

Decentralized risk-sharing: definitions, properties and characterizations

Pooling risks without a central insurer

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Main references¹

- ▶ **Convex order and comonotonic conditional mean risk sharing.** Denuit M. & Dhaene J. (2012). *Insurance: Mathematics and Economics*, 51, 249-256.
- ▶ **Risk-sharing rules and their properties, with applications to peer-to-peer insurance.** Denuit M., Dhaene J. & Robert C.Y. (2022). *Journal of Risk and Insurance*, 89(3), 615-667.
- ▶ **Comonotonicity and Pareto optimality, with application to collaborative insurance.** Denuit M., Dhaene J., Ghossoub M. & Robert C.Y. (2023). *LIDAM Discussion Paper ISBA 2023/05*.
- ▶ **An axiomatic theory for quantile-based risk sharing.** Dhaene J., Robert C.Y., Cheung K.C., Denuit M. (2023). *LIDAM Discussion Paper ISBA 2023/28*.

¹Hereafter, we will refer to these papers as D,D (2012), D,D,R (2022), D,D,G,R (2023) and D,R,C,D (2023).

Agenda

- ▶ 1. Introduction.
- ▶ 2. Risk-sharing and risk-sharing rules.
- ▶ 3. Examples of risk-sharing rules.
- ▶ 4. Internal risk-sharing rules.
- ▶ 5. Aggregate risk-sharing rules.
- ▶ 6. Properties of risk-sharing rules.
- ▶ 7. Axiomatic characterization of the CMRS rule.
- ▶ 8. Further topics (not covered in this talk).
- ▶ References.

1. Introduction

Risk pooling

- ▶ At time 0 (now), consider a pool of agents, each with a random future loss at time 1.
- ▶ **Centralized risk pooling:**
 - ▶ Refers to **risk transfer** mechanisms where the individual losses faced by the members of the pool are passed to a central insurer.
 - ▶ Each **insured** in the **insurance portfolio** is compensated *ex-post* from the insurer for his loss.
 - ▶ In return, the insurer charges an *ex-ante* **premium** to each insured and sets up a **solvency capital** at initiation.
 - ▶ The premiums follow from an appropriate **premium principle**.
 - ▶ Premiums and solvency capital are set such that the probability that their sum at time 1 exceeds the aggregate loss of the insurance portfolio is sufficiently high.

1. Introduction

Risk pooling

- ▶ At time 0 (now), consider a group of agents, each with a random future loss at time 1.
- ▶ Decentralized risk pooling:
 - ▶ Refers to risk-sharing (RS) mechanisms under which the participants in the pool share their risks among each other.
 - ▶ Each participant in the risk-sharing pool is compensated *ex-post* from the pool for his loss.
 - ▶ In return, each participant pays an *ex-post* contribution to the pool.
 - ▶ These contributions follow from a risk-sharing rule, which is chosen such that the sum of all individual contributions to the pool is equal to the aggregate loss of the pool.
 - ▶ A decentralized approach is self-financing, implying that it does not require setting up a solvency capital.

1. Introduction

Risk pooling

▶ Risk-sharing pools of i.i.d. losses:

- ▶ A natural choice for i.i.d.² losses is the **uniform risk-sharing** rule, where each participant contributes ex-post an equal part of the aggregate loss.

▶ Risk-sharing pools of losses which are not i.i.d.:

- ▶ In this case, the choice of an appropriate and simple RS rule is often not straightforward.
- ▶ A possible choice is the **conditional mean RS** (CMRS) rule³, where each participant contributes ex-post the conditional expectation of his loss brought to the pool, given the observed aggregate loss covered by the pool.

²i.i.d. = independent and identically distributed.

³See D,D (2012).

1. Introduction

Decentralized insurance

- ▶ **Decentralized insurance**: 'insurance' based on a decentralized risk pooling strategy:
 - ▶ A group of participants bring their losses together in a RS pool.
 - ▶ The pool covers the losses of its participants,
 - ▶ against a contribution to be paid ex post by each member.
 - ▶ The r.v.'s describing the contributions are agreed upon ex ante,
 - ▶ and chosen such that the self-financing condition is satisfied.
 - ▶ The role of the 'insurer' is restricted to 'administrator'.
- ▶ **Synonyms**: **collaborative insurance**, **crowdsurance**, or **peer-to-peer (P2P) insurance**.
- ▶ **P2P insurance** is sometimes restricted to risk sharing with transfers directly between participants rather than relying on an administrator who first collects the contributions and then redistributes them among participants.

1. Introduction

Decentralized insurance

- ▶ The **group of participants** who form the pool can consist of:
 - ▶ friends, family members, affinity groups, patients suffering from a same disease, etc.
 - ▶ lawyers, farmers, physicians, etc., who form a risk pool to protect themselves against professional risks.
 - ▶ agents exposed to natural catastrophes or major industrial risks (induced by nuclear plants e.g.).
- ▶ Decentralized insurance revives **early forms of mutual insurance** (where the *contributions of the many* are used to cover the *losses of the few*).

1. Introduction

Decentralized insurance

- ▶ Risk-sharing mechanisms have been **studied for decades** in the actuarial literature, starting with the work of **Karl Borch** on equilibrium in reinsurance markets in the 1960's.⁴
- ▶ The **recent interest** in a **sharing economy, collaborative consumption** and **decentralized finance**, together with recent **advances in technology** have made decentralized insurance a viable candidate to disrupt the traditional insurance sector.
- ▶ In response to this recent evolution, **actuarial researchers** have shown a **renewed interest in the math** supporting decentralized insurance⁵.

⁴See e.g. Borch, K. (1962). Equilibrium in a reinsurance market. *Econometrica* 3, 424-444.

⁵See e.g. the website of FADeRiS, a series of workshops on the Foundations and Applications of Decentralized Risk-Sharing, since 2023.

1. Introduction

Conventions and notations

- ▶ Time 0 is 'now'.
- ▶ All r.v.'s are real-valued and defined on the atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- ▶ $\omega \in \Omega$ denotes the realized state-of-the-world at time 1.
- ▶ A random vector is denoted by a bold upper-case letter with subscript indicating its dimension, e.g.

$$\mathbf{X}_n = (X_1, X_2, \dots, X_n),$$

- ▶ while its realization (observed at time 1) is denoted by

$$\mathbf{X}_n(\omega) = (X_1(\omega), X_2(\omega), \dots, X_n(\omega))$$

2. Risk-sharing and risk-sharing rules

Agents and their losses

- ▶ Let χ be an appropriate (sufficiently rich) set of r.v.'s on $(\Omega, \mathcal{F}, \mathbb{P})$, representing possible random losses at time 1.⁶
- ▶ Consider n economic agents, numbered $i = 1, 2, \dots, n$.
- ▶ Each agent i faces a loss $X_i \in \chi$ at the end of the observation period $[0, 1]$.
- ▶ Without pooling, each agent bears his own loss:
At time 1, agent i suffers loss $X_i(\omega)$.

⁶If not explicitly stated differently, we assume that $\chi = L_+^1$.

2. Risk-sharing and risk-sharing rules

Pools of losses

- ▶ The joint cdf of the loss vector \mathbf{X}_n is denoted by $F_{\mathbf{X}_n}$.
- ▶ The marginal cdf's of the individual losses are denoted by $F_{X_1}, F_{X_2}, \dots, F_{X_n}$, respectively.
- ▶ The aggregate loss faced by the n agents with loss vector \mathbf{X}_n is denoted by $S_{\mathbf{X}_n} = \sum_{i=1}^n X_i$.
- ▶ Hereafter, we will often call \mathbf{X}_n the pool (of losses) and each agent a participant.
- ▶ In case no confusion about n is possible, we will write \mathbf{X} instead of \mathbf{X}_n and $S_{\mathbf{X}}$ (or S) instead of $S_{\mathbf{X}_n}$.

2. Risk-sharing and risk-sharing rules

Reallocations

- ▶ **Definition:** For any pool $\mathbf{X} \in \chi^n$ with aggregate loss $S_{\mathbf{X}}$, the set $\mathcal{A}_{\mathbf{X}}(\chi^n)$ is defined by:

$$\mathcal{A}_{\mathbf{X}}(\chi^n) = \left\{ (Y_1, Y_2, \dots, Y_n) \in \chi^n \mid \sum_{i=1}^n Y_i = S_{\mathbf{X}} \right\}$$

- ▶ The elements of $\mathcal{A}_{\mathbf{X}}(\chi^n)$ are called the n -dimensional **reallocations** of \mathbf{X} in χ^n .
- ▶ In case no confusion is possible, we will write $\mathcal{A}_{\mathbf{X}}$ instead of $\mathcal{A}_{\mathbf{X}}(\chi^n)$.

2. Risk-sharing and risk-sharing rules

Risk-sharing

- ▶ **Definition:** Risk-sharing in a pool $\mathbf{X} \in \chi^n$ is a two-stage process.

- ▶ Ex-ante step (at time 0):

The losses X_i in the pool \mathbf{X} are reallocated by transforming \mathbf{X} into another random vector $\mathbf{C}[\mathbf{X}] \in \mathcal{A}_{\mathbf{X}}$:

$$\mathbf{C}[\mathbf{X}] = (C_1[\mathbf{X}], C_2[\mathbf{X}], \dots, C_n[\mathbf{X}])$$

- ▶ Ex-post step (at time 1):

- ▶ Each participant i receives from the pool his loss X_i .
- ▶ In return, he contributes to the pool his reallocated loss $C_i[\mathbf{X}]$.

- ▶ Self-financing condition:

As $\mathbf{C}[\mathbf{X}] \in \mathcal{A}_{\mathbf{X}}$, risk-sharing is **fully reallocating** the loss vector \mathbf{X} :

$$\sum_{i=1}^n C_i[\mathbf{X}] = \sum_{i=1}^n X_i$$

2. Risk-sharing and risk-sharing rules

Risk-sharing rules

- ▶ Consider a group of n agents (participants) and a set of loss vectors χ^n .
- ▶ **Definition:** A **risk-sharing rule** for the group of n participants is a mapping $\mathbf{C} : \chi^n \rightarrow \chi^n$ which transforms any **loss vector \mathbf{X}** into a **contribution vector $\mathbf{C}[\mathbf{X}]$** :

$$\mathbf{X} \in \chi^n \rightarrow \mathbf{C}[\mathbf{X}] \in \mathcal{A}_{\mathbf{X}}(\chi^n)$$

- ▶ **Terminology:**
 - ▶ Consider the n participants and suppose their loss vector is \mathbf{X} .
 - ▶ X_i is the **loss** of participant i (paid by the pool).
 - ▶ $C_i[\mathbf{X}]$ is the **contribution** of participant i (paid to the pool).

2. Risk-sharing and risk-sharing rules

Risk-sharing rules

- ▶ Risk-sharing rules can be complex mappings.
- ▶ The contribution vector $\mathbf{C}[\mathbf{X}]$ may depend on a priori info (i.e. info known at time 0) concerning:
 - ▶ losses: expectations of the X_i and/or cdf of \mathbf{X} .
 - ▶ participants: their ages, their initial wealth, their place-of-residence, their ID number, their utility functions.
 - ▶ other: current state-of-the-economy.
- ▶ The contribution vector $\mathbf{C}[\mathbf{X}]$ may also depend on a posteriori info (i.e. time-1 observable r.v.'s) concerning:
 - ▶ losses: the losses X_i and/or the aggregate claims S .
 - ▶ participants: mileage increase of their cars, telematics info about their driving behavior.
 - ▶ other: inflation index at time 1.

2. Risk-sharing and risk-sharing rules

Risk-sharing rules

Example 2.1:

- ▶ Consider two agents with age vector $\mathbf{x} = (x_1, x_2)$ and loss vector in χ^2 .
- ▶ Consider the RS rule $\mathbf{C} : \chi^2 \rightarrow \chi^2$, where for any pool $\mathbf{X} \in \chi^2$ the contribution vector is given by

$$\mathbf{C}[\mathbf{X}] = \left(\frac{x_1}{x_1 + x_2} S_{\mathbf{X}}, \frac{x_2}{x_1 + x_2} S_{\mathbf{X}} \right)$$

- ▶ Interpretation:
The older participant contributes a higher proportion of the aggregate claims.

2. Risk-sharing and risk-sharing rules

Risk-sharing rules

Example 2.2:

- ▶ Consider a group of n **car drivers** with loss vector in χ^n describing their third party liability claims.
- ▶ Let us denote the **increases in mileage** of the cars of these agents by the random vector $\mathbf{Z} = (Z_1, Z_2, \dots, Z_n)$.
- ▶ For these car drivers, consider the RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$, where for any pool $\mathbf{X} \in \chi^n$, one has

$$\mathbf{C}[\mathbf{X}] = \left(\frac{Z_1}{Z_1 + \dots + Z_n} S_{\mathbf{X}}, \dots, \frac{Z_n}{Z_1 + \dots + Z_n} S_{\mathbf{X}} \right)$$

- ▶ Interpretation:
Participants pay a proportion of the aggregate claims based on the increase in mileage of the car they are driving.

2. Risk-sharing and risk-sharing rules

Risk-sharing rules

Example 2.3:

- ▶ Consider a group of n participants:
 - ▶ These participants can be divided in m age categories, denoted by $1, 2, \dots, m$.
 - ▶ The age category of participant j is denoted by a_j .
- ▶ Consider the RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$, where for any pool $\mathbf{X} \in \chi^n$, the contribution vector follows from

$$C_i[\mathbf{X}] = \sum_{k=1}^m \left(\frac{\sum_{j=1}^n X_j \times 1_{a_j=k}}{\sum_{j=1}^n 1_{a_j=k}} \right) \times 1_{a_i=k}, \quad i = 1, 2, \dots, n$$

2. Risk-sharing and risk-sharing rules

Risk-sharing rules

Example 2.3 (cont'd):

- ▶ The RS rule C is defined by

$$C_i[\mathbf{X}] = \sum_{k=1}^m \left(\frac{\sum_{j=1}^n X_j \times \mathbf{1}_{a_j=k}}{\sum_{j=1}^n \mathbf{1}_{a_j=k}} \right) \times \mathbf{1}_{a_i=k}$$

- ▶ Interpretation: Losses are uniformly shared within each age category.

2. Risk-sharing and risk-sharing rules

Combining risk retention, risk-sharing and risk transfer

- ▶ In order to **reduce his contribution** to the pool, a participant with initial loss Y_i may:
 - ▶ **retain** part of Y_i
 - ▶ **transfer** part of Y_i to an insurer,
 - ▶ **share** the remaining part of Y_i with the members of a pool.
- ▶ **In this presentation:**
 - ▶ X_i denotes the loss that i brings to the the pool, after eventual retention and transfer of risk.
 - ▶ $C_i[\mathbf{X}]$ denotes the contribution of i to the pool.
- ▶ **Example:**
 - ▶ Suppose that participant i *retains* the lower layer $[0, l_i)$ of initial loss $Y_i \geq 0$,
 - ▶ and *transfers* the upper layer $[u_i, \infty)$ of Y_i , with $u_i > l_i$ to an insurer.
 - ▶ The loss *shared* in the pool is then

$$X_i = (Y_i - l_i)_+ - (Y_i - u_i)_+$$

3. Examples of risk-sharing rules

The stand-alone risk-sharing rule

- ▶ **Definition:** $\mathbf{C} : \chi^n \rightarrow \chi^n$ is the **stand-alone RS rule** if

$$\mathbf{C}[\mathbf{X}] = \mathbf{X}, \quad \text{for any } \mathbf{X} \in \chi^n$$

- ▶ **Interpretation:** This RS rule corresponds to the case where the agents decide not to pool their risks.
- ▶ The pool just acts as a register to collect data about individual losses, without re-allocating them among participants.

3. Examples of risk-sharing rules

The uniform risk-sharing rule

- ▶ **Definition:** The **uniform RS rule** $\mathbf{C}^{\text{uni}} : \mathcal{X}^n \rightarrow \mathcal{X}^n$ is defined by

$$C_i^{\text{uni}}[\mathbf{X}] = \frac{S_{\mathbf{X}}}{n}, \quad i = 1, 2, \dots, n,$$

for any $\mathbf{X} \in \mathcal{X}^n$.

- ▶ **Interpretation:** This RS rule equally distributes the aggregate loss $S_{\mathbf{X}}$ over all participants.
- ▶ It is the most simple, non-trivial and well-known RS rule.

3. Examples of risk-sharing rules

The mean-proportional risk-sharing rule

- ▶ **Definition:** The **mean-proportional RS rule**
 $C^{\text{prop}} : \chi^n \rightarrow \chi^n$ is defined by

$$C_i^{\text{prop}}[\mathbf{X}] = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{\mathbf{X}}]} S_{\mathbf{X}}, \quad i = 1, 2, \dots, n.$$

for any $\mathbf{X} \in \chi^n \subseteq (L_+^1)^n$.

- ▶ **Interpretation:** The participants agree to take a fixed percentage of the total loss $S_{\mathbf{X}}$, in accordance with the expected values of the losses they bring to the pool.
- ▶ Proportional rules have been widely used in a risk-sharing context in practice.

3. Examples of risk-sharing rules

The order statistics risk-sharing rule

- ▶ The i -th **order statistic** $X_{(i)}$ of a random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is the i -th smallest value in \mathbf{X} . Hence, $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$.
- ▶ **Definition:** The **order statistics RS rule** $\mathbf{C}^{\text{ord}} : \chi^n \rightarrow \chi^n$ is defined by

$$\mathbf{C}^{\text{ord}}[\mathbf{X}] = (X_{(1)}, X_{(2)}, \dots, X_{(n)})$$

for any $\mathbf{X} \in \chi^n$.

- ▶ **Interpretation:** For participants which are ordered in ascending risk-bearing capacity (e.g. age), a higher risk-bearing capacity leads to a higher contribution.

3. Examples of risk-sharing rules

The conditional mean risk-sharing rule

- ▶ **Definition:** The conditional mean RS rule $\mathbf{C}^{\text{cm}} : \mathcal{X}^n \rightarrow \mathcal{X}^n$ is defined by

$$C_i^{\text{cm}}[\mathbf{X}] = \mathbb{E}[X_i \mid S_{\mathbf{X}}], \quad i = 1, 2, \dots, n,$$

for any $\mathbf{X} \in \mathcal{X}^n = (L_+^1)^n$.

- ▶ **Interpretation:** Each participant contributes the expected value of the loss he brings to the pool, given the aggregate loss experienced by the pool.
- ▶ The CMRS rule was introduced in the actuarial literature in D,D (2012)⁷. It has many nice properties, see D,D,R (2022). Its axiomatic characterization is given in Jiao, Kou, Liu & Wang (2022).

⁷An early reference is Landsberger & Meilijson (1994).

3. Examples of risk-sharing rules

The conditional mean risk-sharing rule

- ▶ A **reshuffle** of the loss vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is a random vector \mathbf{X}^π defined by

$$\mathbf{X}^\pi = (X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)}),$$

where $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ is a permutation of $\{1, \dots, n\}$.

- ▶ **Interpretation:** \mathbf{X} and \mathbf{X}^π are composed of the same individual losses, but only their places are interchanged: After reshuffling, $X_{\pi(i)}$ is the new loss attributed to participant i .
- ▶ **Definition:**⁸ The loss vector \mathbf{X} is **exchangeable** in case for any reshuffle π of \mathbf{X} , one has that $\mathbf{X}^\pi \stackrel{d}{=} \mathbf{X}$.

⁸ $\mathbf{X}^\pi \stackrel{d}{=} \mathbf{X}$ is a notation for ' \mathbf{X}^π and \mathbf{X} have the same cdf'.

3. Examples of risk-sharing rules

The conditional mean risk-sharing rule

- ▶ The individual losses of an exchangeable loss vector are i.d.⁹, but not necessary mutually independent.

- ▶ **Property:**

In case the loss vector $\mathbf{X} \in \mathcal{X}^n$ is exchangeable, then

$$\mathbf{C}^{\text{cm}}[\mathbf{X}] = \mathbb{E}[\mathbf{X} \mid S_{\mathbf{X}}] = \left(\frac{S_{\mathbf{X}}}{n}, \frac{S_{\mathbf{X}}}{n}, \dots, \frac{S_{\mathbf{X}}}{n} \right)$$

- ▶ **Interpretation:**

For exchangeable loss vectors, the conditional mean RS rule reduces to the uniform RS.

⁹The notation 'i.d.' stands for 'identically distributed'.

4. Internal risk-sharing rules

Definition

► Definition:

$\mathbf{C} : \chi^n \rightarrow \chi^n$ is a **internal RS rule** if for any loss vector $\mathbf{X} \in \chi^n$, one has that $\mathbf{C}[\mathbf{X}]$ is \mathbf{X} -measurable.

► Interpretation:

\mathbf{C} is *internal* in the sense that the randomness of any contribution vector $\mathbf{C}[\mathbf{X}]$ is solely due to the randomness of the loss vector \mathbf{X} .

4. Internal risk-sharing rules

Definition

► **Other characterizations:**

$\mathbf{C} : \chi^n \rightarrow \chi^n$ is an **internal RS rule** if and only if any of the following equivalent conditions holds:

- (1) For any $\mathbf{X} \in \chi^n$, there exists a function $\mathbf{h}^{\text{int}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{\text{int}}(\mathbf{X})$$

- (2) For any $\mathbf{X} \in \chi^n$, one has that

$$\mathbf{C}[\mathbf{X}] = \mathbb{E}[\mathbf{C}[\mathbf{X}] \mid \mathbf{X}]$$

► **Remark:**

The function \mathbf{h}^{int} may be pool-specific, in the sense that it may be different for different loss vectors.

4. Internal risk-sharing rules

Examples of internal RS rules

- ▶ Stand-alone RS: internal,

$$C_i[\mathbf{X}] = \mathbf{X}$$

- ▶ Uniform RS: internal,

$$C_i^{\text{uni}}[\mathbf{X}] = \frac{S_{\mathbf{X}}}{n}$$

- ▶ Proportional RS: internal,

$$C_i^{\text{prop}}[\mathbf{X}] = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{\mathbf{X}}]} S_{\mathbf{X}}$$

- ▶ Order statistics RS: internal,

$$C_i^{\text{ord}}[\mathbf{X}] = X_{(i)}$$

- ▶ Conditional mean RS: internal,

$$C_i^{\text{cm}}[\mathbf{X}] = \mathbb{E}[X_i | S_{\mathbf{X}}]$$

4. Internal risk-sharing rules

Counterexample

Example 2.4:

- ▶ At time 1, we flip a coin. The r.v. Z equals 0 if the coin lands on heads and 1 if it lands on tails.
- ▶ Consider the RS rule $\mathbf{C} : \chi^2 \rightarrow \chi^2$, which is defined such that for any pool $\mathbf{X} = (X_1, X_2)$,

$$C_1[\mathbf{X}] = \begin{cases} X_1 + X_2 & : Z = 0 \\ 0 & : Z = 1 \end{cases}$$

and

$$C_2[\mathbf{X}] = \begin{cases} 0 & : Z = 0 \\ X_1 + X_2 & : Z = 1 \end{cases}$$

- ▶ Observations:
 - ▶ It is impossible to find for any pool \mathbf{X} a function $\mathbf{h}^{\text{int}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\mathbf{C}[\mathbf{X}] = \mathbf{h}^{\text{int}}(\mathbf{X})$.
 - ▶ This means that \mathbf{C} is not an internal RS rule.

4. Internal risk-sharing rules

Special classes of internal risk-sharing rules

- ▶ **C** is **type 1 - internal** if there exists a function $\mathbf{h}^{(1)}$ such that for all \mathbf{X} , one has that;

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{(1)}(\mathbf{X}; F_{\mathbf{X}})$$

- ▶ **C** is **type 2 - internal** if there exists a function $\mathbf{h}^{(2)}$ such that for all \mathbf{X} , one has that:

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{(2)}(\mathbf{X}; F_{X_1}, F_{X_2}, \dots, F_{X_n})$$

- ▶ **C** is **type 3 - internal** if there exists a function $\mathbf{h}^{(3)}$ such that for all \mathbf{X} , one has that:

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{(3)}(\mathbf{X})$$

5. Aggregate risk-sharing rules

Definition

▶ **Definition:**

RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ is an **aggregate RS rule** if for any pool $\mathbf{X} \in \chi^n$, one has that $\mathbf{C}[\mathbf{X}]$ is $S_{\mathbf{X}}$ -measurable.

▶ **Interpretation:**

\mathbf{C} is *aggregate* in the sense that the randomness of any contribution vector $\mathbf{C}[\mathbf{X}]$ is solely due to the randomness of aggregate claims $S_{\mathbf{X}}$.

▶ **Remark:**

An aggregate RS rule is **anonymous**, in the sense that it doesn't care about the 'origin' of the aggregate loss $S_{\mathbf{X}}$.

5. Aggregate risk-sharing rules

Definition

► **Other characterizations:**

$\mathbf{C} : \chi^n \rightarrow \chi^n$ is an **aggregate RS rule** if and only if any of the following equivalent conditions holds:

- (1) For any $\mathbf{X} \in \chi^n$, there exists a function $\mathbf{h}^{\text{aggr}} : \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{\text{aggr}}(S_{\mathbf{X}})$$

- (2) For any $\mathbf{X} \in \chi^n$, one has that

$$\mathbf{C}[\mathbf{X}] = \mathbb{E}[\mathbf{C}[\mathbf{X}] \mid S_{\mathbf{X}}]$$

► **Remark:**

The function \mathbf{h}^{aggr} may be pool-specific, in the sense that it may be different for different loss vectors.

5. Aggregate risk-sharing rules

Definition

▶ **Property:**

An **aggregate** RS rule is an **internal** RS rule.

▶ **Proof:**

▶ Let $\mathbf{C} : \chi^n \rightarrow \chi^n$ be an aggregate RS rule and $\mathbf{X} \in \chi^n$.

▶ There exists a function $\mathbf{h}^{\text{aggr}} : \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{\text{aggr}}(S_{\mathbf{X}})$$

▶ Define the function $\mathbf{h}^{\text{int}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\mathbf{h}^{\text{int}}(\mathbf{x}) = \mathbf{h}^{\text{aggr}}(x_1 + \dots + x_n)$$

▶ Then we find that

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{\text{int}}(\mathbf{X})$$

▶ We can conclude that \mathbf{C} is an internal RS rule. ▽

5. Aggregate risk-sharing rules

Examples

- ▶ **Stand-alone RS**: not aggregate,

$$\mathbf{C}[\mathbf{X}] = \mathbf{X}$$

- ▶ **Uniform RS**: aggregate,

$$C_i^{\text{uni}}[\mathbf{X}] = \frac{S_{\mathbf{X}}}{n}$$

- ▶ **Mean-proportional RS**: aggregate,

$$C_i^{\text{prop}}[\mathbf{X}] = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{\mathbf{X}}]} S_{\mathbf{X}}$$

- ▶ **Order statistics RS**: not aggregate,

$$C_i^{\text{ord}}[\mathbf{X}] = X_{(i)}$$

- ▶ **Conditional mean RS**: aggregate,

$$C_i^{\text{cm}}[\mathbf{X}] = \mathbb{E}[X_i | S_{\mathbf{X}}]$$

5. Aggregate risk-sharing rules

Classification of aggregate risk-sharing rules

- ▶ \mathbf{C} is **type 1 - aggregate** if there exists a function $\mathbf{h}^{(1)}$ such that for all \mathbf{X} , one has that:

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{(1)}(S_{\mathbf{X}}; F_{\mathbf{X}})$$

- ▶ \mathbf{C} is **type 2 - aggregate** if there exists a function $\mathbf{h}^{(2)}$ such that for all \mathbf{X} , one has that:

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{(2)}(S_{\mathbf{X}}; F_{X_1}, F_{X_2}, \dots, F_{X_n})$$

- ▶ \mathbf{C} is **type 3 - aggregate** if there exists a function $\mathbf{h}^{(3)}$ such that for all \mathbf{X} , one has that:

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{(3)}(S_{\mathbf{X}})$$

6. Properties of risk-sharing rules

Introduction

- ▶ **Premium principles**: have been studied in detail around the 'eighties'¹⁰.
- ▶ **Risk measures**: During the the 'nineties' and beyond, risk measures and their axiomatic characterizations have been studied extensively¹¹.
- ▶ **Risk-sharing rules**: Recently, the study of decentralized RS has (re-)gained interest.
 - ▶ Hereafter, we present a non-exhaustive list of properties that RS rules should/could satisfy¹².
 - ▶ Some properties are inspired by properties of premium principles or risk measures, other are tailored to RS rules.

¹⁰Goovaerts, De Vijlder & Haezendonck (1984) is a key reference.

¹¹Artzner, Delbaen, Eber & Heath (1997) is a fundamental paper.

¹²See D,D,R (2022).

6. Properties of risk-sharing rules

Classes of properties

- ▶ **Conservation properties**: also hold for the stand-alone RS rule: **reshuffling, normalization, translativity, positive homogeneity, constancy, no rip-off, actuarial fairness, law invariance.**
- ▶ **Improvement properties**: guarantee that the RS rule 'improves' the situation compared to the stand-alone situation: **willingness-to-join, comonotonicity.**
- ▶ **Coalition properties**: guarantee that participants are not affected by coalitions set up between other participants: **coalition fairness, merging fairness.**
- ▶ **Specific pool properties**: should hold (or not) for specific pools only: **stand-alone property for comonotonic pools, uniformity property for exchangeable pools.**

6.1. Conservation properties of risk-sharing rules

The 'reshuffling' property

- ▶ **Definition:**¹³ A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the reshuffling property if for any $\mathbf{X} \in \chi^n$ and any of its reshuffles \mathbf{X}^π , one has that

$$C_i [\mathbf{X}^\pi] = C_{\pi(i)} [\mathbf{X}] \quad \text{for any } i = 1, \dots, n$$

- ▶ **Example:**

- ▶ Consider $\mathbf{X} = (X_1, X_2, X_3)$ and $\mathbf{X}^\pi = (X_3, X_1, X_2)$.
- ▶ As $\pi(1) = 3$, the reshuffling property implies that

$$C_1 [\mathbf{X}^\pi] = C_3 [\mathbf{X}]$$

- ▶ **Interpretation :** If participants exchange their individual losses, then their contributions are changed in the same way.

¹³Jiao, Kou, Liu & Wang (2022), hereafter abbreviated as J,K,L,W (2022), call this property *symmetry*.

6.1. Conservation properties of risk-sharing rules

The 'reshuffling' property

- ▶ **Order statistics RS**: does not satisfy reshuffling,

$$C_1^{\text{ord}}[\mathbf{X}^\pi] = C_1^{\text{ord}}[X_3, X_1, X_2] = \min(X_1, X_2, X_3)$$

while

$$C_{\pi(1)}^{\text{ord}}[\mathbf{X}] = C_3^{\text{ord}}[X_1, X_2, X_3] = \max(X_1, X_2, X_3)$$

- ▶ **Conditional means RS**: satisfies reshuffling,

$$C_i^{\text{cm}}[\mathbf{X}^\pi] = \mathbb{E}\left[X_{\pi(i)} \mid S\right] = C_{\pi(i)}^{\text{cm}}[\mathbf{X}]$$

6.1. Conservation properties of risk-sharing rules

The 'reshuffling' property

► Theorem:

- Consider the RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$.
- \mathbf{C} is the **uniform RS rule** if and only if it satisfies the following axioms:

- (1) \mathbf{C} is a type 3 - aggregate RS rule:

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{(3)}(S_{\mathbf{X}})$$

- (2) \mathbf{C} satisfies reshuffling:

$$C_i[\mathbf{X}^\pi] = C_{\pi(i)}[\mathbf{X}]$$

► Proof:

- \Rightarrow : Straightforward.
- \Leftarrow For any reshuffle \mathbf{X}^π of \mathbf{X} and any participant i , we have:

$$C_i[\mathbf{X}] \stackrel{(\text{ax.1})}{=} h_i^{(3)}(S_{\mathbf{X}}) = h_i^{(3)}(S_{\mathbf{X}^\pi}) \stackrel{(\text{ax.1})}{=} C_i[\mathbf{X}^\pi] \stackrel{(\text{ax.2})}{=} C_{\pi(i)}[\mathbf{X}]$$

We can conclude that all $C_i[\mathbf{X}]$ are equal. ▽

6.1. Conservation properties of risk-sharing rules

The 'reshuffling' property

► Proposition:

The axioms (1) and (2) are **independent**:

For any choice of a particular axiom, there always exists a RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ which satisfies that axiom but not the other.

► Proof:

- The RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$, given by

$$\mathbf{C}[\mathbf{X}] = \left(\frac{X_1}{2} + \frac{S_{\mathbf{X}}}{2n}, \dots, \frac{X_n}{2} + \frac{S_{\mathbf{X}}}{2n} \right)$$

satisfies axiom (2), but not axiom (1).

- For $n \neq 2$, the RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ given by

$$\mathbf{C}[\mathbf{X}] = \left(\frac{S_{\mathbf{X}}}{2}, \frac{S_{\mathbf{X}}}{2(n-1)}, \dots, \frac{S_{\mathbf{X}}}{2(n-1)} \right)$$

satisfies axiom (1), but not axiom (2).



6.1. Conservation properties of risk-sharing rules

The 'normalization' property

- ▶ **Definition:**¹⁴ A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the **normalization** property if for any $\mathbf{X} \in \chi^n$, one has

$$X_i = 0 \Rightarrow C_i[\mathbf{X}] = 0$$

holds for any i .

- ▶ **Interpretation:** A participant with a zero loss r.v. has a zero contribution.
- ▶ **Remark:** If a RS rule satisfies the reshuffling property, then a sufficient condition for the normalization property to hold is that it holds for the last participant.

¹⁴J,K,L,W (2022) call this property *zero preserving*. Normalization as defined here is a weaker property than normalization defined in D,D,R (2022).

6.1. Conservation properties of risk-sharing rules

The 'normalization' property

- ▶ **Uniform RS**: does not satisfy normalization,

$$C_3^{\text{uni}} [X_1, X_2, 0] = \frac{X_1 + X_2}{3}$$

- ▶ **Conditional means RS**: satisfies normalization,

- ▶ The CMRS satisfies reshuffling.
- ▶ Furthermore,

$$C_n^{\text{cm}} [X_1, X_2, \dots, X_{n-1}, 0] = \mathbb{E} [0 \mid S] = 0$$

6.1. Conservation properties of risk-sharing rules

The 'translativity' property

- ▶ **Notation:** \mathbf{e}_j is the n -dimensional unit vector with all components equal to 0, except the j -th component which equals 1.
- ▶ **Definition:** A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the **translativity** property if for any $\mathbf{X} \in \chi^n$ and any participant $j = 1, \dots, n$, one has that

$$C_i[\mathbf{X} + c \mathbf{e}_j] = C_i[\mathbf{X}], \quad \text{for all } i \neq j \text{ and any } c \geq 0$$

- ▶ **Interpretation:** Increasing the loss of participant j by a deterministic amount $c \geq 0$, leaves the contributions of the other participants unchanged.
- ▶ **Consequence:** In case the RS rule \mathbf{C} satisfies the translativity property, then

$$C_j[\mathbf{X} + c \mathbf{e}_j] = C_j[\mathbf{X}] + c$$

6.1. Conservation properties of risk-sharing rules

The 'translativity' property

- ▶ **Remark:** If the RS rule satisfies the reshuffling property, then a sufficient condition for the translativity property to hold is that it holds for the last participant.
- ▶ **Uniform RS:** does not satisfy translativity,

$$C_1^{\text{uni}}[X_1, X_2, X_3 + c] = \frac{S + c}{3} \neq \frac{S}{3} = C_1^{\text{uni}}[X_1, X_2, X_3]$$

- ▶ **Conditional mean RS** satisfies translativity,
 - ▶ The CMRS satisfies reshuffling.
 - ▶ For any $i \neq n$, one has

$$C_i^{\text{cm}}[\mathbf{X} + c \mathbf{1}_n] = \mathbb{E}[X_i \mid |S + c] = \mathbb{E}[X_i \mid S] = C_i^{\text{cm}}[\mathbf{X}]$$

6.1. Conservation properties of risk-sharing rules

The 'positive homogeneity' property

► **Definition:**

A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the positive homogeneity property¹⁵ if for any $\mathbf{X} \in \chi^n$, one has that

$$C_i [c \mathbf{X}] = c \times C_i [\mathbf{X}]$$

holds for any i and any $c \geq 0$.

- **Interpretation:** Multiplying all losses by $c \geq 0$ implies that all contributions change in the same way.

¹⁵ $c \mathbf{X}$ stands for $(c \times X_1, c \times X_2, \dots, c \times X_n)$

6.1. Conservation properties of risk-sharing rules

The 'positive homogeneity' property

- ▶ Mean-proportional RS is positive homogeneous,

$$\begin{aligned} C_i^{\text{prop}} [c \mathbf{X}] &= \frac{\mathbb{E}[c X_i]}{\mathbb{E}[c S]} (c \times S) \\ &= c \frac{\mathbb{E}[X_i]}{\mathbb{E}[S]} S = c \times C_i^{\text{prop}} [\mathbf{X}] \end{aligned}$$

- ▶ Conditional mean RS is positive homogeneous,

$$C_i^{\text{cm}} [c \mathbf{X}] = \mathbb{E}[c \times X_i \mid c \times S] = c \mathbb{E}[X_i \mid S] = c \times C_i^{\text{cm}} [\mathbf{X}]$$

6.1. Conservation properties of risk-sharing rules

The 'constancy' property

- ▶ **Definition:**¹⁶ A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the constancy property if for any $\mathbf{X} \in \chi^n$, one has

$$X_j = c \Rightarrow C_j[\mathbf{X}] = c$$

holds for any j and any $c \geq 0$.

- ▶ **Interpretation:** In case a participant's loss is a non-negative constant, then his contribution is equal to that constant.
- ▶ **Remark:** If the RS rule satisfies the reshuffling property, then a sufficient condition for the constancy property to hold is that it holds for the last participant.

¹⁶Constancy as defined here is a weaker property than the constancy defined in D,D,R (2022). Constancy is called *constant preserving* in J,K,L,W (2022).

6.1. Conservation properties of risk-sharing rules

The 'constancy' property

- ▶ Uniform RS: does not satisfy constancy,

$$C_3^{\text{uni}}[X_1, X_2, c] = \frac{X_1 + X_2 + c}{3}$$

- ▶ Mean-proportional RS: does not satisfy no-ripoff,

$$C_3^{\text{prop}}[X_1, X_2, c] = \frac{c}{\mathbb{E}[S]} S$$

- ▶ Conditional mean RS: satisfies constancy,

$$C_n^{\text{cm}}[X_1, X_2, \dots, X_{n-1}, c] = \mathbb{E}[c | S] = c$$

6.1. Conservation properties of risk-sharing rules

The 'constancy' property

► **Proposition:**

If a RS rule satisfies **translativity** and **normalization**, then it also satisfies **constancy**.

► **Proof:**

- The case where $X_n = c$:

$$\begin{aligned} C_n[X_1, X_2, \dots, X_{n-1}, c] &= C_n[X_1, X_2, \dots, X_{n-1}, 0 + c] \\ &\stackrel{\text{transl.}}{=} C_n[X_1, X_2, \dots, X_{n-1}, 0] + c \\ &\stackrel{\text{normaliz.}}{=} c \end{aligned}$$

- The case where $X_j = c$ for $j \neq n$ is proven similarly. ∇

6.1. Conservation properties of risk-sharing rules

The 'no rip-off' property

- ▶ The 'largest value' of r.v. X :

$$F_X^{-1}(1) = \inf\{x \in \mathbb{R} \mid F_X(x) = 1\}$$

- ▶ Definition:¹⁷ A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the no rip-off property if for any $\mathbf{X} \in \chi^n$, one has that

$$C_i[\mathbf{X}] \leq F_{X_i}^{-1}(1)$$

holds for any $i = 1, 2, \dots, n$.

- ▶ Interpretation: The participant's contribution will never exceed his 'worst-case loss'.

¹⁷J,K,L,W (2022) call this property the 'risk fairness' axiom.

6.1. Conservation properties of risk-sharing rules

The 'no rip-off' property

- ▶ **Uniform RS**: does not satisfy no-riporff,¹⁸

$$C_1^{\text{uni}} [U, 2U] = \frac{3U}{2} \in [0, 1.5]$$

- ▶ **Proportional RS**: does not satisfy no-riporff,

$$C_3^{\text{prop}} [X_1, X_2, c] = \frac{c}{\mathbb{E}[S]} S$$

- ▶ **Conditional mean RS**: satisfies no rip-off,

$$C_i^{\text{cm}} [\mathbf{X}] = \mathbb{E}[X_i | S] \leq \mathbb{E}[F_{X_i}^{-1}(1) | S] = F_{X_i}^{-1}(1).$$

¹⁸In this text, the notation U is used exclusively for a r.v. which is uniformly distributed over $[0, 1]$.

6.1. Conservation properties of risk-sharing rules

The 'actuarial fairness' property

- ▶ **Definition:** A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies **actuarial fairness** if for any $\mathbf{X} \in \chi^n$, one has that

$$\mathbb{E} [C_i[\mathbf{X}]] = \mathbb{E} [X_i]$$

holds for any $i = 1, 2, \dots, n$.

- ▶ **Interpretation:** On average, participants do neither gain nor lose from risk sharing, in the sense that their expected contribution (by joining the pool) is equal to their expected loss (when staying alone).

6.1. Conservation properties of risk-sharing rules

The 'actuarial fairness' property

▶ **Uniform RS**: does not satisfy actuarial fairness,

- ▶ Consider the loss vector $(U, U, 4U)$.
- ▶ Expected loss of first participant:

$$\mathbb{E}[U] = \frac{1}{2}$$

- ▶ Expected contribution of first participant:

$$\mathbb{E}[C_1^{\text{uni}}[U, U, 4U]] = \mathbb{E}\left[\frac{U + U + 4U}{3}\right] = 1$$

▶ **Conditional mean RS**: satisfies actuarial fairness,

$$\mathbb{E}[C_i^{\text{cm}}[\mathbf{X}]] = \mathbb{E}[\mathbb{E}[X_i | S]] = \mathbb{E}[X_i]$$

6.1. Conservation properties of risk-sharing rules

The 'actuarial fairness' property

▶ **Proposition:**

If a RS rule satisfies **actuarial fairness** and **no rip-off**, then it also satisfies **constancy**.

▶ **Proof**

- ▶ Suppose $X_j = c$.
- ▶ Actuarial fairness implies

$$\mathbb{E} [C_i [\mathbf{X}]] = \mathbb{E} [X_j] = c$$

- ▶ No rip-off implies

$$C_i [\mathbf{X}] \leq F_{X_i}^{-1}(1) = c$$

- ▶ We can conclude that

$$C_i [\mathbf{X}] = c$$



6.1. Conservation properties of risk-sharing rules

The 'law-invariance' property

► **Definition:**

A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the **law-invariance** property in case

$$\mathbf{X} \stackrel{d}{=} \mathbf{Y} \Rightarrow \mathbf{C}[\mathbf{X}] \stackrel{d}{=} \mathbf{C}[\mathbf{Y}]$$

holds for any \mathbf{X} and $\mathbf{Y} \in \chi^n$.

► **Interpretation:**

If pools are 'equal in distribution', then also their contribution vectors are 'equal in distribution'.

6.1. Conservation properties of risk-sharing rules

The 'law-invariance' property

- ▶ **Uniform RS**: is law invariant: If $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$, then

$$\left(\frac{X_1 + \dots + X_n}{n}, \dots \right) \stackrel{d}{=} \left(\frac{Y_1 + \dots + Y_n}{n}, \dots \right)$$

- ▶ **Conditional means RS**: is law invariant: If $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$, then

$$(\mathbb{E}[X_1 | S_{\mathbf{X}}], \dots, \mathbb{E}[X_n | S_{\mathbf{X}}]) \stackrel{d}{=} (\mathbb{E}[Y_1 | S_{\mathbf{Y}}], \dots, \mathbb{E}[Y_n | S_{\mathbf{Y}}])$$

6.2. Improvement properties of risk-sharing rules

Classes of properties

- ▶ **Conservation properties**: also hold for the stand-alone RS rule: **reshuffling, normalization, translativity, positive homogeneity, constancy, no rip-off, actuarial fairness, law invariance.**
- ▶ **Improvement properties**: guarantee that the RS rule 'improves' the situation compared to the stand-alone situation: **willingness-to-join, comonotonicity.**
- ▶ **Coalition properties**: guarantee that participants are not affected by coalitions between other participants **coalition fairness, merging fairness.**
- ▶ **Specific pool properties**: should hold (or not) for specific pools only: **stand-alone property for comonotonic pools, uniformity property for exchangeable pools.**

6.2. Improvement properties of risk-sharing rules

Increasing convex order

- ▶ Consider the r.v.'s X and Y .
- ▶ **Definition:**
 X is smaller than Y in stop-loss order¹⁹ if for any non-decreasing concave function $u : \mathbb{R} \rightarrow \mathbb{R}$ and any real w , one has that

$$\mathbb{E}[u(w - X)] \geq \mathbb{E}[u(w - Y)]$$

- ▶ **Interpretation in expected utility theory:**
Risk-averse decision makers prefer loss X over loss Y .
- ▶ **Notation:**

$$X \leq_{sl} Y$$

¹⁹'Stop-loss order' is also called 'increasing convex order'.

6.2. Improvement properties of risk-sharing rules

The 'willingness-to-join' property

► **Definition:**

A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the willingness-to-join property²⁰ if for any $\mathbf{X} \in \chi^n$, one has that

$$C_i[\mathbf{X}] \leq_{sl} X_i$$

holds for any $i = 1, 2, \dots, n$.

- **Interpretation:** Any risk-averse participant i will prefer the contribution $C_i[\mathbf{X}]$ over his initial loss X_i .

²⁰In J,K,L,W (2022) this property is called the *universal improvement property*.

6.2. Improvement properties of risk-sharing rules

The 'willingness-to-join' property

- ▶ **Proposition:** If a RS rule satisfies **willingness-to-join**, then it also satisfies **actuarial fairness**, **no rip-off** and **constancy**.

- ▶ **Proof:**

- ▶ For any i , willingness-to-join implies

$$\mathbb{E}[C_i(\mathbf{X})] \leq \mathbb{E}[X_i]$$

The full allocation condition states that

$$\sum_{i=1}^n \mathbb{E}[C_i(\mathbf{X})] = \sum_{i=1}^n \mathbb{E}[X_i]$$

We can conclude that \mathbf{C} is actuarially fair:

$$\mathbb{E}[C_i(\mathbf{X})] = \mathbb{E}[X_i]$$

- ▶ Willingness-to-join implies no rip-off:

$$C_i(\mathbf{X}) \leq F_{C_i(\mathbf{X})}^{-1}(1) \leq F_{X_i}^{-1}(1)$$

- ▶ Actuarial fairness and no rip-off imply constancy.



6.2. Improvement properties of risk-sharing rules

The 'willingness-to-join' property

- ▶ **Uniform RS**: does not satisfy willingness-to-join, since it is not an actuarially fair RS rule.
- ▶ **Proportional RS**: does not satisfy willingness-to-join, since it does not satisfy the no rip-off property.
- ▶ **Conditional mean RS**: satisfies willingness-to-join:

$$\mathbb{E}[X_i | S] \leq_{sl} X_i$$

6.2. Improvement properties of risk-sharing rules

Comonotonicity

- ▶ Consider the pool $\mathbf{X} = (X_1, \dots, X_n)$ with aggregate loss $S_{\mathbf{X}}$.
- ▶ **Definition:** \mathbf{X} is **comonotonic** if there exist non-decreasing functions $g_j : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\mathbf{X} = (g_1(S_{\mathbf{X}}), \dots, g_n(S_{\mathbf{X}})) \quad (1)$$

- ▶ **Interpretation:** \mathbf{X} is comonotonic if the increase of one of the individual losses implies an increase of all the other individual losses.

6.2. Improvement properties of risk-sharing rules

The 'comonotonicity' property

- ▶ **Definition:** A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ is **comonotonic** if for any $\mathbf{X} \in \chi^n$, one has that $\mathbf{C}[\mathbf{X}]$ is a comonotonic random vector.
- ▶ **Equivalent definition:** A RS rule \mathbf{C} is **comonotonic** if for any $\mathbf{X} \in \chi^n$, there exists a function $\mathbf{h}^{\text{com}} : \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$\mathbf{C}[\mathbf{X}] = \mathbf{h}^{\text{com}}(S_{\mathbf{X}}) = (h_1^{\text{com}}(S_{\mathbf{X}}), \dots, h_n^{\text{com}}(S_{\mathbf{X}}))$$

where all functions h_i^{com} are non-decreasing.

- ▶ **Interpretation:**

Comonotonicity of a RS rule ensures that any participant has an interest in keeping his loss as small as possible²¹.

²¹For this reason, Carlier & Dana (2003) call this property the *no-sabotage condition*.

6.2. Improvement properties of risk-sharing rules

The 'comonotonicity' property

▶ **Remark:**

- ▶ **C** is **comonotonic** \Rightarrow **C** is **aggregate** \Rightarrow **C** is **internal**.
- ▶ **C** is **internal** $\not\Rightarrow$ **C** is **aggregate** $\not\Rightarrow$ **C** is **comonotonic**.

- ▶ **Uniform RS: comonotonic.** Indeed, for any **X**, one has that $\mathbf{C}^{\text{uni}}[\mathbf{X}]$ is comonotonic:

$$\mathbf{C}^{\text{uni}}[\mathbf{X}] = \left(\frac{S_{\mathbf{X}}}{n}, \dots, \frac{S_{\mathbf{X}}}{n} \right)$$

- ▶ **Conditional mean RS: aggregate but not comonotonic** since the contribution vector

$$\mathbf{C}^{\text{cm}}[\mathbf{X}] = (\mathbb{E}[X_1 | S_{\mathbf{X}}], \dots, \mathbb{E}[X_n | S_{\mathbf{X}}])$$

is not necessarily comonotonic.

6.3. Coalition properties of risk-sharing rules

Classes of properties

- ▶ **Conservation properties**: also hold for the stand-alone RS rule: **reshuffling, normalization, translativity, positive homogeneity, constancy, no rip-off, actuarial fairness, law invariance.**
- ▶ **Improvement properties**: guarantee that the RS rule 'improves' the situation compared to the stand-alone situation: **willingness-to-join, comonotonicity.**
- ▶ **Coalition properties**: guarantee that participants are not affected by coalitions between other participants: **coalition fairness, merging fairness.**
- ▶ **Specific pool properties**: should hold (or not) for specific pools only: **stand-alone property for comonotonic pools, uniformity property for exchangeable pools.**

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

- ▶ Suppose that the participants in the pool (X_1, X_2, X_3) agree to use the RS rule **C** to determine their contributions.
- ▶ Next, suppose that participants 2 and 3 decide to form a coalition, under which they equally share their joint loss $(X_2 + X_3)$.
- ▶ This coalition transforms the pool (X_1, X_2, X_3) into

$$\left(X_1, \frac{X_2 + X_3}{2}, \frac{X_2 + X_3}{2}\right)$$

- ▶ It seems reasonable to require that the contribution of participant 1, who is excluded from this coalition, remains unchanged.
- ▶ RS rules satisfying this property will be said to satisfy the coalition fairness property.

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

► **Recall:**

A **reallocation** \mathbf{Y} of a pool $\mathbf{X} \in \chi^n$ is an element of the following set:

$$\mathcal{A}_{\mathbf{X}} = \left\{ (Y_1, Y_2, \dots, Y_n) \in \chi^n \mid \sum_i^n Y_i = S_{\mathbf{X}} \right\}$$

► **Definition:**

- Consider a group of n participants and a pool $\mathbf{X} \in \chi^n$.
- A **coalition** 'excluding participant i ' is a reallocation $\mathbf{Y} \in \mathcal{A}_{\mathbf{X}}$ of \mathbf{X} , with $Y_i = X_i$.

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

▶ **Definition:**

A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the **coalition fairness** property if for any $\mathbf{X} \in \chi^n$ and any $\mathbf{Y} \in \mathcal{A}_{\mathbf{X}}$, one has that

$$Y_i = X_i \Rightarrow C_i[\mathbf{Y}] = C_i[\mathbf{X}]$$

holds for any i .

▶ **Interpretation:**

- ▶ The contribution of a participant in a pool is not affected by coalitions between other participants in this pool.
- ▶ A coalition between a group of participants does not change the joint contributions of this group.

- ▶ **Remark:** If a RS rule satisfies *reshuffling*, then a sufficient condition for *coalition fairness* to hold is that it holds for the last participant.

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

- ▶ Consider the pools \mathbf{X} and \mathbf{Y} in $\mathcal{A}_{\mathbf{X}}$, with $Y_i = X_i$.
- ▶ Uniform RS: is **coalition fair**. Indeed,

$$C_i^{\text{uni}}[\mathbf{Y}] = \frac{S_{\mathbf{Y}}}{n} = \frac{S_{\mathbf{X}}}{n} = C_i^{\text{uni}}[\mathbf{X}]$$

- ▶ Mean-proportional RS: is **coalition fair**. Indeed,

$$C_i^{\text{prop}}[\mathbf{Y}] = \frac{\mathbb{E}[Y_i]}{\mathbb{E}[S_{\mathbf{Y}}]} S_{\mathbf{Y}} = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{\mathbf{X}}]} S_{\mathbf{X}} = C_i^{\text{prop}}[\mathbf{X}]$$

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

- ▶ Consider the pools \mathbf{X} and \mathbf{Y} in $\mathcal{A}_{\mathbf{X}}$, with $Y_i = X_i$.
- ▶ **Conditional Mean RS**: is **coalition fair**. Indeed,

$$C_i^{\text{cm}}[\mathbf{Y}] = \mathbb{E}[Y_i | S_{\mathbf{Y}}] = \mathbb{E}[X_i | S_{\mathbf{X}}] = C_i^{\text{cm}}[\mathbf{X}]$$

- ▶ **Order statistics RS**: is not coalition fair. Indeed, consider

$$(X_1, X_2, X_3) \text{ and } (X_1, X_2 + X_3, 0)$$

One has that

$$C_1^{\text{ord}}[X_1, X_2, X_3] = \min\{X_1, X_2, X_3\}$$

while

$$C_1^{\text{ord}}[X_1, X_2 + X_3, 0] = \min\{X_1, X_2 + X_3, 0\}$$

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

► Proposition:

A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies **coalition fairness**

\Leftrightarrow for any $\mathbf{X} \in \chi^n$ and any participant i , one has that

$$C_i[\mathbf{X}] = C_i[X_i \mathbf{e}_i + (S_{\mathbf{X}} - X_i) \mathbf{e}_j] \quad \text{for any } j \neq i$$

► Proposition:

If a RS rule satisfies **normalization** and **coalition fairness**, then it also satisfies **reshuffling**.

6.4. Specific pool properties of risk-sharing rules

Classes of properties

- ▶ **Conservation properties**: also hold for the stand-alone RS rule: **reshuffling, normalization, translativity, positive homogeneity, constancy, no rip-off, actuarial fairness, law invariance.**
- ▶ **Improvement properties**: guarantee that the RS rule 'improves' the situation compared to the stand-alone situation: **willingness-to-join, comonotonicity.**
- ▶ **Coalition properties**: guarantee that participants are not affected by coalitions between other participants: **coalition fairness, merging fairness.**
- ▶ **Specific pool properties**: should hold (or not) for specific pools only: **stand-alone property for comonotonic pools, uniformity property for exchangeable pools.**

6.4. Specific pool properties of risk-sharing rules

The 'stand-alone for comonotonic pools' property

- ▶ **Definition:** A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the stand-alone property for comonotonic pools if for any comonotonic pool $\mathbf{X} \in \chi^n$, one has that

$$\mathbf{C}[\mathbf{X}] = \mathbf{X}$$

- ▶ **Interpretation:** In a comonotonic pool, no diversification benefit arises from risk-sharing. Therefore, it may be reasonable to require that in such a pool each participant remains with his own risk.

6.4. Specific pool properties of risk-sharing rules

The 'stand-alone for comonotonic pools' property

- ▶ **Uniform RS**: does not satisfy stand-alone for comonotonic pools.

Indeed, for the comonotonic pool $\mathbf{X} = (U, 2U, U)$, one has

$$C_1^{\text{uni}}[U, 2U, U] = \frac{4}{3}U \neq U$$

- ▶ **Conditional mean RS**: satisfies stand-alone for comonotonic pools.

Indeed, for the comonotonic pool $\mathbf{X} = [g_1(S_{\mathbf{X}}), \dots, g_n(S_{\mathbf{X}})]$, with all g_i non-decreasing functions, one has

$$C_i^{\text{cm}}[\mathbf{X}] = \mathbb{E}[X_i | S_{\mathbf{X}}] = \mathbb{E}[g_i(S_{\mathbf{X}}) | S_{\mathbf{X}}] = g_i(S_{\mathbf{X}}) = X_i$$

6.4. Specific pool properties of risk-sharing rules

The 'uniformity for exchangeable pools' property

► **Definition:**

$\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies uniformity for exchangeable pools if for any exchangeable pool $\mathbf{X} \in \chi^n$, one has:

$$\mathbf{C}[\mathbf{X}] = \left(\frac{S_{\mathbf{X}}}{n}, \frac{S_{\mathbf{X}}}{n}, \dots, \frac{S_{\mathbf{X}}}{n} \right)$$

► **Interpretation:**

- The exchangeable pool \mathbf{X} is **homogeneous** in the sense that the joint cdf of \mathbf{X} is not changed by reshuffling.
- In this case, the uniform RS rule seems to be appropriate.

6.4. Specific pool properties of risk-sharing rules

The 'uniformity for exchangeable pools' property

- ▶ **Order statistics RS**: does not satisfy uniformity for exchangeable pools.

Indeed, for any exchangeable pool \mathbf{X} , one has that

$$C_1^{\text{ord}}[\mathbf{X}] = \min \{X_1, \dots, X_n\}$$

which is not necessarily equal to $\frac{S_{\mathbf{X}}}{n}$.

- ▶ **Conditional mean RS**: satisfies uniformity for exchangeable pools.

Indeed, for any exchangeable pool \mathbf{X} and any i one has that

$$C_i^{\text{cm}}[\mathbf{X}] = \mathbb{E}[X_i | S_{\mathbf{X}}] = \frac{S_{\mathbf{X}}}{n}$$

7. Axiomatic characterization of the CMRS rule

- ▶ Consider the **CMRS rule** $\mathbf{C}^{\text{cm}} : (L_+^1)^n \rightarrow (L_+^1)^n$, with

$$\mathbf{C}^{\text{cm}}[\mathbf{X}] = \mathbb{E}[\mathbf{X} \mid S_{\mathbf{X}}]$$

- ▶ We have proven that \mathbf{C}^{cm} satisfies the following properties:
 - ▶ \mathbf{C}^{cm} is an **aggregate RS rule**:

$$\mathbf{C}^{\text{cm}}[\mathbf{X}] \text{ is } S_{\mathbf{X}} \text{ - measurable}$$

- ▶ \mathbf{C}^{cm} satisfies **no rip-off**: For any i ,

$$C_i^{\text{cm}}[\mathbf{X}] \leq F_{X_i}^{-1}(1)$$

- ▶ \mathbf{C}^{cm} satisfies **actuarial fairness**: For any i ,

$$\mathbb{E}[C_i^{\text{cm}}[\mathbf{X}]] = \mathbb{E}[X_i]$$

- ▶ \mathbf{C}^{cm} satisfies **coalition fairness**: For any $\mathbf{Y} \in \mathcal{A}_{\mathbf{X}} \left((L_+^1)^n \right)$,

$$Y_i = X_i \Rightarrow C_i^{\text{cm}}[\mathbf{Y}] = C_i^{\text{cm}}[\mathbf{X}]$$

7. Axiomatic characterization of the CMRS rule

► Theorem:

- Consider the RS rule $\mathbf{C} : (L_+^1)^n \rightarrow (L_+^1)^n$, with $n \geq 3$.
- \mathbf{C} is the **CMRS rule** if and only if it satisfies the following axioms:
 - (1) \mathbf{C} is an aggregate RS rule.
 - (2) \mathbf{C} satisfies no rip-off.
 - (3) \mathbf{C} satisfies actuarial fairness.
 - (4) \mathbf{C} satisfies coalition fairness.

► Proof:

- \Rightarrow : See Denuit, Dhaene & Robert (2022).
- \Leftarrow ²² : See Jiao, Kou, Liu & Wang (2022).



²²This is the hard part of the proof.

7. Axiomatic characterization of the CMRS rule

► Proposition²³:

The axioms (1), (2), (3) and (4) are **independent**.

- Independence means that for any $n \geq 3$ and any choice of 3 of these axioms, there always exists a RS rule $\mathbf{C} : (L_+^1)^n \rightarrow (L_+^1)^n$ which satisfies these 3 axioms, but not the 4th axiom.

²³See J,K,L,W (2022).

7. Axiomatic characterization of the CMRS rule

▶ **Theorem**²⁴:

- ▶ Consider the RS rule $\mathbf{C} : (L_+^1)^n \rightarrow (L_+^1)^n$, with $n \geq 3$.
- ▶ The RS rule $\mathbf{C} : (L_+^1)^n \rightarrow (L_+^1)^n$ is the **CMRS rule** if and only if it satisfies the following axioms:

(1') \mathbf{C} is an aggregate RS rule.

(2') \mathbf{C} satisfies willingness-to-join.

(3') \mathbf{C} satisfies coalition fairness.

▶ **Proof**:

- ▶ \Rightarrow : Follows immediately.
- ▶ \Leftarrow : Follows from property that 'willingness-to-join' implies 'actuarial fairness' and 'no rip-off'.



²⁴See J,K,L,W (2022).

8. Further topics (not covered in this talk)

▶ Quantile risk-sharing²⁵:

- ▶ The QRS rule is defined such that the contribution of each participant i is given by $F_{X_i}^{-1}(p)$.
- ▶ Probability level p is determined such that the full allocation condition is satisfied.

▶ Pareto-optimal risk-sharing²⁶:

- ▶ What is the relation between Pareto-optimal risk-sharing and comonotonicity?

▶ Risk-sharing pool dynamics²⁷:

- ▶ What if we add or withdraw one participant from the pool?²⁸
- ▶ What if the number of participants in the pool approaches infinity?

²⁵See D,D,R (2022) and D,R,C,D (2023).

²⁶see D,D,G,R (2023).

²⁷Work in progress.

²⁸See D,D,R (2022).

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