables. Formulas in $\mathcal{L}$ are defined inductively as well. Atomic formulas are equalities between terms or relations on terms. Formulas are built up from atomic formulas by the logical connectives, e.g. '$\neg$', '$\wedge$', and '$\exists$'. *Basic* formulas are either atomic formulas or negations of atomic formulas. Formulas for which all appearing variables are bound by a quantifier are called sentences.

$\mathcal{T}$ will refer to a theory in a first order language $\mathcal{L}$, i.e. a set of sentences each of $\mathcal{L}$, and $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ will generally refer to $\mathcal{L}$-structures and often times they will be models of $\mathcal{T}$, i.e. for each $\varphi \in \mathcal{T}$, $\mathcal{A} \models \varphi$. The underlying set of a structure $\mathcal{A}$, also called the *universe* of $\mathcal{A}$, will be denoted by A. We will refer to specific kinds of formulas in $\mathcal{L}$, e.g. $\forall$-formulas, which we will also call universal (likewise, $\exists$-formulas are existential), will be formulas of the form $\forall \overline{x}\, \varphi$, where none of the variables in φ are bound by quantifiers other than $\forall$ (and $\exists$, respectively).

We will sometimes be consider expansions of a model $\mathcal{M}$ in a language $\mathcal{L}$

to including constants naming each of the elements in M. In other words, let $\{m_i : i < \omega\}$ be an enumeration of the elements in M, $W_M = \{c_{m_i} : i < \omega\}$ be a set of constants, each corresponding to an element of M, and let $\mathcal{L}^* = \mathcal{L} \cup W_M$. We consider the augmented structure $\mathcal{M}'$ as the structure $\mathcal{M}$ in $\mathcal{L}^*$ which interprets each c_m as the corresponding element $m \in M$. The point is to get a hold on the statements about particular elements of the structure: let the elementary diagram $diag(\mathcal{M})$ be the set of sentences in $\mathcal{L}^*$ which are true in $\mathcal{M}'$. In other words, $diag(\mathcal{M}) = \{\varphi \in \mathcal{L}^* : \mathcal{M}' \models \varphi\}$. If $\varphi \in \mathcal{L}^*$, let $\varphi_{\mathcal{L}} = \varphi \restriction \mathcal{L}$; then $\mathcal{M} \models \varphi_{\mathcal{L}}(\overline{m})$. We will also consider the *atomic* diagram, i.e. the set of atomic sentences true in $\mathcal{M}'$, denoted $diag^+(\mathcal{M})$.

We will be considering groups as well. In general, if G is a group and $S \subseteq G$, $\langle S \rangle$ will denote the subgroup of G *generated* by the elements of S, i.e. the smallest subgroup of G containing S. A presentation of a group G will be denoted as $\langle \overline{a}, \Phi(\overline{a}) \rangle$, where $\overline{a}$ is a set of constants which generate the group and $\Phi(\overline{x})$ is a set of equations which the group G satisfies, i.e. $G \models \bigwedge \Phi(\overline{a})$, and such that if $H(\overline{a}) \models \Phi(\overline{a})$, then $G \cong \langle \overline{a} \rangle_H$ by the map sending $a \mapsto a^H$. We call $\overline{a}$ and $\Phi(\overline{x})$ the set of generators and relations, respectively. To be precise, G will actually be the $\mathcal{L}$-reduct of the canonical model of such a presentation (note that $\langle \overline{a}, \Phi(\overline{a}) \rangle$ is a structure in the language $\mathcal{L} \cup \{\overline{a}\}$). Every group can be presented, simply by taking the generators of a group (every group has a set of generators: $G = \langle G \rangle$) and letting Φ be the set of all equations among them.

2 Forcing and Games

Our task in what follows will be to construct models. How we do that is by playing games- the model theoretic translation of forcing from set theory- with a determined set of rules or 'allowed moves'; what we have at the end of the games will give us the models we want. To get the right models, we need to play the right games and play them the right way, supposing there is a way of playing them which will give us the desired results. Our job will be to give a description of the game and strategy for winning it.

2.1 Games

As with much of mathematics, the idea behind our defined concepts is to capture what they mean and generalize them for further application. Indeed, the notion of game has a precise mathematical sense which is useful for our purposes.

Definition 3. *A game G is defined to be a set $(\alpha, X, (M_1, M_2), (W_1, W_2))$ where α is an ordinal, called the length of game G, $M_1 \subseteq \alpha$ and $M_2 = \alpha \setminus M_1$ partition α, and $W_1 \subseteq X^\alpha$ and $W_2 = X^\alpha \setminus W_1$ partition X^α. A game is played on the ordinal α; at the i-th turn, if $i \in M_1$ then player 1 moves by choosing some $x_i \in X$, and player 2 chooses the x_i otherwise. A play of the game occurs when players 1 and 2 have constructed a sequence $(x_i)_{i<\alpha} \in X^\alpha$, and we will call the sequence $(x_i)_{i<\alpha}$ the outcome. An outcome $(x_i)_{i<\alpha}$ is a win for player 1 if $(x_i)_{i<\alpha} \in W_1$ and a win for player 2 otherwise.*

Naturally the language of winning suggests a notion of competition. In a run of a game G, player 1 *wants* the outcome (x_i) to land in the set W_1, which will correspond to a win for player 1. Similarly player 2 wants (x_i) to land in W_2. In some games, a player can be sure of winning, in the following sense. Let $\sigma = (\sigma_i)_{i<\alpha}$ be a sequence where $\sigma_j : \mathcal{P}(X^j) \to X$. Then σ is said to be a strategy for player 2 if in a play $(x_i)_{i<\alpha}$ of game G, for each $i \in M_2$, σ tells player 2 what move to make next based on the current position; it is a *winning* strategy if $(x_i)_{i<\alpha} \in W_2$ whenever the strategy is used for each of player 2's moves in any play of the game. In other words, σ tells player 2 what to put in the sequence given an initial segment of the sequence; it is a winning strategy if player 2 always wins when it is used. A game is said to be determined if one of the players has a winning strategy.

Not all games are determined but in some games, they *become* determined after an initial segment of the game. All finite games are determined; this follows almost immediately from the definition of winning: if neither player has a strategy then each player can always do something to prolong the game. This uses the fact that if an initial segment is not winning, then the other player can do something to prevent the next move from becoming winning; if he couldn't, then the original segment would be winning.

From now on, we will assume that the length α of any game we are considering to be ω. If M_1 is the set of even integers and M_2 the odd integers, the game is called a *standard* game. We lose nothing by assuming that if $|M_1| = |M_2| = \omega$, then the game is standard.

Definition 4. *A game* $G = (\omega, X^\omega, (M_1, M_2), (W_1, W_2))$ *is said to be topolog-*

ical if the following condition holds:

There are a function $F : X^{<\omega} \to \mathcal{P}(S)$ and a subset $K \subseteq S$ such that $F(\overline{x}^\wedge\overline{y}) \subseteq F(\overline{x})$ whenever $\overline{x}, \overline{y} \in X^{<\omega}$, $F(x_0, \ldots, x_{n-1}) = F(x_{\sigma(0)}, \ldots, x_{\sigma n-1})$ for all $\overline{x} \in X^{<\omega}$ and $\sigma \in S_n$, and finally if $W_2 \subseteq \{(x_i) : \bigcap_{i<\omega} F(x_0, \ldots, x_{i-1}) \subseteq K\}$.

Remark 1. *As long as both $|M_1| = |M_2| = \omega$ we can assume the game is standard and will therefore say that $M_1 = odds$ and $M_2 = evens$. This requires proof, but it is a simple consequence from the preceding definition. Therefore, we will from now on assume that all our games are standard.*

We now introduce forcing. The basis of a notion of forcing is a notion of consistency, whose definition is long and pretty intuitive, so it will not be necessary to give a formal definition. A notion of consistency $\mathscr{N}$ is a set of sentences in $\mathcal{L}$, i.e. $\mathscr{N} \subset \mathcal{P}(Sent(\mathcal{L}))$, where $Sent(\mathcal{L}) \subset \mathcal{L}$ denotes the sentences of $\mathcal{L}$. A notion of consistency $\mathscr{N}$ satisfies conditions which one would ordinarily expect a consistent set to satisfy, e.g. φ and $\neg\varphi$ can't both be in any $p \in \mathscr{N}$. Also, they satisfy a certain closure property: if e.g. $\varphi \wedge \psi \in p$ for some $p \in \mathscr{N}$, then $p \cup \{\varphi, \psi\} \in \mathscr{N}$.

Definition 5. *Let $W = \{c_i : i < \omega\}$ be a set of constants not in $\mathcal{L}$, and let $\mathcal{L}^* = \mathcal{L} \cup W$. A notion of forcing is a notion of consistency $\mathscr{N}$ in a language $\mathcal{L}^*$, i.e. $\mathscr{N} \subset \mathcal{P}(\mathcal{L}^*)$, and it satisfies the following two conditions:*

1. *If p is a set of sentences in $\mathscr{N}$, t is a closed term in $\mathcal{L}^*$, and c is a constant appearing in no sentence in p or in the term t, then $p \cup \{t = c\} \in \mathscr{N}$.*

2. *For every p in $\mathscr{N}$, there are only finitely many constants from W.*

In some notions of forcing the following two properties sometimes hold:

3. For every $p \in \mathscr{N}$ and every atomic sentence $\varphi \in \mathcal{L}^*$, either $p \cup \varphi$ or $p \cup \neg\varphi$ is in $\mathscr{N}$.

4. If we impose a partial ordering $<_N$ on $\mathscr{N}$ where $p <_N q$ for $p, q \in \mathscr{N}$ iff $p \subset q$, then $\mathscr{N}$ has a least element 0_N.

Elements of $\mathscr{N}$ are usually called $\mathscr{N}$-conditions, or simply conditions. The connection between forcing and games is as follows.

Definition 6. *A construction sequence of a notion of forcing $\mathscr{N}$ is a chain of conditions, i.e. an element of $CS(\mathscr{N}) = \{(p_i)_{i<\omega} \in \mathscr{N}^\omega : \forall\, i < \omega\ p_i \subseteq p_{i+1}\}$. We will be playing games where the goal is to produce a certain construction sequence. Suppose $P \subset CS(\mathscr{N})$; P is said to be enforceable if player 2 has a winning strategy for the game $G(\mathscr{N}, P) = (\omega, (Evens, Odds), (CS(\mathscr{N}) \setminus P, P))$.*

2.2 Canonical Model

In a play or outcome of our games, we end up with a construction sequence which either belongs to W_1 or W_2; other than telling us who won, this clearly won't do without giving us something else. What we will do, then, is look at an outcome of a game, and give it a model. The following theorem tells us that we can do this pretty much automatically.

Definition 7. *Suppose $\mathcal{T}$ is a theory in $\mathcal{L}^* = \mathcal{L} \cup W$ where W is a countable set of distinct constants not in $\mathcal{L}$. We say that $\mathcal{T}$ is =-closed if the following two conditions hold:*

1. *For every closed term $t \in \mathcal{L}$, $t = t \in \mathcal{T}$*

2. *For every atomic formula $\varphi(x) \in \mathcal{L}$, if $\varphi(t) \in \mathcal{T}$ and either $t = s \in \mathcal{T}$ or $s = t \in \mathcal{T}$, then $\varphi(s) \in \mathcal{T}$.*

Theorem 8. *[Lemma 1.5.1, Theorem 1.5.2 [2]] Let $\mathcal{T}$ be an =-closed set of atomic sentences in $\mathcal{L}$. Then there is a structure $\mathcal{A}$ such that $\mathcal{T}$ is the set of all atomic sentences which are true in $\mathcal{A}$, and every $a \in A$ is of the form $t^{\mathcal{A}}$ for some closed term t in $\mathcal{L}^*$. Furthermore, if $\mathcal{T}$ is complete for atomic sentences, then $\mathcal{A}$ is unique up to isomorphism.*

Proof. Let W be the set of all closed terms in $\mathcal{L}^*$ and define an equivalence relation $\sim_{\mathcal{T}}$ on W as follows: $s \sim t$ iff $s = t \in \mathcal{T}$. To see that $\sim_{\mathcal{T}}$ is an equivalence relation, first note that $t \sim_{\mathcal{T}} t$ for every closed term $\mathcal{T}$ since $\mathcal{T}$ is =-closed and therefore $t = t \in \mathcal{T}$. Suppose $s = t \in \mathcal{T}$. Let $\varphi_s(x)$ be the atomic formula $x = s$. Since $\varphi_s(s) \in \mathcal{T}$ and $s = t \in \mathcal{T}$, by =-closure of $\mathcal{T}$, $\varphi_s(t) \in \mathcal{T}$, where $\varphi_s(t)$ is simply $t = s$, which shows that $t \sim_{\mathcal{T}} s$. Finally, suppose $r \sim_{\mathcal{T}} s$ and $s \sim_{\mathcal{T}} t$. Then $r = s \in \mathcal{T}$ and $s = t \in \mathcal{T}$. Again, let $\varphi_t(x)$ be $x = t$. So $\varphi_t(s) \in \mathcal{T}$ and $r = s \in \mathcal{T}$ implies, by =-closure that $\varphi_t(r) \in \mathcal{T}$, which is exactly to say that $r = t \in \mathcal{T}$. Thus $\sim_{\mathcal{T}}$ is an equivalence relation.

For each $t \in W$, let $t/{\sim_{\mathcal{T}}}$ be the equivalence class of t under $\sim_{\mathcal{T}}$ and let $W/{\sim_{\mathcal{T}}}$ be the set of equivalence classes. Define A, the universe of $\mathcal{A}$, to be $A = W/{\sim_{\mathcal{T}}}$.

We now define the structure $\mathcal{A}$. For each $c \in \mathcal{L}^*$, $c^{\mathcal{A}} = c/{\sim_{\mathcal{T}}}$. For each function symbol $f \in \mathcal{L}^*$, $f^{\mathcal{A}}(s_0/{\sim_{\mathcal{T}}}, \ldots, s_{n-1}/{\sim_{\mathcal{T}}}) = f(\overline{s})/{\sim_{\mathcal{T}}}$. To see that this definition is well defined, suppose $s_i \sim_{\mathcal{T}} t_i$ for $0 \leq i \leq n-1$. Then $f(\overline{s}) = f(\overline{s}) \in \mathcal{T}$ since $f(\overline{s})$ is a closed term, and $s_0 = t_0, \ldots, s_{n-1} = t_{n-1} \in \mathcal{T}$ implies that therefore $f(\overline{s}) = f(\overline{t}) \in \mathcal{T}$ which implies that $f(\overline{s})/{\sim_{\mathcal{T}}} = f(\overline{t})/{\sim_{\mathcal{T}}}$. Finally, for the relation symbols $R \in \mathcal{L}^*$, we say $(s_0/{\sim_{\mathcal{T}}}, \ldots, s_{n-1}/{\sim_{\mathcal{T}}}) \in R^{\mathcal{A}}$ iff $R(\overline{s}) \in \mathcal{T}$. The same argument used for functions shows that this definition is well defined.

Now we prove by induction on terms that $t^{\mathcal{A}} = t/{\sim_{\mathcal{T}}}$. If $t = c$, then $t^{\mathcal{A}} = c^{\mathcal{A}} = c/{\sim_{\mathcal{T}}}$, as previously defined. If $t = f(\overline{s})$, where $\overline{s}$ is a closed term, then $t^{\mathcal{A}} = f^{\mathcal{A}}(\overline{s/{\sim_{\mathcal{T}}}}) = f(\overline{s})/{\sim_{\mathcal{T}}}$.

Finally, $\mathcal{A}$ needs to satisfy the atomic formulas: $\mathcal{A} \vDash s = t$ iff $s^{\mathcal{A}} = t^{\mathcal{A}}$ iff $s/{\sim_{\mathcal{T}}} = t/{\sim_{\mathcal{T}}}$ iff $s = t \in \mathcal{T}$. Also $\mathcal{A} \vDash R^{\mathcal{A}}(\overline{s})$ iff $\overline{s/{\sim_{\mathcal{T}}}} \in R^{\mathcal{A}}$ iff $R(\overline{s}) \in \mathcal{T}$.

To see that $\mathcal{A}$ is unique, suppose that $\mathcal{B}$ is another such model. We define the isomorphism $\kappa : \mathcal{B} \to \mathcal{A}$ in the intuitive way: for each $b \in B$, consider a $tp_{\mathcal{B}}(b) = \{\varphi \in \mathcal{T} \text{ atomic} : \mathcal{B} \vdash \varphi(b)\}$ and similarly a $tp_{\mathcal{A}}(a) = \{\varphi \in \mathcal{T} \text{ atomic} : \mathcal{A} \vDash \varphi(a)\}$. We map $b \mapsto a$ iff $tp_{\mathcal{B}}(b) = tp_{\mathcal{A}}(a)$. To see that κ is injective, suppose $b \mapsto a$ and that $b' \mapsto a$. Then $tp_{\mathcal{B}}(b) = tp_{\mathcal{A}}(a) = tp_{\mathcal{B}}(b')$. But $x = t_b \in tp_{\mathcal{B}}(b)$ for some closed term $t \in \mathcal{L}^*$, which means that $x = t_b \in tp_{\mathcal{B}}(b')$ too so $\mathcal{B} \vDash b' = b$. To see that κ is surjective, let $a \in A$. Suppose there is no $b \in B$ whose type in $\mathcal{B}$ is not the same $tp_{\mathcal{A}}(a)$. Then there is an formula $\varphi(x) \in \mathcal{L}^*$ such that $\mathcal{A} \vDash \varphi(a)$ for which $\mathcal{B} \nvDash \varphi(b)$ for every $b \in B$. Passing $\varphi(a)$ to the sentence $\varphi(t_a)$, where t_a is a closed term in $\mathcal{L}^*$ which names a, this means that $\mathcal{A} \vDash \varphi(t_a^{\mathcal{A}})$ but $\mathcal{B} \nvDash \varphi(t_a^{\mathcal{B}})$ contradicting that $\mathcal{B}$ is such a model of

$\mathcal{T}$. Satisfying the homomorphism properties is similar, and taken care of for us by the definition of 'type' together with the appropriate formulas. $\square$

2.3 Example 1: Compactness Theorem

The Compactness Theorem says that if all the finite subsets $\mathcal{T}_f$ of a theory $\mathcal{T}$ are satisfiable, then $\mathcal{T}$ itself is satisfiable. It is powerful in that it tends to be easier (translation: possible) to analyze and understand finite theories than infinite ones. Thus with Compactness, once we get a solid grasp on all the finite sub theories, we can cross the infinite boundary to the theory we are actually interested in. We restate the theorem from above.

Theorem 9. *Let $\mathcal{T}$ be a finitely satisfiable set of sentences in a countable language $\mathcal{L}$. Then there is a structure $\mathcal{A}$ which satisfies the theory, i.e. $\mathcal{A} \vDash \mathcal{T}$.*

The proof we will give uses the machinery of forcing which we have outlined so far, but it is by no means the only one out there. We will prove the theorem for complete theories, theories in which either φ or $\neg\varphi$ is in them for every sentence $\varphi \in \mathcal{L}$. It works because if $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{A} \vDash \mathcal{T}'$ then $\mathcal{A} \vDash \mathcal{T}$. It is necessary, then, to show that every theory is contained inside a complete one.

Lemma 10 (Lemma 2.1.9 [3])**.** *Let $\mathcal{T}$ be a finitely satisfiable set of sentences in a countable language $\mathcal{L}$. Then there is a complete $\mathcal{L}$-theory $\mathcal{T}' \supseteq \mathcal{T}$ which is also finitely satisfiable.*

Proof. First we claim that if $\mathcal{T}$ is finitely satisfiable then either $\mathcal{T} \cup \{\varphi\}$ is finitely satisfiable or $\mathcal{T} \cup \{\neg\varphi\}$ is finitely satisfiable. Suppose that one of them is not finitely satisfiable, say $\mathcal{T} \cup \{\neg\varphi\}$; then we want to show that $\mathcal{T} \cup \{\varphi\}$ is.

That $\mathcal{T} \cup \{\neg\varphi\}$ is not finitely satisfiable means that $\mathcal{T} \nvDash \varphi$, or that $\mathcal{T} \vDash \neg\varphi$. So there is a finite subset $\mathcal{T}_f \subseteq \mathcal{T}$ such that $\mathcal{T}_f \vDash \varphi$. Let $S \subset \mathcal{T}$ be a finite subset of $\mathcal{T}$. Then $S \cup \mathcal{T}_f$ is a finite subset of $\mathcal{T}$ and therefore satisfiable. But $\mathcal{T}_f \vDash \varphi$ implies that $\mathcal{T}_f \cup S \vDash \varphi$ and hence that $S \cup \{\varphi\}$ is satisfiable, provided that $\mathcal{T}_f \cup S$ itself is satisfiable. Since the finite subset $S \subseteq \mathcal{T}$ was arbitrary, $\mathcal{T}$ is finitely satisfiable.

Now consider an enumeration of the sentences in $\mathcal{L}$ not in $\mathcal{T}$, say $(\varphi_n)_{n<\omega}$. We construct a complete finitely satisfiable theory as follows:

0. At stage 0., let $\mathcal{T}_0 = \mathcal{T}$.

n+1. At stage $n+1$., let $\mathcal{T}^+_{n+1} = \mathcal{T}_n \cup \{\varphi\}$ and $\mathcal{T}^-_{n+1} = \mathcal{T}_n \cup \{\neg\varphi\}$. As we just claimed, together with the induction hypothesis that $\mathcal{T}_n$ is finitely satisfiable, at least one of $\mathcal{T}^\pm_{n+1}$ is finitely satisfiable. Let $\mathcal{T}_{n+1}$ be $\mathcal{T}^+_{n+1}$ if $\mathcal{T}^+_{n+1}$ is and $\mathcal{T}^-_{n+1}$ otherwise.

ω. For the limiting stage, let $\mathcal{T}_\omega = \bigcup_{n<\omega} \mathcal{T}_n$.

Define $\mathcal{T}' = \mathcal{T}_\omega$, and we claim that it is complete and finitely satisfiable. To see that it is finitely satisfiable, let $S \subset \mathcal{T}'$ be a finite subset of n sentences. For each $\varphi \in S$, there is some $k_\varphi < \omega$ for which $\varphi \in \mathcal{T}_{k_\varphi}$. Let $N = \max\{k_\varphi : \varphi \in S\}$; then $S \subseteq \mathcal{T}_N$ which is finitely satisfiable. That it is complete is clear from its definition: if $\varphi \in \mathcal{L}$ is a sentence then either it is in $\mathcal{T}$ in which case it is in $\mathcal{T}_0 \subseteq \mathcal{T}'$ or it is not, in which case it is some φ_n in the enumeration of the sentences in $\mathcal{L}$ not in $\mathcal{T}$, and hence it is in $\mathcal{T}_n \subseteq \mathcal{T}'$. □

We are now in a position to prove the Compactness Theorem.

Proof of Compactness, Theorem 2.1.2 [1], Theorem 9. Recall that we are proving compactness for a complete finitely satisfiable theory $\mathcal{T}$ in the countable first order language $\mathcal{L}$. Recall also that $\mathcal{L}^*$ is the augmented language $\mathcal{L} \cup W$ where W is a countably infinite set of constants not in $\mathcal{L}$.

In order to prove compactness with forcing, it is first necessary to come up with the right notion of forcing. Let $\mathscr{N} = \{p(\bar{c}) \subset_{\text{fin}} Sent(\mathcal{L}^*) : \exists \bar{x} \bigwedge p(\bar{x}) \in \mathcal{T}\}$, and $\mathcal{P} = \{\bar{p} = (p_i)_{i<\omega} : \mathcal{A}^+(\bar{p}) \vDash \mathcal{T}\}$, again where $\mathcal{A}^+(\bar{p})$ refers to the canonical model of the atomic sentences in $\bar{p}$.

We need to see that $\mathscr{N}$ is a notion of forcing. Now if $p(\bar{c}) \in \mathscr{N}$, then $\exists \bar{x} \bigwedge p(\bar{x}) \in \mathcal{T}$. Let $\mathcal{M}$ be some model of $\mathcal{T}$. Then there is some $\bar{a} \in M^{|\bar{x}|}$ such that $\mathcal{M} \vDash \bigwedge p(\bar{a})$; if $t = t(\bar{c})$ and $c' \in W$ not in p or $\bar{c}$, then let $b = t^{\mathcal{M}}(\bar{a})$, so with this model $\mathcal{M} \vDash \mathcal{T}$, we get that $p \cup \{t = c'\} \in \mathscr{N}$. Finally, since $|p| < \omega$, there are only finitely many constants from $\mathcal{L}^*$ appearing in p, and therefore only finitely many constants from W. Thus $\mathscr{N}$ is a notion of forcing.

Let $(t_i)_{i<\omega}$ be some enumeration of the closed terms of $\mathcal{L}^*$ and let $\{\psi_i\}_{i<\omega}$ enumerate the sentences of $\mathcal{T}$. The theorem is proven if we can show that Player 2 has a winning strategy for the game $G(\omega, \mathscr{N}, (Evens, Odds), (\mathscr{N}^\omega \setminus \mathcal{P}, \mathcal{P}))$.

At turn $2i + 1$, the initial sequence of the game's play is $p_0, \ldots, p_{2i}$. The strategy will tell her what move to make according to certain requirements, as follows:

1. $R^{=}_{2i} = \{t_j = t_j : j \leq 2i\} \cup \{\varphi(t_j) : \{t_j = t, \varphi(t)\} \subseteq \bigcup_{j \leq 2i} p_j,\ k \leq 2i\}$.
2. $R^{\forall}_{2i} = \{\varphi(c_k) : k \leq 2i \text{ and } \forall x\, \varphi(x) \in \bigcup_{j \leq 2i} p_j\}$.

3. $R^{\wedge}_{2i} = \{\varphi : \varphi \wedge \psi \in \bigcup_{j \leq 2i} p_j \text{ or } \psi \wedge \varphi \in \bigcup_{j \leq 2i} p_j\}$.

4. $R^{\exists}_{2i}$: Let $\exists x_0 \varphi_0(x_0), \ldots, \exists x_{n-1}\varphi_{n-1}(x_{n-1})$ enumerate the sentences in $\bigcup_{j \leq 2i} p_j$ that begin with an existential quantifier. Let $c'_0, \ldots, c'_{n-1}$ be distinct constants from W which are not in $\{c_0, \ldots, c_{2i}\}$ and do not appear in $\bigcup_{j \leq 2i} p_j$. Then set $R^{\exists}_{2i} = \{\varphi_i(c'_i) : 0 \leq i \leq n-1\}$.

5. $R^{\mathcal{T}}_{2i} = \{\psi_j : 0 \leq j \leq 2i\}$.

Since we've enforced $R^{=}_{2i}$ for every $i < \omega$, we know that $\mathcal{T}_{at} = \{\varphi \in \bigcup_{i<\omega} p_i : \varphi \text{ is atomic}\}$ is =-closed, so there is a canonical model $\mathcal{A}^+(\overline{P})$.

These requirements are used as the strategy for Player 2, as follows. At move $2i+1$, Player 2 puts in whatever sentences are needed to satisfy the requirements. Since each p_j is finite, the finite set of p_j's are too, so satisfying them all will keep p_{2i+1} finite.

We claim that $\mathcal{A}(\overline{p}) = \mathcal{A}^+(\overline{p}) \upharpoonright \mathcal{L}$ is a model of $\mathcal{T}$. To this end, we prove the following claim:

Claim 1. *For every sentence $\varphi \in \bigcup_{i<\omega} p_i$, $\mathcal{A}^+(\overline{p}) \models \varphi$.*

Proof of Claim. We prove the claim by induction on formulas. The argument for atomic formulas is given by Theorem 8.

If $\neg\varphi \in \bigcup p_i$, then $\varphi \notin \bigcup p_i$ and so by induction $\mathcal{A}^+ \not\models \varphi$ iff $\mathcal{A}^+ \models \neg\varphi$. For conjunction, by satisfaction of requirements $R^{\wedge}_{2i}$ for each i, if $\varphi \wedge \psi \in \bigcup_{i<\omega} p_i$, then both φ and ψ are in $\bigcup_{i<\omega} p_i$, so by induction $\mathcal{A}^+ \models \varphi$ and $\mathcal{A} \models \psi$ which implies that $\mathcal{A}^+ \models \varphi \wedge \psi$. For negation, if $\neg\varphi \in \bigcup_{i<\omega} p_i$, then $\varphi \notin \bigcup_{i<\omega} p_i$. So by induction

$A^+ \not\vDash \varphi \Leftrightarrow A^+ \vDash \neg\varphi$. Finally, suppose $\exists x\varphi \in \bigcup p_i$, then by satisfaction of $R^{\exists}_{2i}$ for each $i < \omega$, $\varphi(c) \in \bigcup_{i<\omega} p_i$ for some $c \in W$, and $\mathcal{A}^+ \vDash \varphi(c^{\mathcal{A}})$ by induction. Then $\mathcal{A}^+ \vDash \varphi(c^{\mathcal{A}})$ implies that $\mathcal{A}^+ \vDash \exists x\varphi$. □

We have thus shown that $\mathcal{A}^+ \vDash \bigcup p_i$. Now $\mathcal{T} \subseteq \bigcup p_i$, which implies, obviously, that $\bigcup p_i \vDash \mathcal{T}$ and therefore that $\mathcal{A}^+(\overline{p}) \vDash \mathcal{T}$. Note however, that the model $\mathcal{A}^+$ is in the language $\mathcal{L}^*$, with all the constants to name the elements of A. Therefore, just take the $\mathcal{L}$-reduct to get a model in $\mathcal{L}$, namely $\mathcal{A}^+ \restriction \mathcal{L}$.

□

2.4 Example 2: The Random Graph

We used forcing to give an alternative proof of a standard theorem of model theory, i.e. we applied forcing to model theory. But model theory itself is widely applicable to other areas of mathematics. For example, model theoretic arguments demonstrate the existence and uniqueness of the random graph; we will give an argument of that here using forcing.

Definition 11. *A structure $\mathcal{A}$ is called ultrahomogeneous if every isomorphism between finitely generated substructures of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$ itself. We say that a structure $\mathcal{C}$ is weakly homogeneous if it satisfies the following: if $\mathcal{A} \subseteq \mathcal{B}$ are finitely generated substructures of $\mathcal{C}$ for which there is an embedding $\kappa : \mathcal{A} \hookrightarrow \mathcal{C}$, then there is an embedding $\kappa^* \supseteq \kappa : \mathcal{B} \hookrightarrow \mathcal{C}$.*

Definition 12. *For a class K of structures in language $\mathcal{L}$ we have the following properties:*

0. K is closed under isomorphism.

HP. *If* $\mathcal{A} \in K$ *and* $\mathcal{B} = \langle S \rangle$ *for some finite subset* $S \subseteq A$*, then* $\mathcal{B} \cong \mathcal{C}$ *for some* $\mathcal{C} \in K$*.*

JEP. *If* $\mathcal{A} \in K$ *and* $\mathcal{B} \in K$ *then there is a* $\mathcal{C} \in K$ *such that* $\mathcal{A} \hookrightarrow \mathcal{C}$ *and* $\mathcal{B} \hookrightarrow \mathcal{C}$*.*

AP. *If* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$*, and* $e : \mathcal{A} \hookrightarrow \mathcal{B}$ *and* $f : \mathcal{A} \hookrightarrow \mathcal{C}$ *are embeddings then there are* $\mathcal{M} \in K$ *and embeddings* $g : \mathcal{B} \hookrightarrow \mathcal{M}$ *and* $h : \mathcal{C} \hookrightarrow \mathcal{M}$ *such that* $g \circ e = h \circ f$*.*

The final three properties above are called the heredity property, joint embedding property, and amalgamation property, respectively.

It turns out that a class K with all three properties yields an interesting structure, which we will call the Fraïssé limit of K:

Theorem 13 (Theorem 6.1.2 [2])**.** *Suppose a class* K *of* $\mathcal{L}$ *structures is nonempty and not uncountable with properties HP, JEP, and AP. Then there is a countable structure* $\mathcal{M}$*, unique up to isomorphism, which is ultrahomogeneous, which every finitely generated structure in* K *can be embedded into, and every finitely generated substructure is isomorphic to a member of* K

In this subsection, we will give a proof of Theorem 5 using the machinery of forcing. (A different argument- which may be a little bit more straightforward but does not use forcing- is given in [2].)

Now we consider an example of the Fraïssé limit, the random graph. First we should get a handle on the graph.

Definition 14. *A graph is a structure $\mathcal{G}$ in the language of graphs $\mathcal{L}_G = \{R\}$ (consisting only only a binary relation) which satisfies the following two sentences:*

1. $\forall x(\neg R(x,x))$.

2. $\forall x \forall y(R(x,y) \to R(y,x))$.

   Elements of $\mathcal{G}$ are called vertices and pairs for which the relation holds are called edges.

It is easy to see that the class of all finite graphs has properties JP, JEP, and AP, and therefore by the previous theorem also has a Fraïssé limit $\mathcal{G}$, in this case called the random graph. We divide the proof of the theorem into existence and uniqueness, with the existence argument given first in the following proposition.

Proposition 15. *Let $A \subset W$, where W is a set of distinct constants, and let $E \subseteq A \times A$ be an equivalence relation. Let $S = R^{A/E} \subset A/E \times A/E$ and $diag(A,E,s) = p^S = \{\pm R(a,b) : \pm(a/E, b/E) \in R^{A/E} = \pm S(a,b)\} \cup \{\pm a = b : \pm(a,b) \in E\}$. Then $\mathcal{N} = \{p : p \subset_{fin} diag(A,E,s)\}$ is a notion of forcing. Moreover, Player 2 has a winning strategy for $G(\omega, (Evens, Odds), (\mathscr{N}^\omega \setminus \mathcal{P}, \mathcal{P}))$, where $\mathcal{P}$ is the following property: $\mathcal{P} = \{\overline{p} \in \mathscr{N}^\omega : \mathcal{A}(\overline{p}) \models \epsilon_n \text{ for all } n < \omega\}$, where for each $n < \omega$, ϵ_n is the sentence*

$$\forall x_0, \ldots, x_{2n-1}\Big(\bigwedge_{i<j<2n} x_i \neq x_j \to \exists y\Big(\bigwedge_{i<2n} y \neq x_i \wedge \bigwedge_{i<n} R(y,x_i) \wedge \bigwedge_{n \leq i < 2n} \neg R(y,x_i)\Big)\Big)$$

where the formula inside the $\forall$-quantification we will denote as $\epsilon_n^0(x_0, \ldots, x_{2n-1})$.

Proof. First we show that $\mathscr{N}$ is notion of forcing: let $p \in \mathscr{N}$, then $p \in diag(A, E, s)$. Since $diag(A, E, s)$ is satisfiable, let $\mathcal{M} \models diag(A, E, s)$, and in particular, $\mathcal{M} \models \bigwedge p$. Since p is a finite set of ordered pairs which satisfy the relation, there are elements $(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})$ such that $\mathcal{M} \models R^{A/E}(a_i, b_i)$ for $0 \leq i \leq n-1$. If $t = t(\overline{a}\overline{b})$, and $c \in W$ not in $\overline{a}, \overline{b}$ or p, then let $b = t^{\mathcal{M}}(\overline{a}\overline{b})$, so we get from $\mathcal{M} \models diag(A, E, s)$ that $p \cup \{t = c\} \in \mathscr{N}$. Again, since $|p| < \omega$, there are only finitely many constants from W, and therefore $\mathscr{N}$ is a notion of forcing.

Now we prove that Player 2 has a winning strategy for the property $\mathcal{P}$.

At the stage $2i + 1$ we will do the following:

1. Enforce $\epsilon_n^0(c'_0, \ldots, c'_{2n-1})$ for all $c'_0, \ldots, c'_{2n-1}$ appearing in $\bigcup_{j \leq 2i} p_j$.
2. Add the following sentences $\{c_j = c_j : j \leq 2i\}$.
3. Pick $c''_0, \ldots, c''_{i-1} \notin \{c_0, \ldots, c_{2i}\} \cup \bigcup_{j \leq 2i} p_j$ and not used in 1. Then we add $\{c''_t = c''_t : t < i\} \cup \{c''_s \neq c''_t : s < t < i\}$.

Then we are going pick $X = \{x_0, \ldots, x_{n-1}\}$, and $X^{-1} = \{x_0^{-1}, \ldots, x_{n-1}^{-1}\}$ such that $X \cap X^{-1} = \emptyset$. Then we have $R_<(x, y_1, y_2)$, and $R_a(x, y)$ with $a \in G$ with the following holding $\forall x, y_1, y_2\, (R_<(x, y_1, y_2) \to \neg R_<(x, y_2, y_1))$, $\forall x, y(R_a(x, y) \to \neg \bigvee_{b \neq a} R_b(x, y))$.

□

We now proceed to the uniqueness proof.

Lemma 16 (Theorem 6.4.4 [2])**.** *The following are equivalent, for a countably infinite graph $\mathcal{A}$.*

1. *$\mathcal{A}$ is isomorphic to the random graph $\mathcal{G}$.*

2. *Let X and Y be disjoint finite subsets of A. Then there is an element $a \notin X \cup Y$ such that $\mathcal{A} \vDash R(a, x) \wedge \neg R(a, y)$ for every $x \in X$ and $y \in Y$.*

Proof. ($1 \Rightarrow 2$): There is a finite graph $\mathcal{H}$ composed of the union of X and Y, together with another vertex $v \notin X \cup Y$, with $\mathcal{H} \vDash R(a, x) \wedge \neg R(a, y)$ for every $x \in X$ and $y \in Y$. Vertices of $X \cup Y$ will be adjacent in $\mathcal{H}$ if they are adjacent in the random graph $\mathcal{G}$. Since $\mathcal{G}$ is the Fraïssé limit, there is an embedding $\kappa : \mathcal{H} \to \mathcal{G}$. The restriction $\kappa \restriction X \cup Y$ is an isomorphism between finite substructures of $\mathcal{G}$ so it extends to κ^* an automorphism of $\mathcal{G}$. Then $((\kappa^*)^{-1} \circ \kappa)(v)$ will be the element in 2.

$2 \Rightarrow 1$: We begin with a claim.

Claim 2. *If $\mathcal{H} \subseteq \mathcal{H}'$ are finite graphs in $\mathcal{A}$ and $f : \mathcal{H} \to \mathcal{A}$ is an embedding then f extends to an embedding $f' : \mathcal{H}' \to \mathcal{A}$.*

Proof of Claim. This is proved by induction on $|H' \setminus H|$. The case where H' has only one extra vertex is the only case we need to consider (because the induction argument for $n + 1$ assuming n will be the exact same). Let $w \in H' \setminus H$. Let X be the set of vertices $f(x)$ such that $x \in A$ and $\mathcal{H}' \vDash R(x, w)$ and let Y be the set of vertices $f(y)$ such that $y \in A$ but $\mathcal{H}' \vdash \neg R(y, w)$. By assumption there is a $v \in \mathcal{G}$ which is adjacent to all X and to none of Y. Extend f by putting $f'(w) = v$. □

Finally suppose that $\mathcal{A}$ is a finite graph; we want to show that it embeds into $\mathcal{G}$. But note that $\emptyset \subseteq A$ and certainly there is an embedding $f : \emptyset \hookrightarrow \mathcal{G}$. By Claim 2, f extends to some $f^* : \mathcal{A} \hookrightarrow \mathcal{G}$ embedding $\mathcal{A}$ into $\mathcal{G}$.

□

Now we verify the three defining properties of the random graph, i.e. the Fraïssé limit of the class of all finite graphs. To this end, we need to establish that it is ultrahomogeneous, that every finite graph $\mathcal{H}$ embeds into $\mathcal{G}$ and finally that every finite substructure of $\mathcal{A}$ is in the class of all finite graphs. The last one is obvious because $\mathcal{A}$ is a graph and any finite substructure of a graph will also be a graph; if it is finite then it will be in the class of finite graphs.

We construct an isomorphism $f : \mathcal{A} \to \mathcal{G}$. Let $(a_i)_{i<\omega}$, $(v_i)_{i<\omega}$ be an enumeration of A and G, respectively. Also, set $A_n = (a_i)_{i<n}$ and $G_n = (v_i)_{i<n}$, for each $n < \omega$. At each stage of the construction of f, we will have a finite partial isomorphism f_n with $dom(f_n) \supseteq A_n$ and $im(f_n) \supseteq G_n$.

Here is the construction:

0. At stage 0, set $f_0 = \emptyset$.

n+1. At stage $n+1$, if $a_n \in dom(f_n)$, then do nothing. Therefore, assume $a_n \notin dom(f_n)$. We need to find a $v \in G$ such that $f_n \cup \{(a_n, v)\}$ is still an isomorphism. By Claim 2, there is an embedding $h : A_n \cup \{a_n\} \to \mathcal{G}$ extending f_n so we can just take $v = h(a_n)$.

Conversely, we can again assume that $v_n \notin im(h)$. (Otherwise, we would just set $f_{n+1} = h$ and go on to the next stage.) Let $X = \{h^{-1}(w) : w \in im(h) \wedge R(w, v_n) \text{ holds }\}$ and $Y = \{h^{-1}(w) : w \in im(h) \wedge R(w, v_n) \text{ does not hold }\}$. By 2 we get an $a \in A$ such that $R(a, x)$ holds for all $x \in X$ and $\neg R(a, y)$ holds for all $y \in Y$. Set $f_{n+1} = h \cup \{(a, v_n)\}$. The isomorphism will then be $f = \bigcup_{n<\omega} f_n$.

Finally suppose that $\mathcal{H}$ is a finite graph; we want to show that it embeds into $\mathcal{G}$. But note that $\emptyset \subseteq H$ and certainly there is an embedding $f : \emptyset \hookrightarrow \mathcal{A}$. By Claim 2, f extends to some $f^* : \mathcal{H} \hookrightarrow \mathcal{A}$ embedding $\mathcal{H}$ into $\mathcal{A}$.

2.5 Further Characteristics of Forcing

We now return to characteristics of forcing. Recall that a property P is said to be enforceable if Player 2 has a strategy for ensuring that the sequence $\overline{p}$ has P at the end of the construction. Let $q \in \mathscr{N}$ and suppose that Player 2 has a winning strategy for any game starting at q; then we say that q forces P, or $q \Vdash P$.

Definition 17. *Let $\mathscr{N}$ be a notion of forcing and p a condition. Then $\mathscr{N}/p = \{q \in \mathscr{N} : p \subseteq q\}$.*

Proposition 18 (Lemma 2.3.3 [1])**.** *Let $\mathscr{N}$ be a notion of forcing, p a condition, and P a property.*

1. *$q \Vdash P$ iff P is $\mathscr{N}/q$ enforceable.*

2. *P is enforceable iff every condition forces P.*

3. *If $q \Vdash P$ and $p \supseteq q$ then $p \Vdash P$.*

4. *If for all $p \supseteq q$ there is an $r \supseteq p$ such that $r \Vdash P$, then $q \Vdash P$.*

5. *Suppose $P = \bigcap_{i<\omega} P_i$. Then $q \Vdash P$ iff $q \Vdash P_i$ for every $i < \omega$.*

Proof. The first three are obvious. To prove 4., suppose Player 1 plays some $p_0 \supseteq q$. Then by assumption there is an $r \supseteq p_0$ forcing P. Let Player 2 play r thus forcing P.

For 5, let $P = \bigcap_{i<\omega} P_i$. ($\Rightarrow$): Suppose P is enforceable, and let σ be a winning strategy for Player 2. Let $\overline{p}$ be an outcome of the game $\mathscr{N}/q$ in which Player 2 plays according to each σ. Then $\overline{p} \in P = \bigcap_{i<\omega} P_i \subseteq P_{i_0}$, so σ demonstrates that each P_i is enforceable over q.

($\Leftarrow$): Suppose that for each i, Player 2 has a winning strategy σ_i in the game $\mathscr{N}/q$ demonstrating that $q \Vdash P_i$. Let $f : \omega \to \omega$ be some surjective function such that for every $k < \omega$, $S_k = \{i < \omega : f(i) = k\}$ is infinite. Define σ as follows: at Player 2's i-th turn to move, she plays according to $\sigma_{f(i)}$. Let $\overline{p}$ be an outcome of the game in which Player 2 plays according to σ. For some fixed $k_0 < \omega$, $\overline{p}$ is an outcome of the game $(\omega, (\omega \setminus S_k, S_k), (\mathscr{N}^\omega \setminus P_{k_0}, P_{k_0}))$ in which Player 2 has played according to σ_k; so $\overline{p} \in P_k$. Since this is true for any $k < \omega$ (k was arbitrary), $\overline{p} \in \bigcap_{k<\omega} P_k$.

□

Proposition 19. *If $\mathscr{N}$ has a least element 0_N, P is enforceable iff $0_N \Vdash P$.*

Proof. Suppose $0_N \Vdash P$; then as soon as 0_N is put in the construction sequence, Player 1 has a strategy for winning and thus can enforce P. But 0_N is the least element, which means $0_N \subseteq p$ for all $p \in \mathscr{N}$. So anything Player 1 puts in the sequence will be winning for Player 2. The other way is just as easy: if P is enforceable then no matter what, $p \Vdash P$. But if $p \Vdash P$ for all $p \in P$, then $0_N \Vdash P$. □

Lemma 20 (From Proof of Theorem 2.3.4 [1])**.** *If φ is an atomic sentence and p is a condition, then $p \Vdash \neg\varphi$ iff for every condition $q \supseteq p$, $q \nVdash \varphi$.*

Proof. $(\Rightarrow)$: For this direction, suppose there is a condition $q \supseteq p$ with $q \Vdash \varphi$. By Proposition 2.3, $p \Vdash \varphi$, so $p \nVdash \neg\varphi$.

$(\Leftarrow)$: Suppose that $q \nVdash \varphi$ whenever $p \subseteq q$. We show that therefore $p \Vdash \neg\varphi$. Let σ be any strategy for the game of $\mathscr{N}/p$ enforcing that if $\overline{p}$ is the outcome, then $\bigcup_{i<\omega} p_i$ is $=$-closed (which we can do as in the proof of Compactness with the $R^{=}_{2i}$ requirement, which provides player 2 with a winning strategy for $=$-closedness.)

Let $\overline{p}$ be an outcome in which Player 2 plays according to σ. We claim that $\mathcal{A}^+(\overline{p}) \models \neg\varphi$. If $\mathcal{A}^+(\overline{p}) \models \varphi$, then $\varphi \in p_i$, then φp_i for some $i < \omega$ in which case $p \subseteq p_i$ and $p_i \Vdash \varphi$, contradiction. □

This leads to a final characterization of ordinary forcing.

Theorem 21 (Theorem 2.3.4 [1])**.** *Let $\mathscr{N}$ be a notion of forcing and q a condition. Then the following statements hold.*

1. *If φ is an atomic sentence of $\mathcal{L}^*$, then $q \Vdash \varphi$ iff for every condition $p \supseteq q$, there is a condition $r \supseteq p$ containing φ.*
2. *If $\varphi = \bigwedge_{i<\omega} \varphi_i$, then $q \Vdash \varphi$ iff for every $i < \omega$, $q \Vdash \varphi_i$.*
3. *If $\varphi(\overline{x})$ is a formula then $q \Vdash \forall\overline{x}\varphi$ iff for every tuple $\overline{c}$ of constants, $q \Vdash \varphi(\overline{c})$.*

Proof. 1: This follows immediately from Lemma 20 and Proposition 18 part 4.

2: This follows from Proposition 18 part 5.

3: $\Rightarrow$: Note that if $\varphi \models \psi$, and $p \Vdash \varphi$, then $p \Vdash \psi$. Then it is easy to see that $q \Vdash \varphi(\overline{c})$ since $\forall \overline{x} \varphi(\overline{x}) \models \varphi(\overline{c})$.

$\Leftarrow$: In the other direction, suppose every sentence $\varphi(\overline{c})$ is enforceable. Then by Proposition 18 part 5, the property 'All sentences $\varphi(\overline{c})$ are true in $\mathcal{A}(\overline{p})$ and every element of $\mathcal{A}(\overline{p})$ is named by a witness' is enforceable too. But if the property is true of $\mathcal{A}$, then $\mathcal{A}(\overline{p}) \models \forall \overline{x} \varphi$.

□

2.6 Robinson Forcing

We define a new notion of forcing which will be used for application, called Robinson forcing. We have already seen it before, as the proof of Compactness was an instance. In this section, we will use $\mathscr{R}$ to denote our notion of forcing instead of the usual $\mathscr{N}$.

Definition 22. *Let $\mathcal{T}$ be a fixed theory of $\mathcal{L}$. Robinson forcing is the notion of forcing $\mathscr{R}$ for $\mathcal{L}^*$ satisfying the property that $p \in \mathscr{R}$ iff $|p| < \omega$, each $\varphi \in p$ is a basic sentence, and $\mathcal{T} \cup p$ is satisfiable.*

We will consider the universal consequences of a theory $\mathcal{T}$, denoted $\mathcal{T}_\forall = \{\psi \in \mathcal{L}_\forall : \mathcal{T} \models \psi\}$.

Proposition 23 (Lemma 3.4.1 [1])**.** *1. $\mathscr{R}$ is a notion of forcing for $\mathcal{L}^*$ with the following two properties: there is a least element $0_{\mathscr{R}}$, and for every $p \in \mathscr{R}$ and atomic $\varphi \in \mathcal{L}^*$, either $p \cup \{\varphi\} \in \mathscr{R}$ or $p \cup \{\neg\varphi\} \in \mathscr{R}$ (and possibly both).*

2. 'The compiled structure is a model of $\mathcal{T}_\forall$' is enforceable.

3. If $p \in \mathscr{R}$ and φ is an existential sentence of $\mathcal{L}^*$, and $\mathcal{T} \cup p \cup \{\varphi\}$ is satisfiable, then there is a condition $q \supset p$ such that $q \Vdash p$.

Proof. For the first, we define a partial order $\leq_{\mathscr{R}}$ as $p \leq_{\mathscr{R}} q$ iff $p \subseteq q$. It is easy enough to see that $\leq_{\mathscr{R}}$ is a linear order. Then let $0_{\mathscr{R}} = \emptyset$. Obviously, $0_{\mathscr{R}} \leq p$ for every $p \in \mathscr{R}$. For the second, suppose that $p \cup \{\varphi\}$ is not satisfiable. Since $p \in \mathscr{R}$, $\mathcal{T} \cup p$ is satisfiable. Suppose $\mathcal{M} \vDash \mathcal{T} \cup p$. Then $\mathcal{M} \vDash \neg\varphi$ because $p \vDash \neg\varphi$, so $\mathcal{M}$ shows that $\mathcal{T} \cup p \cup \{\neg\varphi\}$ is satisfiable.

For the second, let $\{\varphi_i\}_{i<\omega}$ be an enumeration of $\mathcal{T}_\forall$, possibly with repetitions.

At stage $2i+1$, suppose $\varphi_i = \forall\overline{x}\theta(\overline{x})$ where $\overline{x} = (x_0, \ldots, x_{m-1})$ and θ is quantifier free. We may assume that θ is in disjunctive normal form: $\bigvee\limits_{r<n} \bigwedge\limits_{s<k_r} \theta_{r,s}(\overline{x})$, where each $\theta_{r,s}$ is a basic formula.

Since $p = \bigcup\limits_{j \leq 2i} p_j$ is in $\mathscr{R}$, we may choose a model $\mathcal{M}^+ \vDash \mathcal{T} \cup p$. Then for each m-tuple $\overline{t} = (t_0, \ldots, t_{m-1})$ of closed terms appearing in p, there is an $r(\overline{t}) < n$ such that $\mathcal{M} \vDash \bigwedge\limits_{s<k_r} \theta_{r,s}(t_0^{\mathcal{M}^+}, \ldots, t_{m-1}^{\mathcal{M}^+})$. Player 2 then puts in $p_{2i+1} = \bigcup\limits_{j \leq 2i} \cup \bigcup\limits_{\overline{t}} \{\theta_{r(\overline{t}),s}(\overline{t}) :\, s < k_{r(\overline{t})}\}$, so that $q \vDash \varphi(\overline{t})$, and therefore Player 2 can enforce that $\mathcal{M}^+(\overline{p}) \vDash \varphi(\overline{t})$ by playing $p_1 = q$.

For 3., suppose that φ is $\exists\overline{x}\theta(\overline{x}, \overline{c})$ where θ is of the form $\bigvee\limits_{i<n} \bigwedge\limits_{j<k_i} \theta_{i,j}(\overline{x}, \overline{c})$, where each $\theta_{i,j}$ is basic, and $\overline{c}$ is a tuple of constants. Since $\mathcal{T} \cup p \cup \{\varphi\}$ is satisfiable, pick some $\mathcal{M}^+ \vDash \mathcal{T} \cup p \cup \{\varphi\}$, and so that $\mathcal{M}^+ \vDash \exists\overline{x}\theta(\overline{x}, \overline{c})$, so $\mathcal{M}^+ \vDash \theta(\overline{a}, \overline{c})$ for some $\overline{a} \in M^{|\overline{x}|}$; in particular, $\mathcal{M}^+ \vDash \bigwedge\limits_{j<k_{i_0}} \theta_{i_0,j}(\overline{a}, \overline{c})$ for some

$i_0 < n$. Let $\overline{c}'$ be constants not in p or $\overline{c}$ and put $q = p \cup \{\theta_{i_0,j}(\overline{a}, \overline{c}) : j < k_{i_0}\}$. We have just seen that $\mathcal{T} \cup q$ is satisfiable, and so $q \in \mathscr{R}$; therefore $q \Vdash \varphi$. $\square$

Theorem 24 (Theorem 3.4.2 [1])**.** *Let p be a condition of Robinson forcing and $\varphi(\overline{x})$ a quantifier free formula of $\mathcal{L}^*$. Then $p \Vdash \forall \overline{x} \varphi$ iff $\mathcal{T} \cup p \vDash \forall \overline{x} \varphi$.*

Proof. $(\Rightarrow)$: For this direction, suppose that $\mathcal{T} \cup p \nvDash \forall \overline{x} \varphi(\overline{x})$. Since $\mathcal{T} \cup p$ is satisfiable, we get that $\mathcal{M}^+ \vDash \mathcal{T} \cup p \cup \{\exists \overline{x} \neg \varphi(\overline{x})\}$. Since φ is quantifier free, $\neg \varphi$ is logically equivalent to a disjunctive normal form formula $\bigvee_{i<n} \bigwedge_{j<k_i} \theta_{i,j}(\overline{x})$ where each $\theta_{i,j}$ is basic. Then, $\mathcal{M}^+ \vDash \bigwedge_{j<k_{i_0}} \theta_{i_0,j}(\overline{a})$ for some $\overline{a} \in M^{|\overline{x}|}$ and $i_0 < n$. Choosing a tuple of constants $\overline{c}$ from W appropriately, we put $q = p \cup \{\theta_{i_0,j}(\overline{c}) : j < k_{i_0}\}$. Then $q \nVdash \varphi(\overline{c})$ and by Theorem 6.3, this means that $p \nVdash \forall \overline{x} \varphi(\overline{x})$.

$(\Leftarrow)$: Suppose that $\mathcal{T} \cup p \vDash \forall \overline{x} \varphi(\overline{x})$. We show that $p \Vdash \forall \overline{x} \varphi(\overline{x})$. By Theorem 21 part 3, it is sufficient to show that $p \Vdash \varphi(\overline{c})$ for every tuple $\overline{c}$ of constants in $\mathcal{L}^*$. Suppose not, that $p \nVdash \varphi(\overline{c})$. Then Player 2 does not have a winning strategy in $\mathscr{R}/p$ to guarantee that an outcome $\overline{p}$ has the following two properties:

1. $\bigcup_{i<\omega} p_i$ is $=$-closed (so it has a canonical model).
2. $\mathcal{A}^+(\overline{p}) \vDash \varphi(\overline{c})$.

Since $=$-closedness *is* enforceable, let $\overline{p}$ be an outcome for which 1 holds but 2 fails. Then $\mathcal{A}^+(\overline{p}) \vDash \mathcal{T} \cup p$ but $\mathcal{A}^+(\overline{p}) \vDash \neg \varphi(\overline{c})$ showing that $\mathcal{T} \cup p \nvDash \forall \overline{x} \varphi(\overline{x})$, contradiction. $\square$

3 Existentially Closed Groups

This section is going to run through a generalization of algebraic closure, which guarantees the existence of solutions to polynomial equations in various rings, in model theoretic terms to arbitrary structures. Recall that an existential sentence is one of the form $\exists\overline{x}\varphi(\overline{x})$, where $\varphi(\overline{x})$ is atomic. We will want to ask when a given model $\mathcal{M}$ satisfies an existential sentence. When it doesn't, when is there an extension which does? Furthermore, what properties of a model $\mathcal{M}$ can we explicate when we know that anytime there is an extension of $\mathcal{M}$ satisfying an existential statement, $\mathcal{M}$ satisfies it already? This section is devoted to addressing this question.

3.1 Existentially Closed Structures

The point of this section is to give an account of the extension problem: given a theory $\mathcal{T}$, and $\mathcal{A} \vDash \mathcal{T}$, and given some existential formula $\varphi(\overline{x})$ and a tuple $\overline{a} \subseteq A$, when is there a model $\mathcal{B} \supseteq \mathcal{A}$ of $\mathcal{T}$ such that $\mathcal{B} \vDash \varphi(\overline{a})$? As an example, consider a template for a polynomial in $\mathbb{R}[x]$, say $f(x, \overline{y}) = \sum_{j=0}^{n} y_j x^j$ where $\overline{y} = (y_0, \ldots, y_n)$. For $\overline{a} \in \mathbb{R}^{n+1}$, our question is whether $\mathcal{T} \cup diag(\mathbb{R}) \cup \{\exists x f(x, \overline{a}) = 0\}$ is satisfiable? The answer to this question is a basic fact of complex analysis: for every polynomial over $\mathbb{R}[x]$ (and also over $\mathbb{C}[x]$), there is a solution in the complex numbers; i.e. $\mathbb{C} \vDash \varphi(\overline{a})$. In fact, $\mathbb{C} \vDash \forall\overline{x}\,\varphi(\overline{x})$, for each $n < \omega$. The translation of this in algebraic terms is that $\mathbb{C}$ is algebraically closed. We will see that the notion of algebraic closure generalizes for formulas in $\mathcal{L}$, and that every model in class of models which is closed under unions of chains is contained in a model which satisfies that notion of closure.

The following notion will give us a better understanding of how to tackle extension problems.

Definition 25. *Given an existential formula $\varphi(\overline{x}) \in \mathcal{L}$, let the resultant $Res_\varphi(\overline{x})$ of φ be the set of all universal formulas $\psi(\overline{x})$ which are implied by φ, i.e. $Res_\varphi(\overline{x}) = \{\psi \in \mathcal{L}_\forall : \mathcal{T} \vDash \forall\overline{x}(\varphi \to \psi)\}$.*

The following theorem gives a characterization of the relation between the resultant and the extension problem.

Theorem 26 (Theorem 3.1.1 [1])**.** *Let $\mathcal{T}$ be a theory in $\mathcal{L}$ and $\mathcal{A}$ an $\mathcal{L}$ structure. Let $\varphi(\overline{x})$ be an existential formula and $\overline{a} \in A^{|\overline{x}|}$ be a tuple of elements in A. Then the following are equivalent:*

1. *There is some model $\mathcal{B}$ of $\mathcal{T}$ containing $\mathcal{A}$ such that $\mathcal{B} \vDash \varphi(\overline{a})$*
2. *$\mathcal{A} \vDash \bigwedge Res_\varphi(\overline{a})$.*

Proof. In the forward direction, if there is such a model $\mathcal{B}$, then for each formula $\psi \in Res_\varphi$, $\mathcal{B} \vDash \forall x\,(\varphi \to \psi)$ since $\mathcal{B} \vDash \mathcal{T}$. Since $\mathcal{B} \vDash \varphi(\overline{a})$, $\mathcal{B} \vDash \psi(\overline{a})$, and since $\mathcal{A} \subseteq \mathcal{B}$ and ψ is universal, $\mathcal{A} \vDash \psi(\overline{a})$.

In the other direction, let $\mathcal{A} \vDash \bigwedge Res_\varphi(\overline{a})$. Expand the structure $\mathcal{A}$ to $\mathcal{A}(\overline{c}) = \mathcal{A} \cup \{\overline{c}\}$, where each c_i is a constant for element a_i in $\overline{a}$. Then write φ as $\exists\overline{y}\gamma(\overline{x}, \overline{y})$, where γ is quantifier free. To find the necessary $\mathcal{B}$, it is enough to show that $\mathcal{T}' = \mathcal{T} \cup diag(\mathcal{A}(\overline{c})) \cup \{\gamma(\overline{c}, \overline{d})\}$ is satisfiable, where $\overline{d}$ is a tuple of distinct new constants. If there is no model for $\mathcal{T}'$, then by compactness there is a finite conjunction $\theta(\overline{c}, \overline{b})$ of sentences in $diag(\mathcal{A}(\overline{c}))$ such that $\mathcal{T} \cup \gamma(\bar{c}, \bar{d}) \vDash \neg\theta(\bar{c}, \bar{b})$, in which case $\mathcal{T} \cup \gamma(\bar{c}, \bar{d}) \vDash \forall \bar{z}\, \neg\theta(\bar{c}, \bar{z})$; therefore $\mathcal{T} \vDash \exists \bar{y} \gamma(\bar{c}, \bar{y}) \to \forall \bar{z} \neg\theta(\bar{c}, \bar{z})$ and finally since $\varphi = \exists \bar{y} \gamma(\bar{x}, \bar{y})$, $\mathcal{T} \vDash \forall \bar{x}\, (\varphi \to \forall \bar{z}\, \neg\theta(\bar{x}, \bar{z}))$. Thus $\forall \bar{z}\, (\neg\theta(\bar{c}, \bar{z})) \in Res_{\varphi}$, and therefore $\mathcal{A} \vDash \forall \bar{z}\, (\neg\theta(\bar{a}, \bar{z}))$, by assumption. Therefore $\mathcal{A}(\bar{c}) \vDash \forall \bar{z}\, (\neg\theta(\bar{c}, \bar{z}))$ which implies that $\mathcal{A}(\bar{c}) \vDash \neg\theta(\bar{d}, \bar{b})$, contradicting that $\theta(\bar{d}, \bar{b}) \in diag(\mathcal{A}(\bar{c}))$.

□

Recall that all the universal consequences of a theory $\mathcal{T}$ is denoted as $\mathcal{T}_{\forall} = \{\psi \in \mathcal{L}_{\forall} : \mathcal{T} \vDash \psi\}$. Then $\mathcal{A} \vDash \mathcal{T}_{\forall}$ iff there is an extension $\mathcal{B} \supseteq \mathcal{A}$ which is a model of $\mathcal{T}$. This is an immediate consequence from Theorem 26 for a $\varphi = \exists x\, (x = x)$, for which $Res_{\varphi} = \mathcal{T}_{\forall}$.

The logical apparatus we are forming is a machinery for demarcating and classifying the relations between different kinds of formulas in $\mathcal{L}$. Recall that formulas are defined by induction on atomic formulas, those which are of the form $\bar{s} = \bar{t}$ or $R(\bar{s})$ where $\bar{s}$ and $\bar{t}$ are terms. But there are other kinds of formulas:

Definition 27. *A strict universal horn formula φ in $\mathcal{L}$ is a formula of the form $\forall \bar{x}((\bigwedge_{i=1}^{n} \chi_i) \to \chi_0)$, where each χ_i is atomic. Of course, $\bar{x}$ can equal $\emptyset$ or n can be zero, which would make $\varphi = \forall \bar{x} \chi_0$.*

A universal horn formula φ in $\mathcal{L}$ is a formula which is either strict universal Horn or of the form $\forall \bar{x} \neg \bigwedge_{i=1}^{n} \chi_i$, where each χ is atomic.

A primitive formula φ in $\mathcal{L}$ is a formula of the form $\exists \bar{x}\, (\bigwedge_{i=1}^{n} \chi_i)$, where each χ_i is either an atomic formula or the negation of an atomic formula.

Finally, a positive primitive formula φ is a primitive formula which is *composed only of atomic formulas.*

Lemma 28. *Let $\mathcal{T}_0$ be a strict universal Horn theory, and let D be a collection of atomic sentences of $\mathcal{L}$. Set $\mathcal{T} = \mathcal{T}_0 \cup D$. Let Γ be the collection of atomic sentences θ of $\mathcal{L}^*$ such that $\mathcal{T} \vDash \theta$. Then the following property is enforceable in $\mathscr{R}$ with respect to $\mathcal{T}$:*

$$\{\overline{p} : \mathcal{A}^+(\overline{p}) \vDash \mathcal{T}_0\} \cap \{\overline{p} : \forall\theta \in \mathcal{L}_{at}, \mathcal{T} \nvDash \theta \Rightarrow \mathcal{A}^+(\overline{p}) \vDash \neg\theta\}.$$

Proof. It suffices to show, by proposition 18 part 5, that for every finite set $F \subseteq_{fin} \mathcal{T}_0$, the property $\{\overline{p} : \mathcal{A}^+(\overline{p}) \vDash F\}$ is enforceable in $\mathscr{R}$. Let $(\theta_i)_{i<\omega}$ be some enumeration of atomic sentences of $\mathcal{L}^*$. At stage $2s+1$, for each $\varphi = \forall\overline{x}(\bigwedge_{i<n} \theta_i(\overline{x}) \to \theta'(\overline{x}))$, and each tuple $\overline{c}$ of constants of $\mathcal{L}^*$, if $\{\theta_i(\overline{c})\}_{i<n} \subseteq \bigcup_{j\leq 2s} p_j$, then put $\theta'(\overline{c})$ into X. Set $Y = \{\neg\theta_i : i \leq 2s, \mathcal{T} \nvDash \theta_i\}$, and let p_{2s+1} be $p_{2s} \cup X \cup Y$. To see that $p_{2s+1} \cup \mathcal{T}$ is satisfiable, it suffices to show, by Compactness, that every finite subset is satisfiable. Therefore, let $\mathcal{T}_f \cup p_f \subseteq_{fin} p_{2s+1} \cup \mathcal{T}$, where p_f is some finite collection of sentences from $p_{2s} \cup X \cup Y$. But if not there is some $\theta \in Y$ such that $\mathcal{T}_f \cup p_f \vDash \neg\theta$. But Y is defined so that $\theta \notin Y$. □

Theorem 29 (Theorem 3.1.3 [1]). *Let $\mathcal{T}$ be a strict universal Horn theory in $\mathcal{L}$.*

If $\varphi(\overline{x})$ is a positive primitive formula in $\mathcal{L}$, then Res_φ is equivalent modulo $\mathcal{T}$ to a set of strict universal Horn formulas in $\mathcal{L}$. In other words, there is a set S of strict universal Horn sentences for which $\psi \in S$ iff $\mathcal{T} \cup Res_\varphi \vDash \psi$ and similarly $\chi \in Res_\varphi$ iff $\mathcal{T} \cup S \vDash \chi$.

Proof. Let $\varphi(\overline{x})$ be a positive primitive formula $\exists\overline{y}\bigwedge_{i=1}^{n}\psi_i(\overline{x},\overline{y})$, where each ψ_i is atomic. Let $\Phi(\overline{x})$ be the set of strict universal Horn formulas in Res_φ, the resultant of φ. We want to show that if $\mathcal{A}\vDash\mathcal{T}\wedge\bigwedge\Phi(\overline{a})$ then there is a $\mathcal{B}\supseteq\mathcal{A}$ which is a model of $\mathcal{T}$ for which $\mathcal{B}\vDash\varphi(\overline{a})$. This is sufficient, if $\theta(\overline{x})\in Res_\varphi$, then $\mathcal{B}\vDash\theta(\overline{a})$ and, therefore, also $\mathcal{A}\vDash\theta(\overline{a})$ because θ is universal.

Suppose that $\mathcal{A}\vDash\mathcal{T}\wedge\bigwedge\Phi(\overline{a})$, but that there is no such $\mathcal{B}$; we will show that this implies a contradiction. We may assume that $\mathcal{A}$ is countable.

We first add a solution of $\displaystyle\bigwedge_{i=1}^{n}\psi_i(\overline{a},\overline{y})$ into $\mathcal{A}$. It will be a 'free' solution. So let $\overline{c}$ denote a tuple of constants which will name the solution, and define $\mathcal{C}^+$ to be the canonical model of the set of atomic sentences $\{\theta :$ $\mathcal{T}\cup diag^+(\mathcal{A})\cup\{\displaystyle\bigwedge_{i=1}^{n}\psi_i(\overline{a},\overline{c})\}\vDash\theta\}$, where θ ranges over atomic sentences of $\mathcal{L}(A,\overline{c})^*=\mathcal{L}(A,\overline{c})\cup W$ with W a countably infinite set of new constant symbols. Let $\mathcal{C}^+$ be the canonical model of this set of sentences, and let $\mathcal{C}=\mathcal{C}^+\restriction\mathcal{L}$. Since $\mathcal{C}^+\vDash diag(\mathcal{A})$, we may fix a homomorphism $g:\mathcal{A}\to\mathcal{C}$. Also, $\mathcal{C}\vDash\mathcal{T}$ by Lemma 28.

Therefore, $\mathcal{C}\vDash\psi(g(\overline{a}),\overline{c})$. Of course, if g is an embedding then $g[\mathcal{A}]\subseteq\mathcal{C}$, and therefore it provides us with an extension of $\mathcal{A}$ with a solution to $\psi(\overline{a},\overline{y})$, but we assumed there is no such extension. So g cannot be an embedding and $g[\mathcal{A}]\nsubseteq\mathcal{C}$. That means that there is a negated atomic formula θ which g doesn't preserve, i.e. $\mathcal{A}\nvDash\theta(\overline{a},\overline{d})$ but $\mathcal{C}\vDash\theta(g(\overline{a}),\overline{d})$. Since $\mathcal{C}^+$ is the canonical model as before, by Compactness there is a finite conjunction $\chi(\overline{a},\overline{d},\overline{e})$ of sentences in $diag^+(\mathcal{A})$ such that $\mathcal{T}\vDash\chi(\overline{a},\overline{d},\overline{e})\wedge\psi(\overline{a},\overline{c})\to\theta(\overline{a},\overline{d})$. Recall that for a set of constants $\overline{c}$ not in $\mathcal{L}$, $\mathcal{T}\vDash\varphi(\overline{c})$ iff $\mathcal{T}\vDash\forall\overline{x}\varphi(\overline{x})$, which means here that $\mathcal{T}\vDash\forall\overline{x}\,(\exists\overline{y}\psi(\overline{x},\overline{y})\to\forall\overline{zw}(\chi(\overline{x},\overline{z},\overline{w})\to\theta(\overline{x},\overline{z})))$. But $\forall\overline{zw}(\chi(\overline{x},\overline{z},\overline{w})\to$

$\theta(\overline{x}, \overline{z}))$ is strict universal Horn, and we have thus shown that it is in Φ. Now, $\mathcal{A} \vDash \neg\forall\overline{zw}(\chi(\overline{a}, \overline{z}, \overline{w}) \to \theta(\overline{x}, \overline{w}))$, since $\mathcal{A} \vDash \chi(\overline{a}, \overline{d}, \overline{e}) \wedge \neg\theta(\overline{a}, \overline{d})$. We assumed that $\mathcal{A} \vDash \bigwedge \Phi(\overline{a})$, contradiction. □

Now we have a background of the extension problem. The next step is to consider structures which have superstructures satisfying the extension problem.

Definition 30. *Let K be a class of structures in a first order language $\mathcal{L}$. We say that a structure $\mathcal{A} \in K$ is existentially closed in K if for every existential formula $\varphi(x)$, and every $\overline{a} \subseteq A$, if there is a structure $\mathcal{B} \supseteq \mathcal{A}$ with $\mathcal{B} \vDash \varphi(\overline{a})$, then $\mathcal{A} \vDash \varphi(\overline{a})$.*

Recall that every field is contained in an algebraically closed field. The proof can be generalized, model theoretically, to other classes of structures. Suppose K is a class of structures; we say that K is inductive if for every ascending sequence of structures $(A_i)_{i<\omega} \subseteq K$, the union $\bigcup_{i<\omega} A_i$ is also in K.

Theorem 31 (Theorem 3.2.1 [1])**.** *Let $\mathcal{L}$ be a first order language and K an inductive class of $\mathcal{L}$ structures. Then for every $\mathcal{A} \in K$, there is an existentially closed $\mathcal{B} \in K$ containing $\mathcal{A}$.*

Proof. There are two major steps to this proof: first we show that for every structure $\mathcal{A} \in K$, there is a structure $\mathcal{A}' \in K$ extending $\mathcal{A}$ such that for every existential formula φ in $\mathcal{L}$ and tuple $\overline{a}$ of A, if there is a $\mathcal{C} \supseteq \mathcal{A}'$ in K satisfying $\varphi(\overline{a})$ then $\mathcal{A}' \vDash \varphi(\overline{a})$ already. This step is almost what we want, except that $\overline{a}$ ranges over tuples from A, not all of A'; therefore in the next step we need to extend the proof for tuples $\overline{a}'$ from $\mathcal{A}'$.

For the first task, consider the set of all pairs $(\varphi, \overline{a})$ of existential formulas and tuples from A. We can well order the set as $(\varphi_i, \overline{a}_i)_{i<\lambda}$ for some ordinal λ. We define a chain of $\mathcal{L}$ structures in K as follows:

0. $\mathcal{A}_0 = \mathcal{A}$.

i+1. $\mathcal{A}_{i+1}$ is any $\mathcal{C} \in K$ containing $\mathcal{A}_i$ which satisfies $\varphi_i(\overline{a}_i)$ if it exists; $\mathcal{A}_i$ otherwise.

δ. $\mathcal{A}_\delta = \bigcup_{i<\delta} \mathcal{A}_i$, when δ is a limit ordinal.

Then define $\mathcal{A}'$ to be $\mathcal{A}_\lambda$. Let $\varphi(\overline{x})$ be an existential formula of $\mathcal{L}$, $\overline{a}$ a tuple of A, and suppose that $\mathcal{C} \vDash \varphi(\overline{a})$ for some $\mathcal{C} \supseteq \mathcal{A}'$. Since φ is an existential formula and $\overline{a}$ is a tuple there is some $i < \lambda$ for which $(\varphi, \overline{a}) = (\varphi_i, \overline{a}_i)$. Therefore $\mathcal{C} \supseteq \mathcal{A}' \supseteq \mathcal{A}_i$, but by our definition $\mathcal{A}_{i+1} \vDash \varphi(\overline{a})$, and therefore $\mathcal{A}' \vDash \varphi(\overline{a})$.

To show the next step, let $\mathcal{A} \in K$, and define a new $\mathcal{A}_i$ inductively for each $i < \omega$. Let $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_{i+1} = \mathcal{A}'_i$, and finally $\mathcal{A}_\omega = \bigcup_{i<\omega} \mathcal{A}_i$. Then let $\mathcal{B} = \mathcal{A}_\omega$. To see that $\mathcal{B}$ is existentially closed, let $\overline{b}$ be a tuple of B and φ an existential formula as before, and suppose that $\mathcal{C} \vDash \varphi(\overline{b})$ for some $\mathcal{C} \supseteq \mathcal{B}$. Since $\overline{b}$ is finite, $\overline{b} \subset A_i$ for some $i < \omega$. But $\mathcal{A}_i \subseteq \mathcal{C}$, so $\mathcal{A}_{i+1}$ satisfies $\varphi(\overline{b})$ by definition; since $\mathcal{B} \supseteq \mathcal{A}_{i+1}$, $\mathcal{B} \vDash \varphi(\overline{b})$ also. $\square$

Theorem 32 (Theorem 3.2.3 [1]). *Let $\mathcal{T}$ be a $\forall_2$ theory in $\mathcal{L}$ and suppose that $\mathcal{A}$ is an $\mathcal{L}$-structure. Then the following are equivalent:*

1. $\mathcal{A}$ is an existentially closed model of $\mathcal{T}$.

2. $\mathcal{A} \vDash \mathcal{T}_\forall$, *and for every existential formula* $\varphi(\overline{x})$ *in* $\mathcal{L}$, $\mathcal{A} \vDash \forall \overline{x}(\bigwedge Res_\varphi(\overline{x}) \rightarrow \varphi)$.

3. $\mathcal{A}$ *is an existentially closed model of* $\mathcal{T}_\forall$.

Proof. $(1 \Rightarrow 2)$: Suppose that $\mathcal{A}$ is an existentially closed model of $\mathcal{T}$. Then $\mathcal{A} \vDash \mathcal{T}_\forall$. Let $\varphi(\overline{x})$ be an existential formula and $\overline{a} \in A^{|\overline{x}|}$ such that $\mathcal{A} \vDash Res_\varphi(\overline{a})$. We want to show that $\mathcal{A} \vDash \varphi(\overline{a})$. But we know from Theorem 10 that there is a model $\mathcal{B} \vDash \mathcal{T}$ extending $\mathcal{A}$ such that $\mathcal{B} \vDash \varphi(\overline{a})$. Since $\mathcal{A}$ is existentially closed, $\mathcal{A} \vDash \varphi(\overline{a})$, as desired.

$(2 \Rightarrow 3)$: Assume 2. Then $\mathcal{A} \vDash \mathcal{T}_\forall$. Suppose that $\mathcal{B}$ extends $\mathcal{A}$ and is a model of $\mathcal{T}_\forall$. Let $\varphi(\overline{x})$ be an existential formula and $\overline{a}$ a tuple from A such that $\mathcal{B} \vDash \varphi(\overline{a})$. We want to show that $\mathcal{A} \vDash \varphi(\overline{a})$. But we know that there is a model $\mathcal{C} \vDash \mathcal{T}$ extending $\mathcal{B}$ such that $\mathcal{C} \vDash \varphi(\overline{a})$ and that $\mathcal{C} \vDash \varphi(\overline{a})$. Then $\mathcal{A} \vDash \bigwedge Res_\varphi(\overline{a})$ by Theorem 26, and so $\mathcal{A} \vDash \varphi(\overline{a})$ by 2.

$(3 \Rightarrow 1)$: This implication reduces to showing that $\mathcal{A} \vDash \mathcal{T}$, because if $\mathcal{B} \vDash \exists \overline{x} \varphi$ for some existential formula in $\mathcal{T}$ and $\mathcal{B}$ extends $\mathcal{A}$, then $\mathcal{B} \vDash \varphi(\bar{b})$, but $\mathcal{T}_\forall \subseteq \mathcal{T}$ and $\mathcal{A} \vDash \mathcal{T}_\forall$ is e.c., so $\mathcal{A} \vDash \exists \overline{x} \varphi$. Assume, then, that $\mathcal{A} \vDash \mathcal{T}_\forall$ is e.c. We know that there is a model $\mathcal{B} \vdash \mathcal{T}$ extending $\mathcal{A}$; since $\mathcal{T}$ is $\forall_2$, a given sentence $\varphi \in \mathcal{T}$ will look like $\forall \overline{x} \exists \overline{y} \psi(\overline{x}, \overline{y})$, where ψ is quantifier free. For any tuple $\overline{a}$ from A, $\mathcal{B} \vDash \exists \overline{y} \psi(\overline{a}, \overline{y})$ since $\mathcal{B} \vDash \mathcal{T}$. But $\mathcal{A}$ is existentially closed and a model of $\mathcal{T}_\forall$ so $\mathcal{A} \vDash \exists \overline{y} \psi(\overline{a}, \overline{y})$. Since $\overline{a}$ is arbitrary, $\mathcal{A} \vDash \forall \overline{x} \exists \overline{y} \psi(\overline{x}, \overline{y})$, and thus $\mathcal{A} \vDash \mathcal{T}$. □

3.2 Existentially Closed Groups and Word Problems

Now we turn our attention to groups in particular. If a group is finitely generated, say by $\overline{a} = \{a_i\}_{i=1}^n$, then consider an enumeration of the closed terms of $\mathcal{L}(\overline{a})$, say $\{t_i : i < \omega\}$. Then $\{i : t_i(\overline{a}) = 1\}$ is called the *word problem* of a group G and is *solvable* if the set is computable. A group is *finitely presented* if it is both finitely generated and the set of relations on those generators is finite. A group is computably presented if both the set of generators and relations are computable. Recall that a set is computable (equivalently, recursive) if its characteristic function can be computed by a Turing machine.

The conclusions of this section combine machinery both from model theory and algebra.

Theorem 33 (Theorem 3.3.4 [1]). *Let G be an existentially closed group in the class of groups. Then G is not the trivial group and for every existential formula $\varphi(\overline{x})$ in $\mathcal{L}$ and every $\overline{a} \subset G$, $G \vDash \varphi(\overline{a})$ iff $G \vDash Res_{\varphi}(\overline{a})$.*

Proof. No matter what G is, there is an extension of G which is a group and satisfies the sentence $\exists x\, x \neq 1$, so likewise $G \vDash \exists x\, x \neq 1$ since G is e.c. For the second part, recall from Theorem 32 that for any $\forall_2$ theory T in the language of $\mathcal{L}$, $\mathcal{M} \vDash T$ is existentially closed iff $\mathcal{M} \vDash T_{\forall}$ and for each existential formula $\varphi(\overline{x})$ of $\mathcal{L}$, $\mathcal{M} \vDash \forall\overline{x}(\bigwedge Res_{\varphi}(\overline{x}) \to \varphi(\overline{x}))$. Here T is the theory of groups and $\mathcal{M} = G$. $\square$

Corollary 34. *If $\mathcal{T}$ is a $\forall_2$ theory in $\mathcal{L}$ then the property that the compiled structure is an e.c. model of T is enforceable.*

Proof. By Theorem 32, an existentially closed model $\mathcal{A} \models \mathcal{T}_\forall$ is equivalently an existentially closed model $\mathcal{A} \models \mathcal{T}$.

Let property P be 'For every $\forall$-formula $\varphi(\overline{x})$ and any tuple $\overline{c}$ of constants in $\mathcal{L}^*$, if $\mathcal{A}^+(\overline{p}) \models \varphi(\overline{c})$, then there is an $\exists$-formula $\psi(\overline{x})$ such that $\mathcal{T}_\forall \models \forall\overline{x}(\psi(\overline{x}) \to \varphi(\overline{x}))$ and $\mathcal{A}^+(\overline{p}) \models \psi(\overline{c})$'. Let $X_{\varphi(\overline{x})} = \{\varphi(\overline{x}) : \psi$ is existential and $\mathcal{T}_\forall \models \forall\overline{x}(\psi(\overline{x}) \to \varphi(\overline{x}))\}$. for each $\forall$-formula $\varphi(\overline{x})$, let $P_{\varphi(\overline{x})}$ be 'For every tuple $\overline{c}$ of constants of $\mathcal{L}^*$, if $\mathcal{A}^+(\overline{p}) \models \varphi(\overline{c})$, then there is a $\psi \in X_{\varphi(\overline{x})}$ such that $\mathcal{A}^+(\overline{p}) \models \psi(\overline{c})$'. By Proposition 2.5, it suffices to show that each P_φ is enforceable since $P = \bigcup\limits_{\varphi} P_\varphi$.

At stage $2s+1$: If there is no $\overline{c}$ such that $\varphi(\overline{c}) \in \bigcup\limits_{i\leq 2s} p_i$, then put $p_{2s+1} = \bigcup\limits_{i\leq 2s} p_i$. Otherwise, given some enumeration of the constants of $\mathcal{L}^*$, let $\overline{c}$ be the first such tuple. Then for some $\psi(\overline{x}) \in X_{\varphi(\overline{x})}$, $\mathcal{T}_\forall \cup \bigcup\limits_{i\leq 2s} p_i \cup \{\psi(\overline{c})\}$ is satisfiable. It is easy to verify, so we omit the proof. Put $p_{2s+1} = \bigcup\limits_{i\leq 2s} p_i \cup \{\psi(\overline{c})\}$ for one of the appropriate $\psi(\overline{x})$ from X_φ. □

At this point it is worth inserting a variant of omitting types, as we will use it for the final theorem in the next section.

Definition 35. *For a set of formulas $\Phi(\overline{x})$ in $\mathcal{L}$, we say that a tuple $\overline{a} \subset A$ realizes $\Phi(\overline{x})$ if $A \models \Phi(\overline{a})$. If on the other hand there is no tuple to realize Φ, then we say A omits Φ.*

Definition 36. *Let T be an $\mathcal{L}$-theory and $\Phi(\overline{x})$ a set of formulas. A support of Φ is any existential formula $\exists\overline{x}\psi \in \mathcal{L}$, where ψ is quantifier free, such that $T \cup \{\exists\overline{x}\psi\}$ is satisfiable and $\mathcal{T} \models \forall\overline{x}(\psi \to \bigwedge \Phi)$.*

Theorem 37 (Omitting $\forall$-Types). *Let $\mathcal{T}$ be an $\mathcal{L}$-theory such that for each $i < \omega$, Φ_i is a set of $\forall$-formulas with no support. Then there is a model $\mathcal{M} \models \mathcal{T}$ which is existentially closed and omits every Φ_i.*

All the work up to now has been to prove the following theorem:

Theorem 38 (Corollary 3.3.8 and Theorem 3.4.6 [1]). *Let G be an existentially closed group. Then every finitely generated group with solvable word problem is embeddable in G, but there is a finitely generated subgroup $H < G$ which has an unsolvable word problem. On the other hand, if H is a finitely generated group with unsolvable word problem, then there is an existentially closed group G into which H is not embeddable.*

We will make use of the following two theorem, and omit the proof which can be found in [1]:

Theorem 39 (Fact 3.3.3 [1]). *Let G be a group and suppose that $(g_i)_{i<\omega}$ enumerates the elements of G. Then there is a group $G^* \supseteq G$ containing elements a and b such that for each $i < \omega$, $g_i = [[a, b^{2i+1}], a]$, where $[a, b]$ denotes the commutator of elements a and b, i.e. $aba^{-1}b^{-1}$.*

Theorem 40 (Theorem 3.3.7 [1]). *If $\overline{x}$ is a tuple of variables and $\Phi(\overline{x})$ a set of $\mathcal{L}$-formulas, then the following are equivalent: 1. $\Phi \sim_T Res_\phi$ for some positive primitive formula ϕ in $\mathcal{L}$, and 2. $\Phi \sim_T \Psi$ where Ψ is some c.e. set of strict q.f. Horn formulas of $\mathcal{L}$.*

Before proving the final theorem, recall that two sets X and Y are *computably inseparably* if there does not exist a computable set Z such that $X \subseteq Z$ and $Z \cap Y = \emptyset$.

Proof of Theorem 14. First we show that any every finitely generated group with solvable word problem is embeddable in G. Recall that if G is e.c. then it is nontrivial. That means there is some $g \neq 1$ with $g \in G$. Therefore, in G $s \neq t$ for any terms s, t, is equivalent to $s = t \rightarrow g = 1$. In general, any q.f. Horn formula $(\neg \bigwedge \chi_i)$ is equivalent to strict quantifier free Horn formula using g, i.e. $\bigwedge \chi_i \rightarrow g = 1$.

Let H be any finitely generated group with solvable word problem and let $H = \langle \overline{a} \rangle$. Since H has a solvable word problem, both $X^+ = \{i < \omega : H \vDash t_i(\overline{a}) = 1\}$ and $X^- = \omega \setminus X^+$ are computable. Let $\Phi(\overline{x}, g)$ be the set of formulas $\{t_i(\overline{x}) : i \in X^+\} \cup \{t_i(\overline{x}) = 1 \rightarrow g = 1 : i \in X^-\}$. Since $\Phi(\overline{x}, g)$ is a computable set of strict q.f. Horn formulas, there is some p.p. formula $\varphi(\overline{x}, g)$ such that $\Phi_(\overline{x}, g) = Res_\varphi$. Now $G \times H \vDash \Phi(\overline{a}, g)$ and therefore there is a group $G' \supseteq G \times H$ such that $G' \vDash \exists \overline{x} \varphi(\overline{x}, g)$, but G e.c. means there is a $\overline{b} \subset G$ such that $G \vDash \varphi(\overline{b}, g)$ and therefore $G \vDash \bigwedge \Phi(\overline{b}, g)$. Since $\Phi(\overline{b}, g)$ describes $\langle \overline{b} \rangle$, this means that $H \cong \langle \overline{b} \rangle_G$ which means exactly that $H \hookrightarrow G$.

For the second part, recall from computability theory that there are ly inseparable pairs of c.e. sets. Let X, Y be such a pair and $g \neq 1$ as before; let $\Phi(x, y, g)$ be the set of formulas $\{[[x, y^{2i+1}], x] = 1 : i \in X\} \cup \{[[x, y^{2i+1}], x] = 1 \rightarrow g = 1 : i \in Y\}$. That there are such elements comes straight from Theorem 39. Using the same exact argument as in the preceding paragraph, we can find elements $a, b \in G$ such that $G \vDash \Phi(a, b, g)$. If the word problem *were* solvable, that would mean these two sets are computable, but that contradicts that X, Y are inseparable.

As for the converse, let T be the theory of groups, and $H = \langle \overline{a} \rangle$; let $\Phi(\overline{x})$

be the set of all equations and inequations such that $H \vDash \Phi(\overline{a})$. All we need to do is construct an e.c. group omitting Φ. And that amounts to showing that Φ has no support.

Suppose, for a contradiction, that $\psi(\overline{x})$ supports Φ. Then for every equation $\phi \in \mathcal{L}$, $\phi \in \Phi$ iff $T \vdash \forall \overline{x} \psi \to \phi$. Similarly, $\phi \notin \Phi$ iff $T \vdash \forall \overline{x}(\psi \to \neg\phi)$. Note that these equalities imply exactly that the set of equations such that $H \vDash \phi(\overline{a})$ and $H \nvDash \phi(\overline{a})$ are both c.e., and therefore, H does in fact have a solvable word problem. Therefore, Φ can't have a support.

□

References

[1] Wilfrid Hodges. *Building Models by Games*. Cambridge UP Archive, 1985.

[2] Wilfrid Hodges. *A Shorter Model Theory*. Cambridge University Press, 1997.

[3] David Marker. *Model Theory: An Introduction*. Number 217 in Graduate Texts in Mathematics. Springer, 2002.