INTEGRATION BEE SOLUTIONS

1. $$\int_{\pi^e}^{e^\pi} 1^x \, dx$$

$$1^x = 1 \quad \forall \quad x \in (\pi^e, e^\pi)$$

$$\Rightarrow I = \int_{\pi^e}^{e^\pi} dx = \boxed{e^\pi - \pi^e}$$

2. $\dfrac{dA}{dx} = \boxed{0}$ since A is value of a definite integral.

3. $$\int \frac{61}{60\sin x + 11\cos x} \, dx$$

$$= \int \frac{1}{\frac{60}{61}\sin x + \frac{11}{61}\cos x} \, dx$$

Let $\dfrac{60}{61} = \sin\theta \Rightarrow \cos\theta = \dfrac{11}{61}$ $\quad [11^2 + 60^2 = 61^2]$

$$I = \int \frac{1}{\sin\theta \sin x + \cos\theta\cos x} \, dx$$

$$= \int \frac{1}{\cos(x-\theta)} \, dx$$

$$= \int \sec(x-\theta) \, dx$$

$= \log_e |\sec(x-\theta) + \tan(x-\theta)| + C$

$\sin\theta = \frac{60}{61} \Rightarrow \theta = \sin^{-1}\left(\frac{60}{61}\right)$

$$I = \boxed{\log_e \left| \sec\left[x - \sin^{-1}\left(\frac{60}{61}\right)\right] + \tan\left[x - \sin^{-1}\left(\frac{60}{61}\right)\right] \right| + C}$$

$$\int \frac{1}{x^{1729} + x} \, dx$$

$$= \int \frac{1}{x^{1729}\left(1 + \frac{1}{x^{1728}}\right)} \, dx$$

Let

$1 + \frac{1}{x^{1728}} = t$

$\Rightarrow \frac{-1728}{x^{1729}} \, dx = dt$

$\frac{1}{x^{1729}} \, dx = \frac{-1}{1728} \, dt$

$I = \frac{-1}{1728} \int \frac{1}{t} \, dt$

$= \frac{-1}{1728} \log_e |t| + C$

$$I = \boxed{-\frac{1}{1728}\log_e\left|1+\frac{1}{x^{1728}}\right| + C}$$

$$\int \sin(\sin(\sin(\ldots(x)))) \, dx$$

Let $\sin(\sin(\sin(\ldots(x)))) = k$

$\Rightarrow \quad \sin k = k$

only one solution exists $(k=0)$.

$\Rightarrow$ ~~sin(sin(~~ $\sin(\sin(\sin(\ldots(x)))) = 0$

$$I = \int 0 \, dx = C$$

$$\int \frac{2000x^{2018} - 18}{x^{2019} + x} \, dx$$

$$\int \frac{2019x^{2018} - 19x^{2018} - 19 + 1}{x^{2019} + x} \, dx$$

$$= \int \frac{2019x^{2018} + 1}{x^{2019} + x} \, dx - 19\int \frac{\cancel{(x^{2018}+1)}}{x\cancel{(x^{2018}+1)}} \, dx$$

$$= \log_e\left|x^{2019} + x\right| - 19\log_e|x| + C$$

$$= \boxed{\log_e\left|\frac{x^{2019}+x}{x^{19}}\right| + C}$$

7. $\int \frac{x^{-7/12}}{\sqrt[3]{x} + \sqrt[4]{x}} \, dx$

Let $x^{1/12} = t \Rightarrow t^{-7} = x^{-7/12}$

$\Rightarrow x = t^{12} \Rightarrow \sqrt[3]{x} = t^4, \; \sqrt[4]{x} = t^3$

$dx = 12 t^{11}$

$$I = 12 \int \frac{t^{11} \cdot t^{-7}}{t^4 + t^3} \, dt$$

$$= 12 \int \frac{t^{\cancel{4}}}{\cancel{t^3}(t+1)} \, dt$$

$$= 12 \int \frac{t+1-1}{t+1} \, dt$$

$$= 12 \int dt - \int \frac{1}{t+1} \, dt$$

$$= 12t - 12 \log_e |t+1| + C$$

$$= \boxed{12 x^{1/12} - 12 \log_e |x^{\frac{1}{12}} + 1| + C}$$

8. $$\int \frac{\sec(x-A)}{\sqrt{1-\sin^2(x-B)}}\,dx$$

$$= \int \frac{1}{\cos(x-A)\cos(x-B)}\,dx$$

$$= \frac{1}{\sin(B-A)} \int \frac{\sin(B-A)}{\cos(x-A)\cos(x-B)}\,dx$$

$$= \frac{1}{\sin(B-A)} \int \frac{\sin[(x-A)-(x-B)]}{\cos(x-A)\cos(x-B)}\,dx$$

$$= \frac{1}{\sin(B-A)} \int \frac{\sin(x-A)\cos(x-B) - \cos(x-A)\sin(x-B)}{\cos(x-A)\cos(x-B)}\,dx$$

$$= \frac{1}{\sin(B-A)} \int [\tan(x-A) - \tan(x-B)]\,dx$$

$$\boxed{= \frac{1}{\sin(B-A)}\left[\log_e\left|\frac{\sec(x-A)}{\sec(x-B)}\right|\right] + C}$$

9. $\int \operatorname{cosec}^3(x)\sec(x)\,dx$

$= \int \frac{1}{\sin^3 x \cos x}\,dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos x}\,dx$

$= \int \frac{\sin^2 x}{\sin^3 x \cos x}\,dx + \int \frac{\cos^2 x}{\sin^3 x \cos x}\,dx$

$= 2\int \frac{1}{2\sin x \cos x}\,dx + \int \frac{1}{\sin^3 x}\cos x\,dx$

$= 2\int \operatorname{cosec}(2x)\,dx + \int \frac{1}{t^3}\,dt$ $\left[\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array}\right]$

$= \boxed{\log_e|\tan x| - \frac{1}{2\sin^2 x} + C}$

10. $\int \frac{x\log_e x\,\operatorname{sech}^2(x) - \tanh(x)}{x(\log x)^2}\,dx$

$= \int \frac{\log_e x\,\operatorname{sech}^2(x) - \frac{\tanh(x)}{x}}{(\log x)^2}\,dx$

This is of form $\frac{fg' - gf'}{(f)^2} = \left(\frac{g}{f}\right)'$ with $f(x) = \log x$, $g(x) = \tanh(x)$

$\therefore I = \boxed{\frac{\tanh(x)}{\log_e(x)} + C}$

11. $\int e^{nix}\,dx = \frac{1}{ni} e^{nix} + c$

$= \frac{i^3}{n} (e^{ix})^n + c$

$= \frac{-i}{n} [\cos x + i\sin x]^n + c$ [Euler's formul

$= \frac{-i}{n} [\cos(nx) + i\sin(nx)] + c$ [De-Moivre]

$= \frac{1}{n} [-i\cos(nx) - i^2 \sin(nx)] + c$

$= \frac{1}{n} [\sin(nx) - i\cos(nx)] + c$

Hence proved

Q.E.D.

12. $f(x) = \sum_{j=1}^{n} x^{j-1} = \frac{1-x^n}{1-x}$

$g(x) = \sum_{k=1}^{n} \left[kx^{k-2} - \frac{h'(x)}{(k+1)x^2} \right]$

$h(x) = x^{k+1}$

$\rightarrow h'(x) = (k+1)\,x^k$

$\rightarrow g(x) = \sum_{k=1}^{n} \left[kx^{k-2} - \frac{\cancel{(k+1)}\,x^k}{\cancel{(k+1)}\,x^2} \right]$

$$\therefore\ g(x) = \sum_{k=1}^{n} x^{k-2}(k-1)$$

Observe

$$f'(x) = \sum_{j=1}^{n} (j-1)\, x^{j-2}$$

$$\Rightarrow f'(x) = g(x)$$

$$\therefore \int g(x)\,dx = \int f'(x)\,dx$$

$$= f(x) + C$$

$$\cancel{g(x)} = f(x) = 1 + x + x^2 + \cdots + x^{n-1}$$

$$g(x) = 0 + 1 + 2x + \cdots + (n-1)x^{n-2}$$

$$\Rightarrow 1 + g(x) = 1 + [\overbrace{1 + 2x + \cdots + (n-1)x^{n-2}}^{f'(x)}]$$

$$\therefore 1 + \int g(x)\,dx = \boxed{\frac{1-x^n}{1-x}}$$

13. $\displaystyle\int_0^1 \frac{(x-x^2)^4}{1+x^2}\,dx$

$$= \int_0^1 \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{1+X^2}\,dx.$$

$$\begin{array}{r|l}
 & X^6 - 4X^5 + 5X^4 - 4X^2 + 4 \\
\hline
1+X^2 & \cancel{x^8} - 4x^7 + 6x^6 - 4x^5 + X^4 \\
 & \cancel{X^8} \quad \overset{(-)}{+} X^6 \\
\hline
 & -\cancel{4X^7} + 5X^6 - \cancel{4X^5} + X^4 \\
 & -\cancel{4X^7} \quad -\cancel{4X^5} \\
\hline
 & \cancel{5X^6} + X^4 \\
 & \cancel{5X^6} \overset{(-)}{+} 5X^4 \\
\hline
 & -\cancel{4X^4} \\
 & -\cancel{4X^4} \overset{(+)}{-} 4X^2 \\
\hline
 & \cancel{4X^2} \\
 & \cancel{4X^2} + 4 \\
\hline
 & -4 \\
\hline
\end{array}$$

$$I = \int_0^1 \left[x^6 - 4X^5 + 5X^4 - 4X^2 + 4 - \frac{4}{1+X^2}\right] dx$$

$$= \frac{1}{7} - \frac{4}{6} + \frac{5}{5} - \frac{4}{3} + 4 - 4\tan^{-1}(1)$$

$$= \frac{1}{7} - \frac{2}{3} - \frac{4}{3} + 5 - \cancel{4}\frac{\pi}{\cancel{4}}$$

$$= \frac{1}{7} + 3 - \pi \quad = \boxed{\frac{22}{7} - \pi}$$

14. $$I = \int_3^7 \frac{\log(x+2)}{\log(24+10x-x^2)}\,dx$$

$$I = \int_3^7 \frac{\log(x+2)}{\log[(12-x)(2+x)]}\,dx$$

$$= \int_3^7 \frac{\log(x+2)}{\log(x+2)+\log(12-x)}\,dx \quad — (1)$$

$$I = \int_3^7 \frac{\log(7+3+2-x)}{\log(7+3+2-x)+\log(12-7-3+x)}\,dx$$

$$= \int_3^7 \frac{\log(12-x)}{\log(12-x)+\log(2+x)}\,dx \quad — (2)$$

(1) + (2),

$$2I = \int_3^7 \frac{\cancel{\log(12-x)+\log(x+2)}}{\cancel{\log(12-x)+\log(x+2)}}\,dx$$

$$I = \frac{1}{2}\int_3^7 dx$$

$$= \frac{1}{2}(7-3)$$

$$\boxed{I = 2}$$

15. $\int_{-\infty}^{\infty} e^{b+at-t^2}\, dt$

$$= e^b \int_{-\infty}^{\infty} e^{at-t^2}\, dt$$

$$= e^b \int_{-\infty}^{\infty} e^{\frac{a^2}{4} - [t^2 - 2(\frac{a}{2})t + \frac{a^2}{4}]}\, dt$$

$$= e^b \int_{-\infty}^{\infty} e^{\frac{a^2}{4} - (t - a/2)^2}\, dt$$

$$t - a/2 = z$$
$$dt = dz$$
$$t: -\infty \to \infty$$
$$z: -\infty \to \infty$$

$$I = e^b \int_{-\infty}^{\infty} e^{\frac{a^2}{4} - z^2}\, dz$$

$$= e^b \cdot e^{\frac{a^2}{4}} \int_{-\infty}^{\infty} e^{-z^2}\, dz$$

Given $\operatorname{erf}(\infty) = 1$

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2}\, dt$$

$$\operatorname{erf}(\infty) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-t^2}\, dt = \int_{-\infty}^{\infty} e^{-z^2}\, dz \quad \left[\int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right.$$

$$\therefore I = \left(e^{b+\frac{a^2}{4}}\right)\sqrt{\pi}$$

$e^{b+\frac{a^2}{4}} \in \mathbb{R}_+ \ \forall\ a, b \in \mathbb{R}$ $\therefore$ I is always real, +ve multiple of $\sqrt{\pi}$.