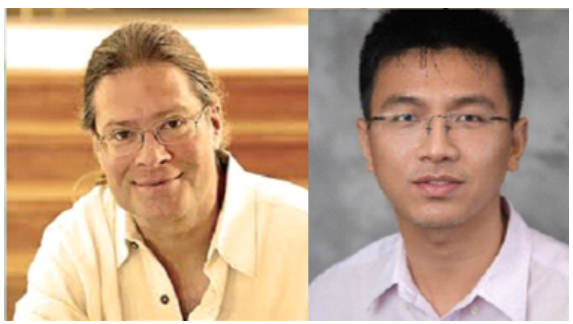
The Planted Matching Problem: Phase Transition and Exact Results

Mehrdad Moharrami

Collaborators: Cristopher Moore and Jiaming Xu



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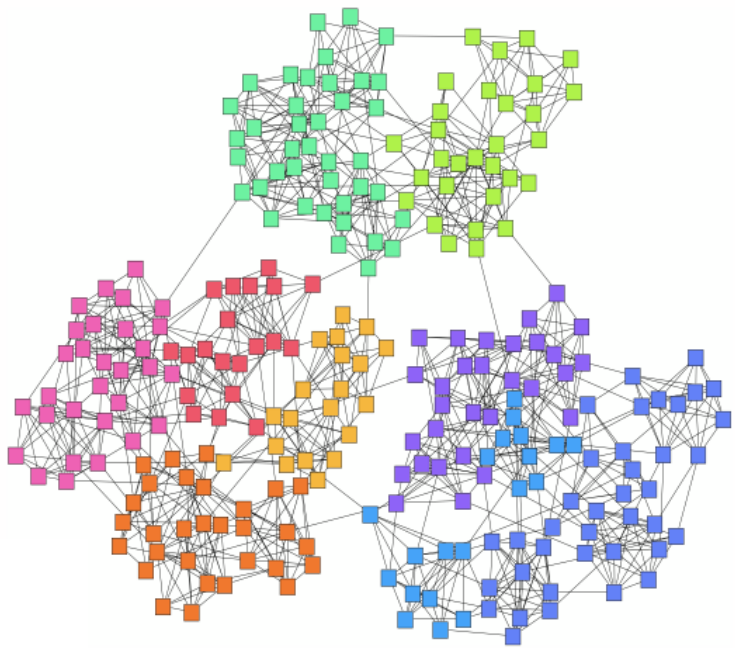

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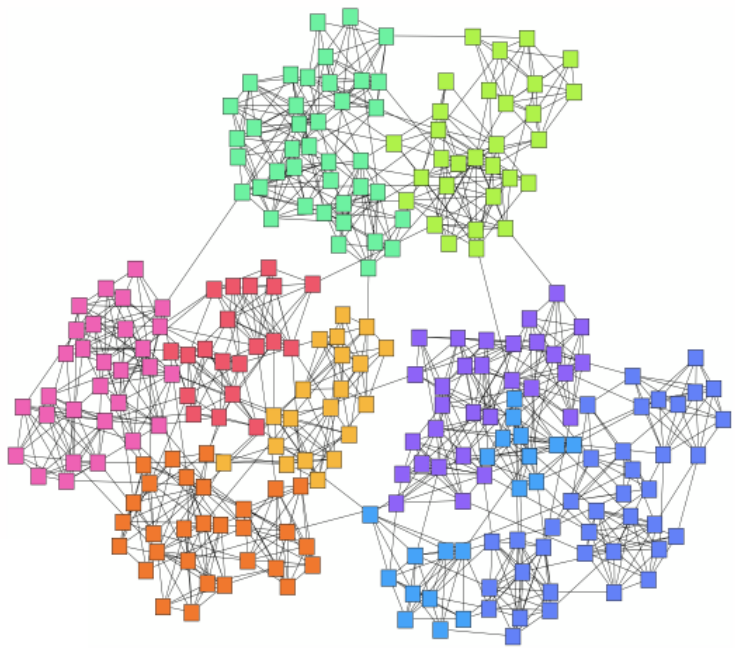
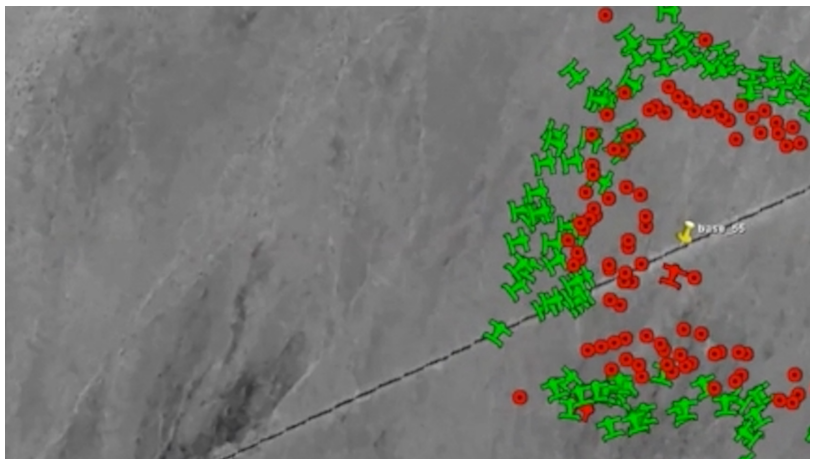
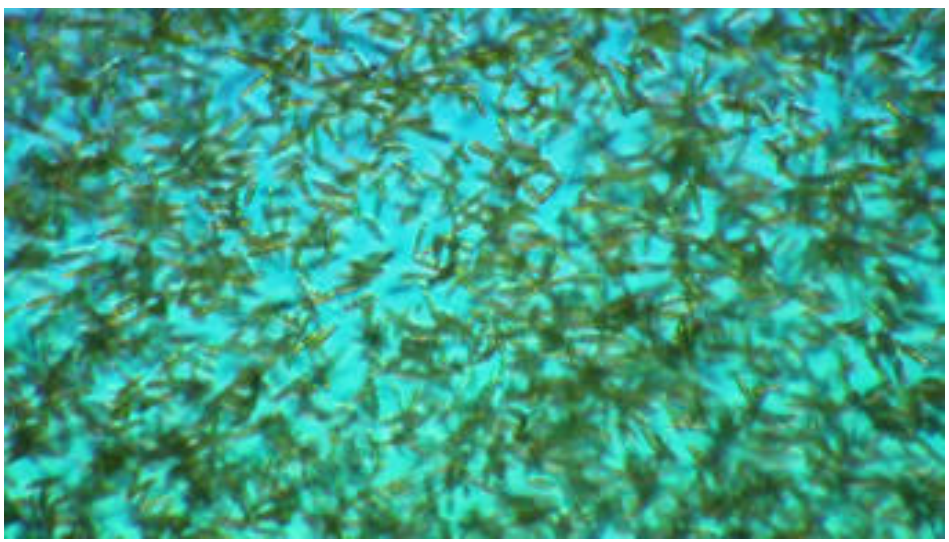


Information-theoretic (a.k.a. statistical) and computational barriers

Particle Tracking/Mobile Object





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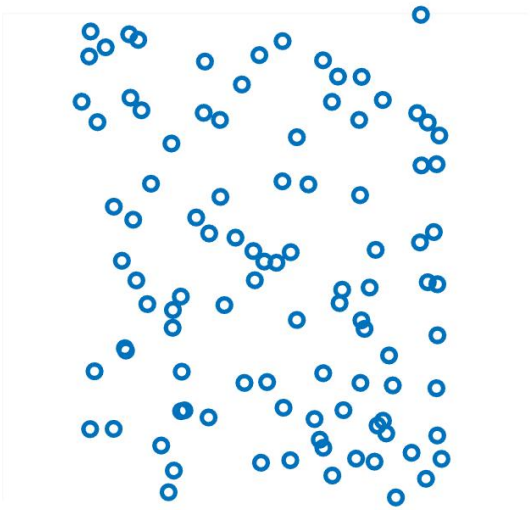

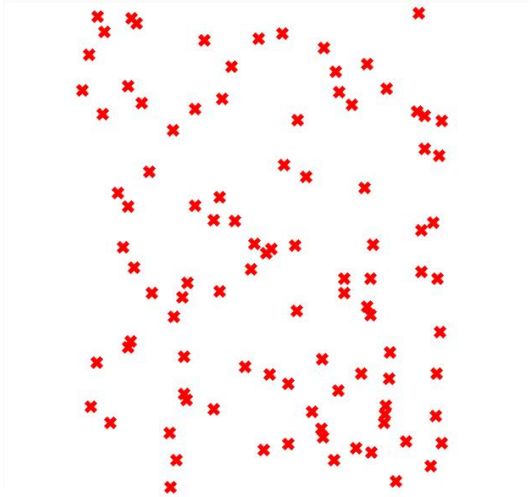

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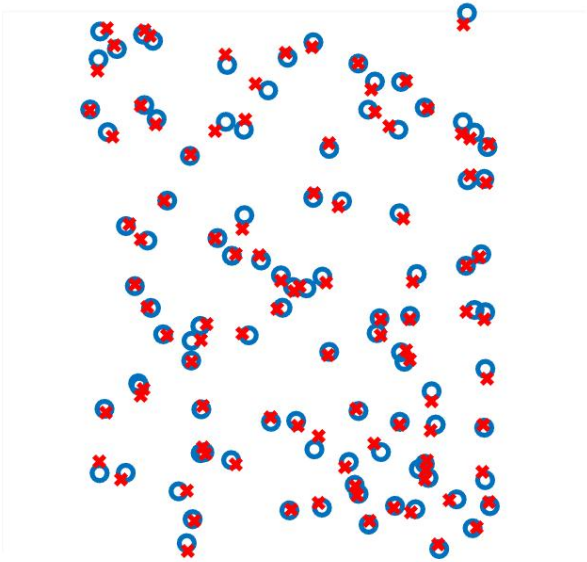

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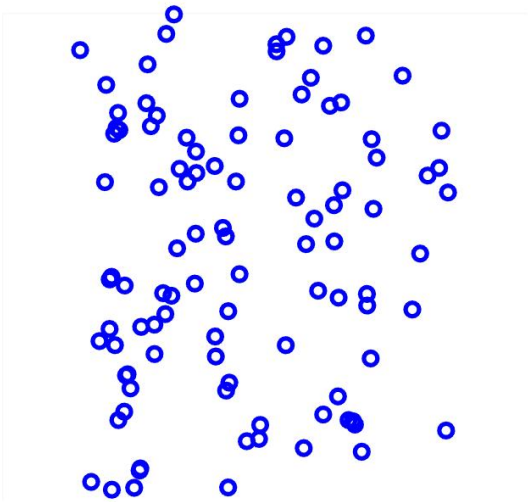

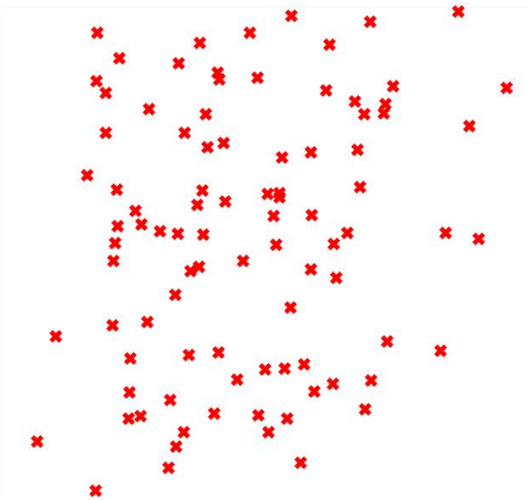

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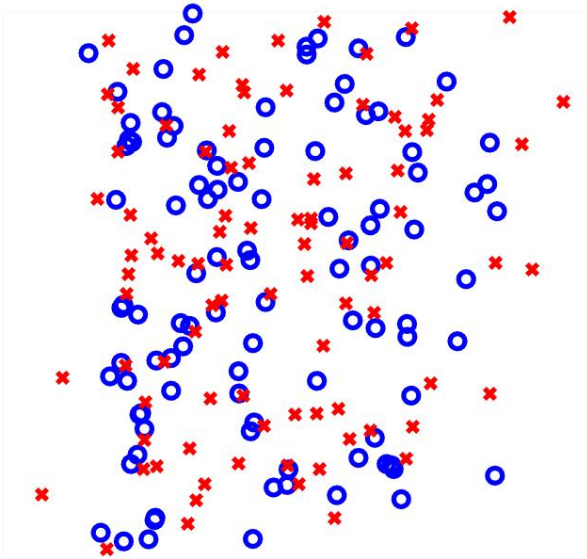

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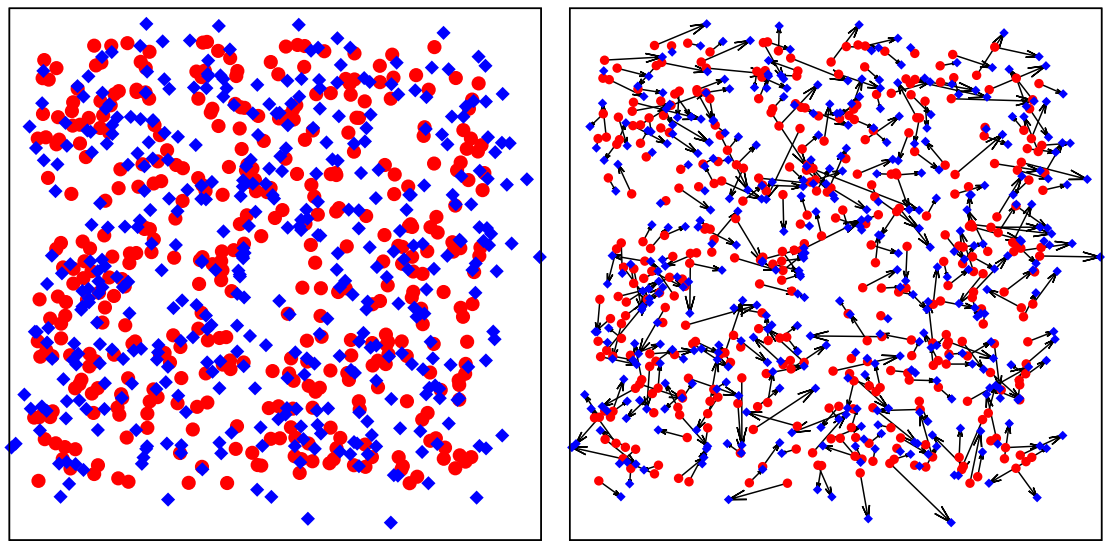

Goal: Recover the underlying true one-to-one mapping of the particles.

Particle Tracking/Mobile Object: Challenges

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Second attempt: n^2 independent distances d_i^j between particle i in first image and particle j in second image:

- $i = j$: distribution P that is fixed
- $i \neq j$: distribution Q that scales with n

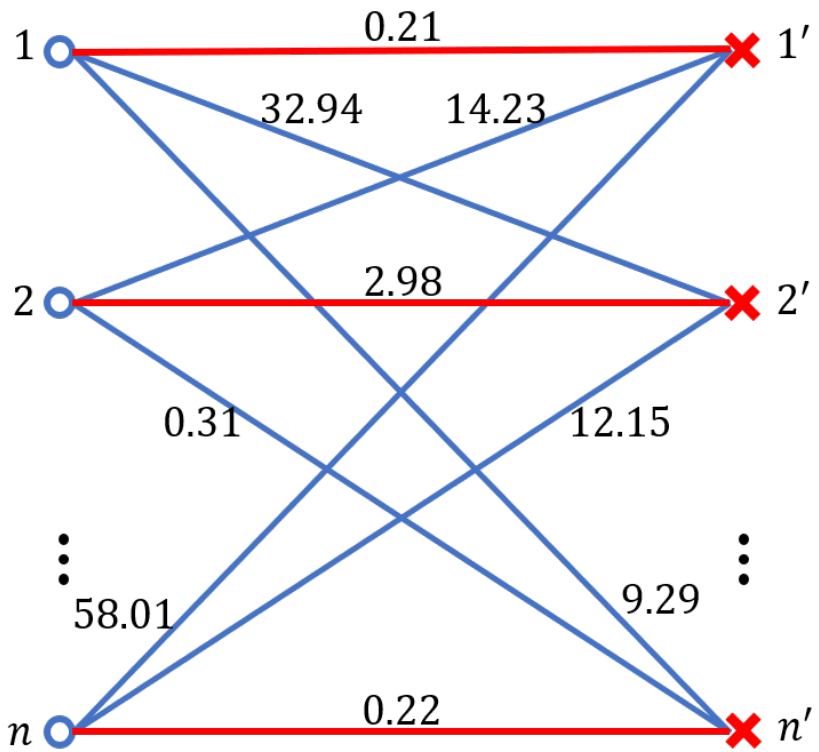
Drawback: triangular inequality fails

Model

- A complete bipartite graph
- A planted matching M^*
- Edge weight

$$W_{ij} \overset{\text{ind.}}{\sim} \begin{cases} P & e \in M^* \\ Q & e \notin M^* \end{cases}$$

- Goal: recover M^* from W



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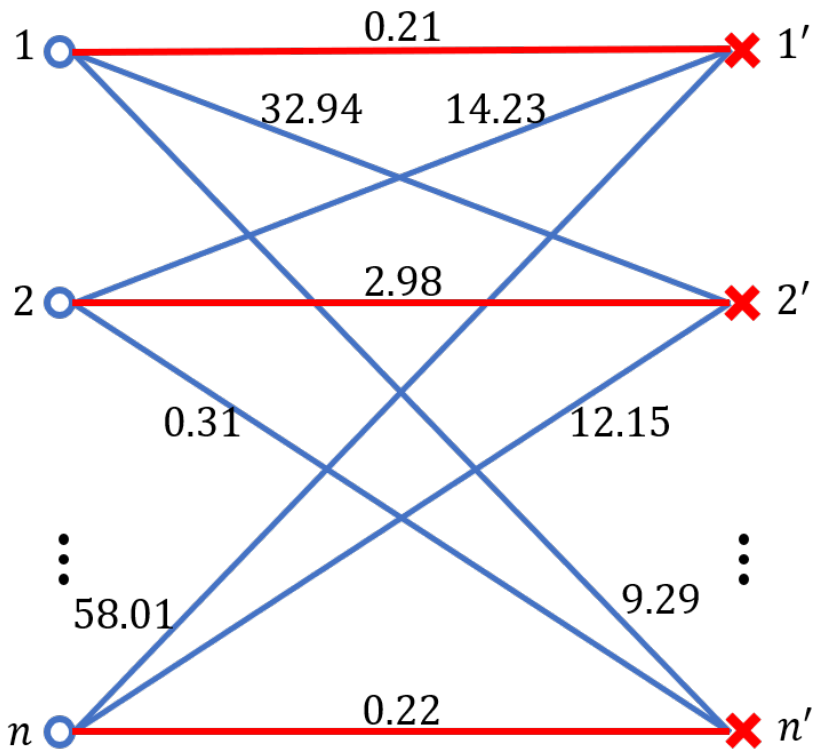

- Our work: $P = \exp(\lambda)$, $Q = \exp(1/n)$ (mean $1/\lambda$ vs. n)

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Question: What is the overlap between $\widehat{M}$ and M^*?

Main Results: Phase Transition at $\lambda = 4$

Theorem

$$\text{overlap: } \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left|\widehat{M} \cap M^*\right| = \begin{cases} 1 & \text{if } \lambda \geq 4 \\ \alpha(\lambda) & \text{if } 0 < \lambda < 4 \end{cases}$$

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and F, G, V, W is the unique solution to a system of ODEs:

$$\frac{\mathrm{d}F}{\mathrm{d}x} = (1-F(x))(1-G(x))V(x), \qquad \frac{\mathrm{d}V}{\mathrm{d}x} = \lambda(V(x)-F(x)),$$
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Boundary conditions: $F(x), V(x), G(-x), W(-x) \to \begin{cases} 1 & x \to +\infty \\ 0 & x \to -\infty \end{cases}$

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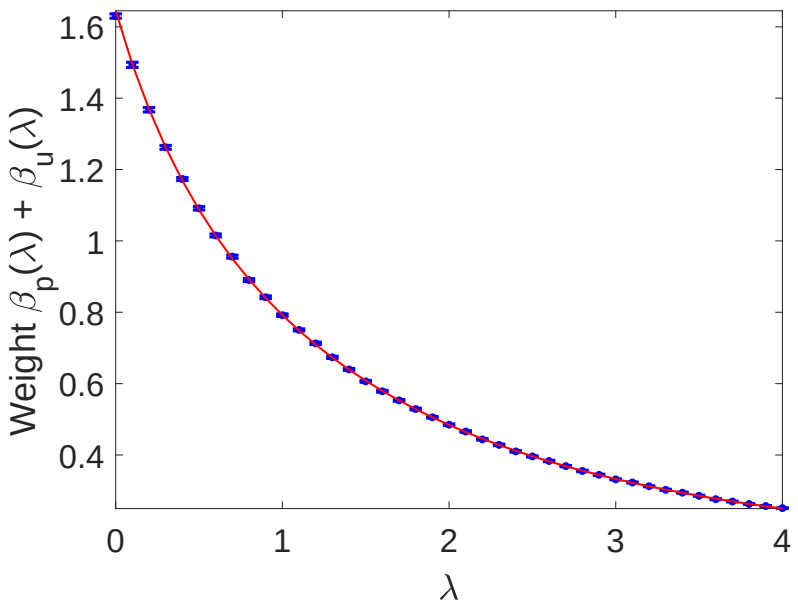

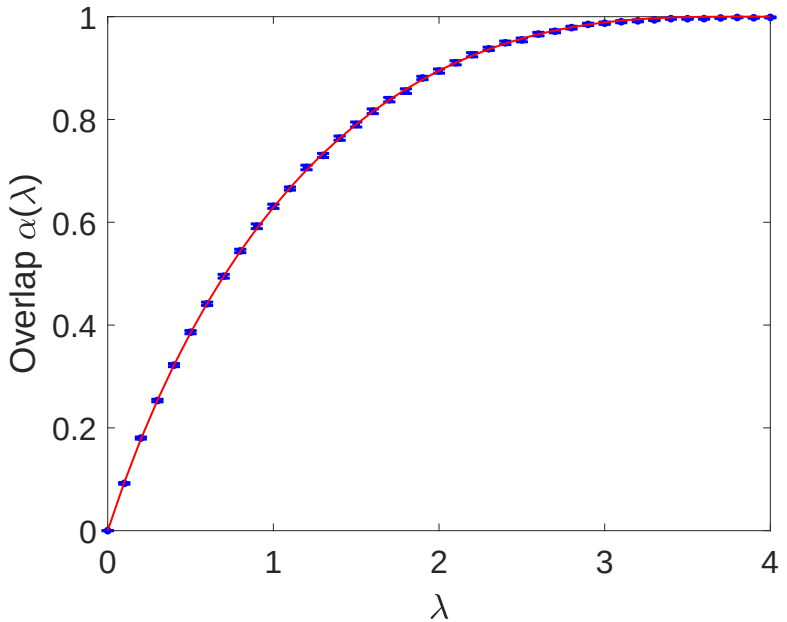

When $\lambda \geq 4$: Count Augmenting Cycles

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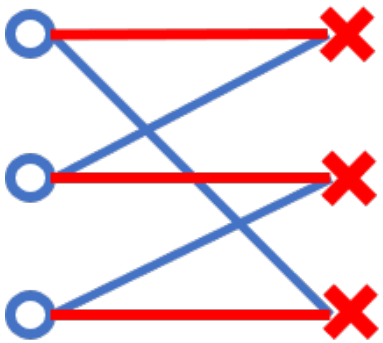

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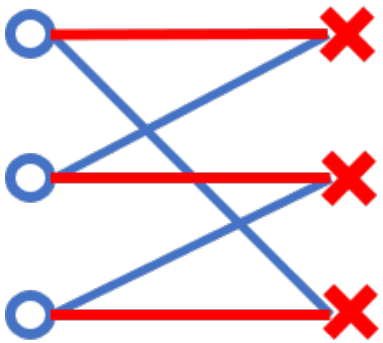

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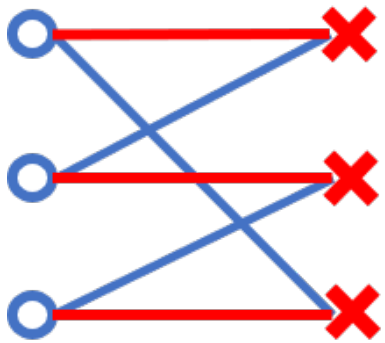

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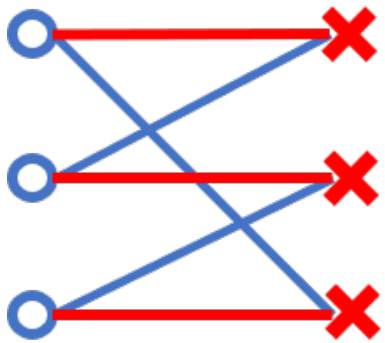

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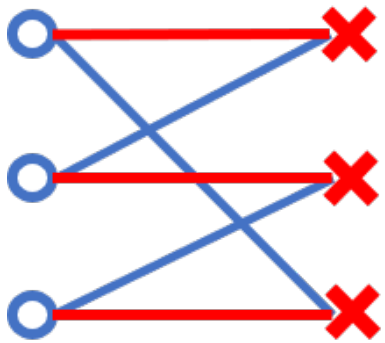

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$\Rightarrow$ Sum over ℓ: total probability a planted edge is in an augmenting cycle is $o(1)$ if $\lambda \geq 4$

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Challenges: What sort of infinite object we are looking at? How to justify the interchange of limit?

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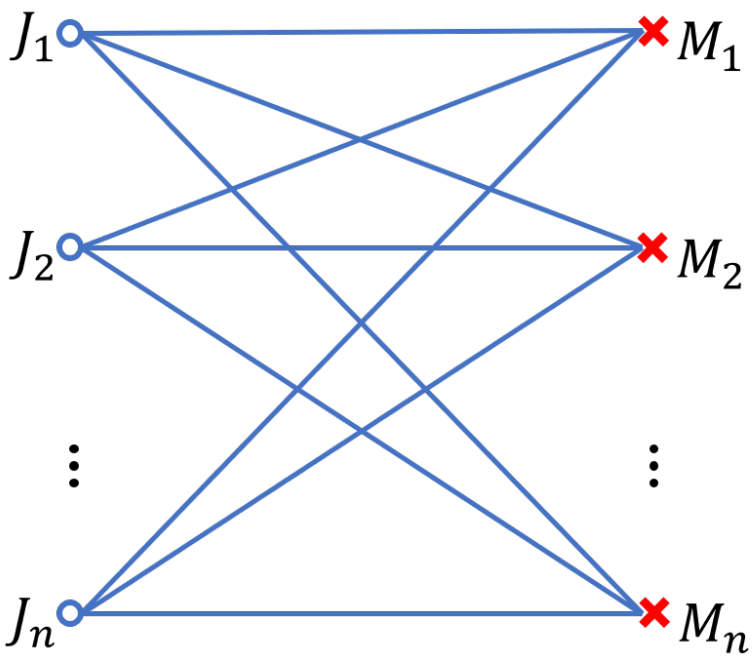

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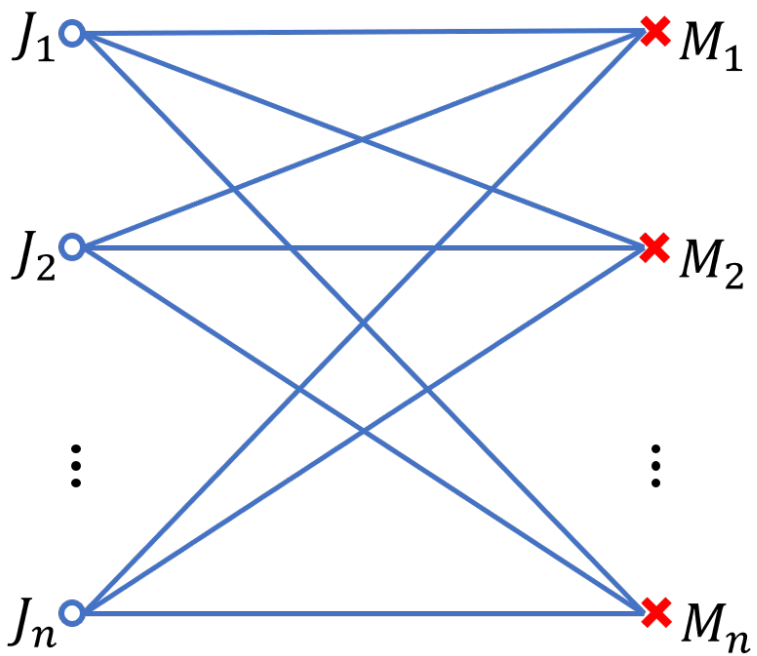

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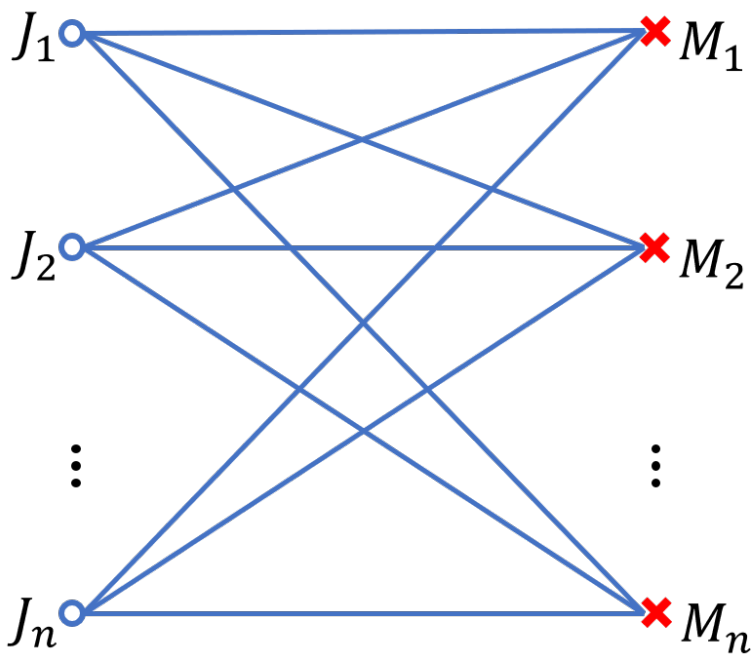

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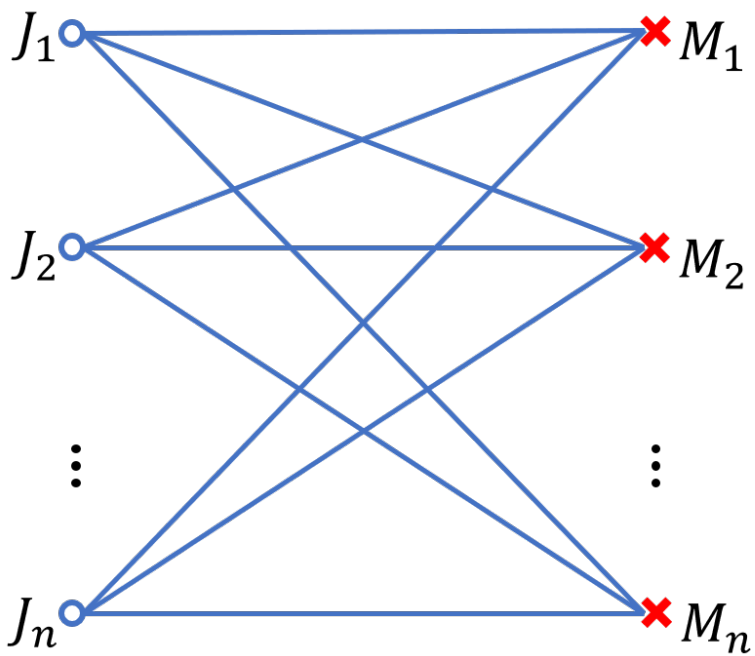

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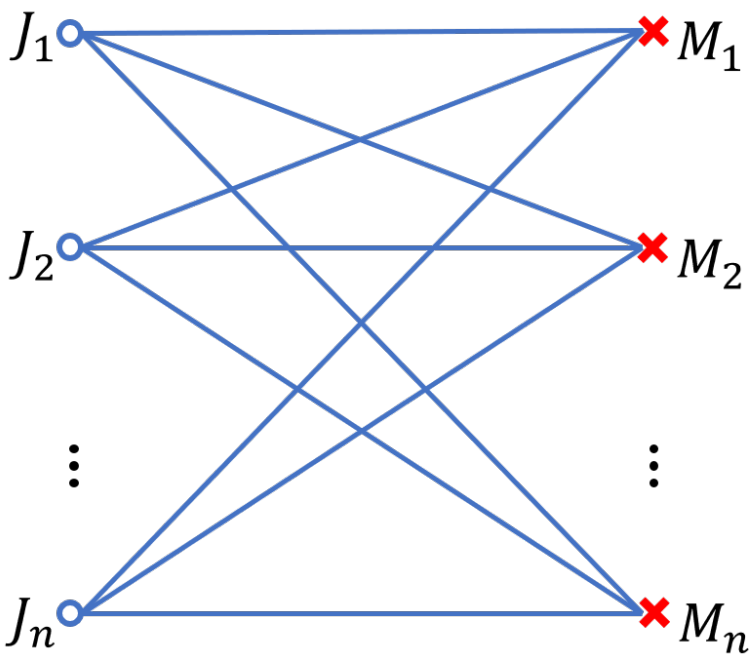

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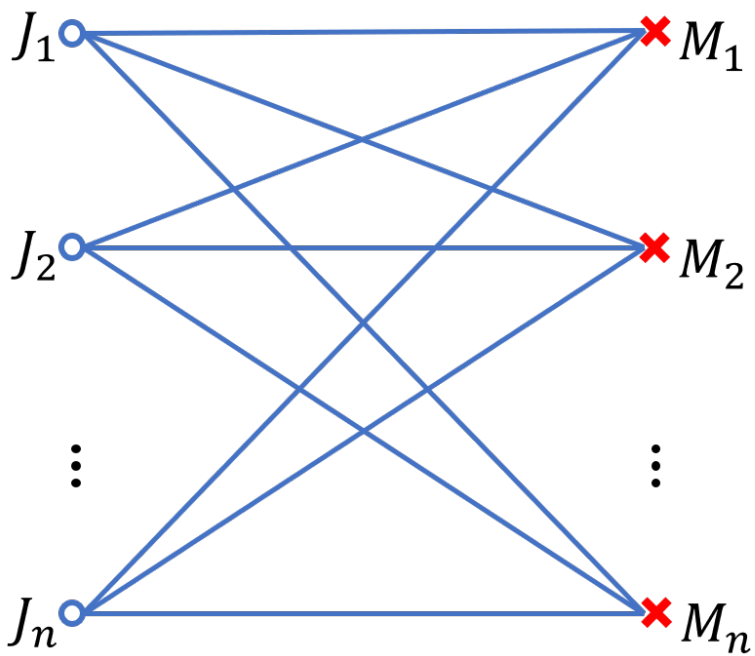

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Aldous and Steele [3]: The Objective Method

– Identify an infinite probabilistic object

– Solve the optimization problem on this infinite object

– Construct a near-optimal solution for finite problems

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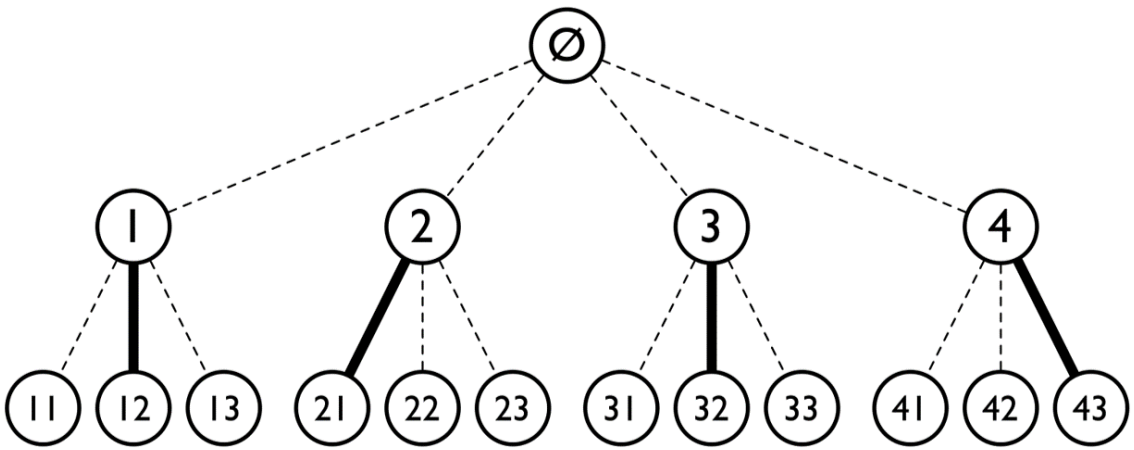

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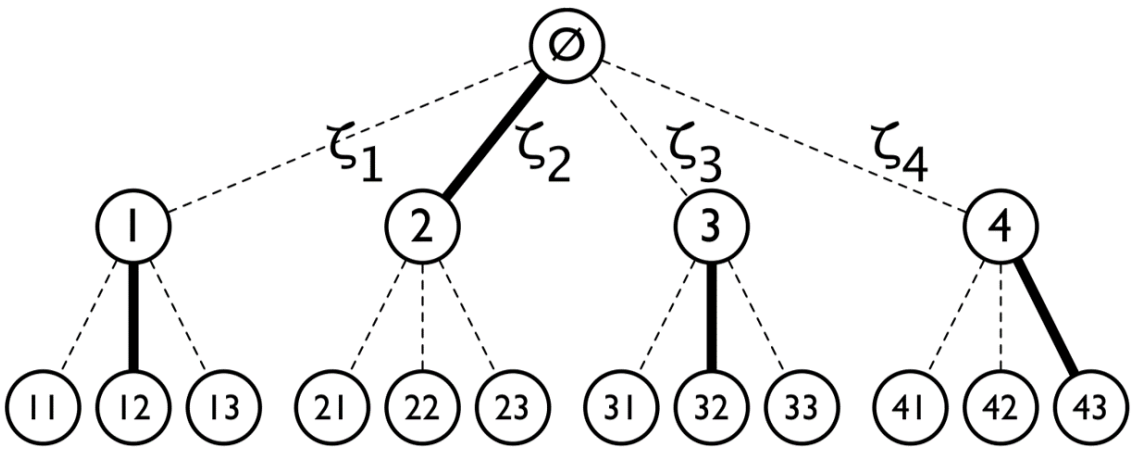

Warmup: The Job Assignment Problem

Converges locally weakly to Poisson Weighted Infinite Tree
Solving minimum matching problem on a tree:

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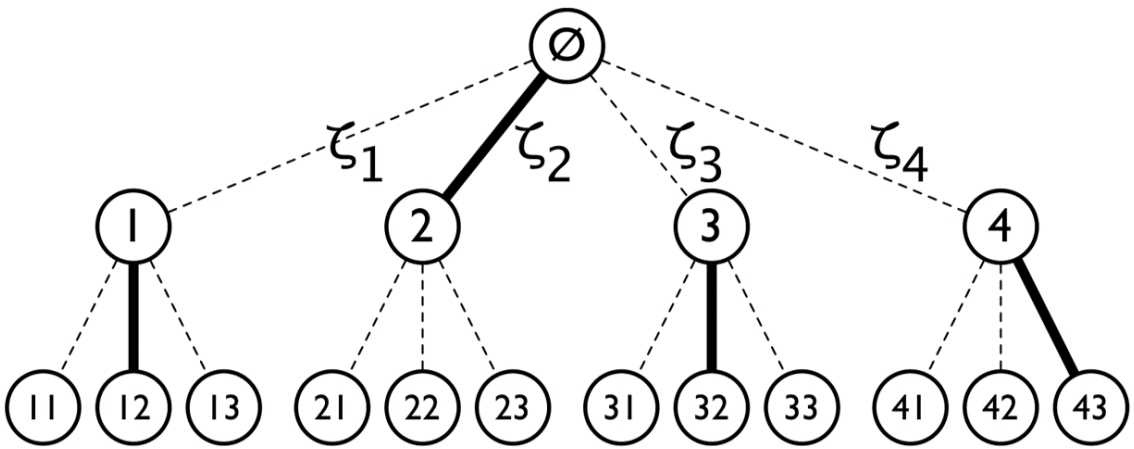

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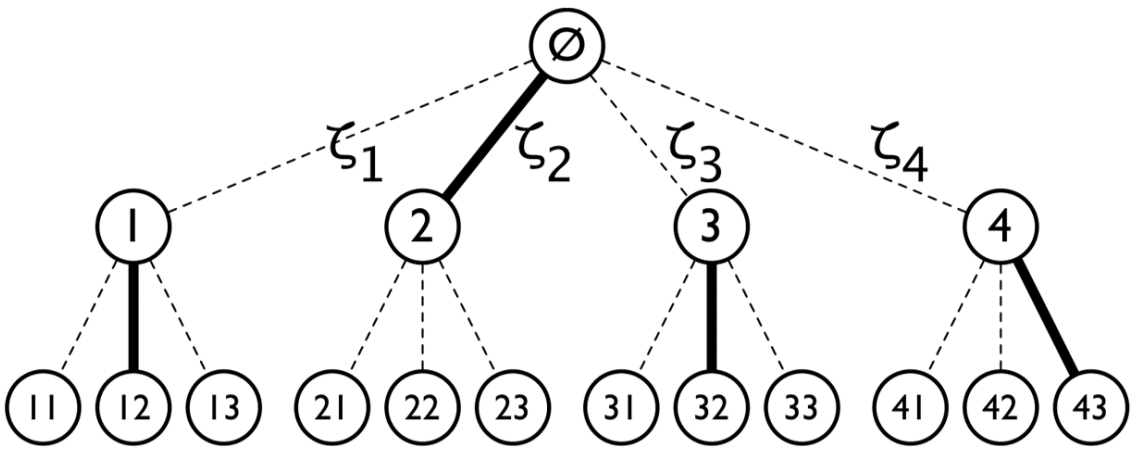

$$X_\emptyset = \text{cost of min matching on } T_\emptyset \; - \text{cost of min matching on } T_\emptyset \setminus \{\emptyset\}$$
$$= \min\{\zeta_i - X_i\}$$

Sorted weights: $\zeta_1, \zeta_2, \ldots$ are arrivals of a Poisson process with rate 1

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Pretend everything is finite:

$$I(T_v) := \text{Cost of matching on the tree } T_v \text{ rooted at } v$$
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We have:

$$\begin{aligned} I(T_\emptyset) &= \min_{i=1,2,\cdots} \{\zeta_i + I(T_i \setminus \{i\}) + \sum_{j \neq i} I(T_j)\} \\ &= \sum_{j \geq 1} I(T_j) + \min_{i=1,2,\cdots} \{\zeta_i + I(T_i \setminus \{i\}) - I(T_i)\} \end{aligned}$$

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Hence,

$$X_\emptyset = I(T_\emptyset) - I(T_\emptyset \setminus \{\emptyset\}) = \min\{\zeta_i - (I(T_i) - I(T_i \setminus \{i\}))\} = \min\{\zeta_i - X_i\}$$

Warmup: From Distributional to Differential Equations

$X \stackrel{d}{=} \min\{\zeta_i - X_i\}$ where ζ_i are Poisson arrivals

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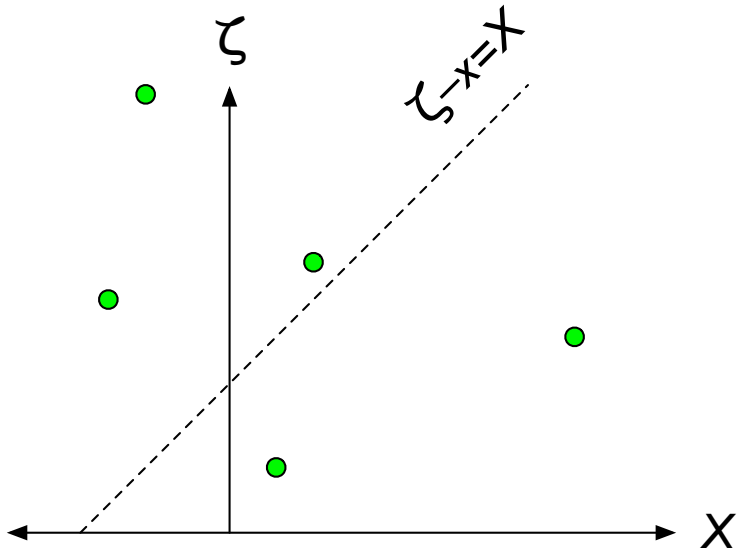
Generate pairs (ζ, x): two-dimensional Poisson process with density $f(x)$



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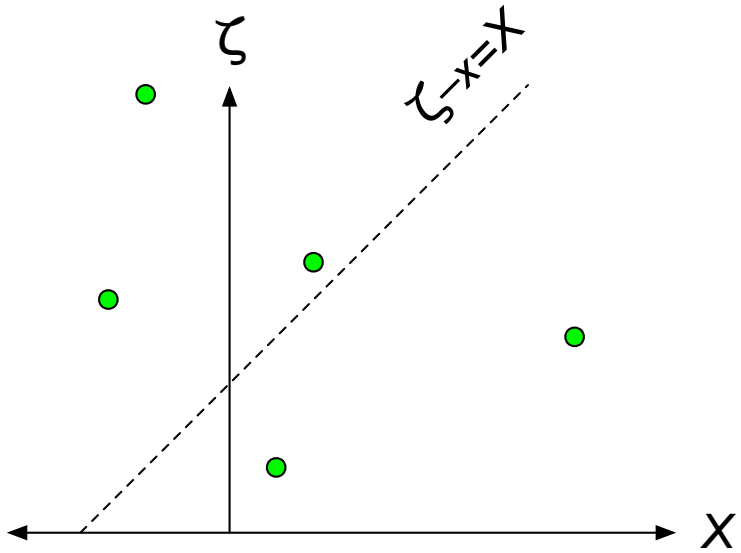

Define the complementary cdf

$$\bar{F}(x) = 1 - F(x) = \mathbb{P}(X > x) = \mathbb{P}(\forall i : \zeta_i - x > X_i)$$

$$\bar{F}(x) = \exp\left(-\int_{-x}^{\infty} \bar{F}(t)\, dt\right) \quad \Rightarrow \quad \frac{\mathrm{d}F(x)}{\mathrm{d}x} = F(x)F(-x)$$

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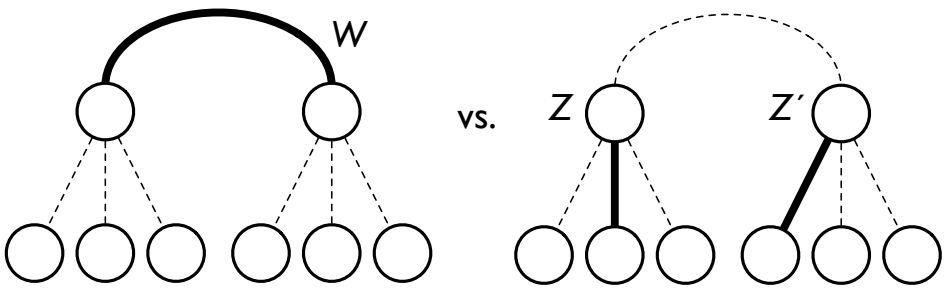

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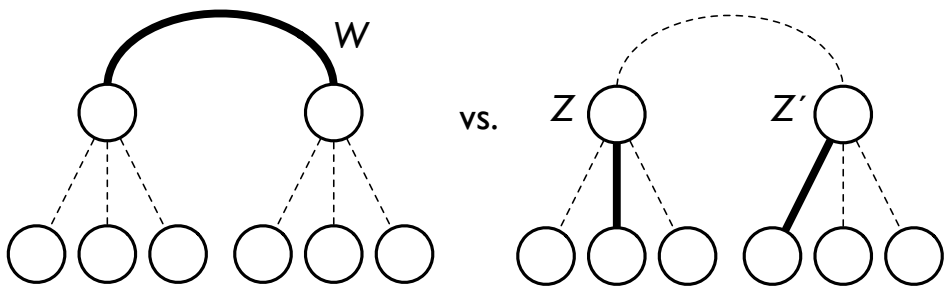
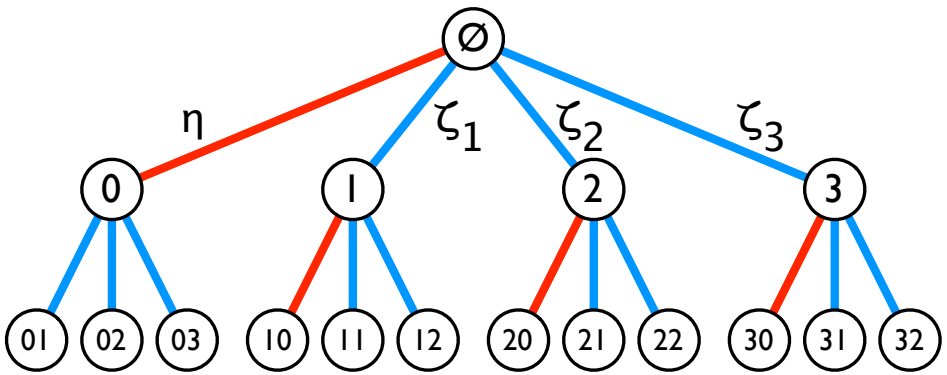


Contribution of a single edge: $\mathbb{E}\big[W\,1[W < X + X']\big]$

$$= \frac{1}{4}\,\mathrm{Var}[X+X'] = \frac{1}{2}\,\mathrm{Var}[X] = \frac{\pi^2}{6}$$

Planted Matching Problem

Partner in planted matching is either parent or child 0, other children sorted $1, 2, 3, \ldots$



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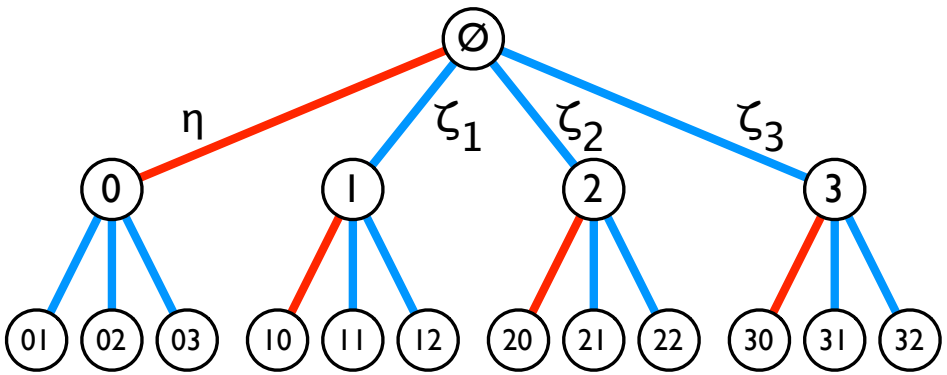

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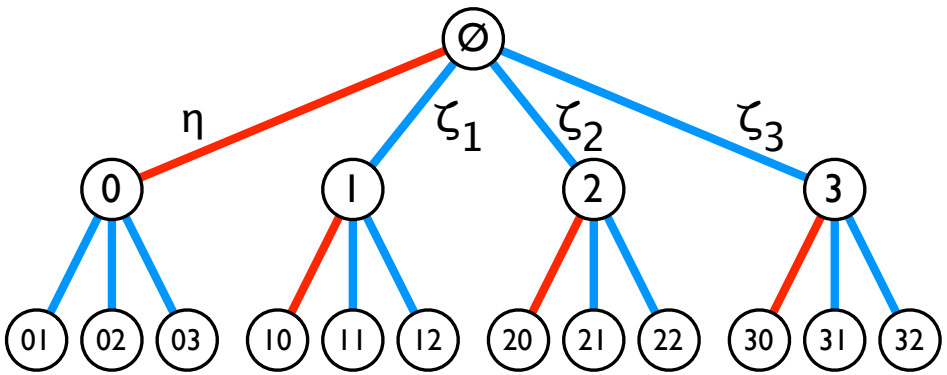

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$$Y \stackrel{d}{=} \min\{\eta - Z, Z'\}$$

$$Z \stackrel{d}{=} \min_i\{\zeta_i - Y_i\}$$

From Distributional to Differential Equations

$$\boxed{\begin{aligned} Y &\stackrel{d}{=} \min\left\{\eta - Z, Z'\right\} \\ Z &\stackrel{d}{=} \min\left\{\zeta_i - Y_i\right\}_{i=1}^{\infty} \end{aligned}}$$

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Boundary conditions: $F(+\infty) = V(+\infty) = 1,\ F(-\infty) = V(-\infty) = 0$

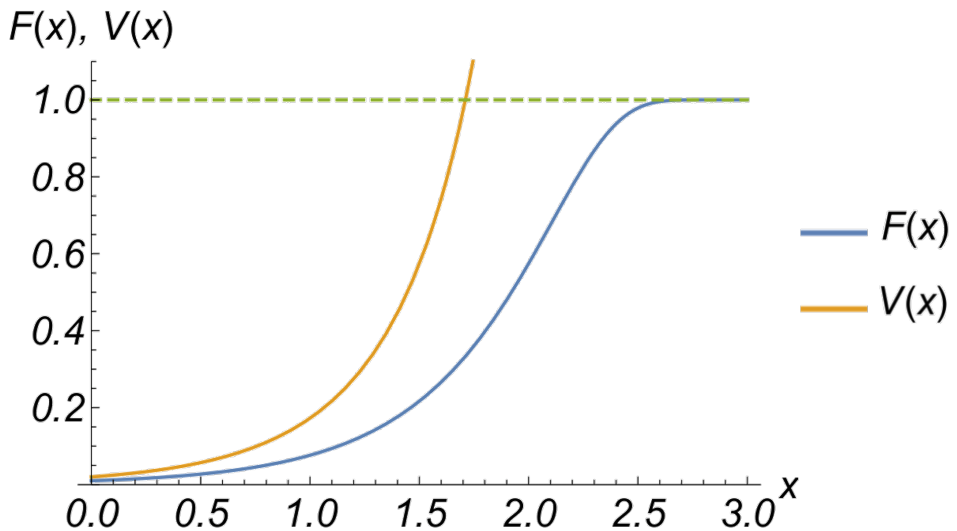
No Solution for $\lambda \geq 4$

At least no sensible one. . .

No Solution for $\lambda \geq 4$

At least no sensible one...

Want $F(+\infty) = V(+\infty) = 1$. But...



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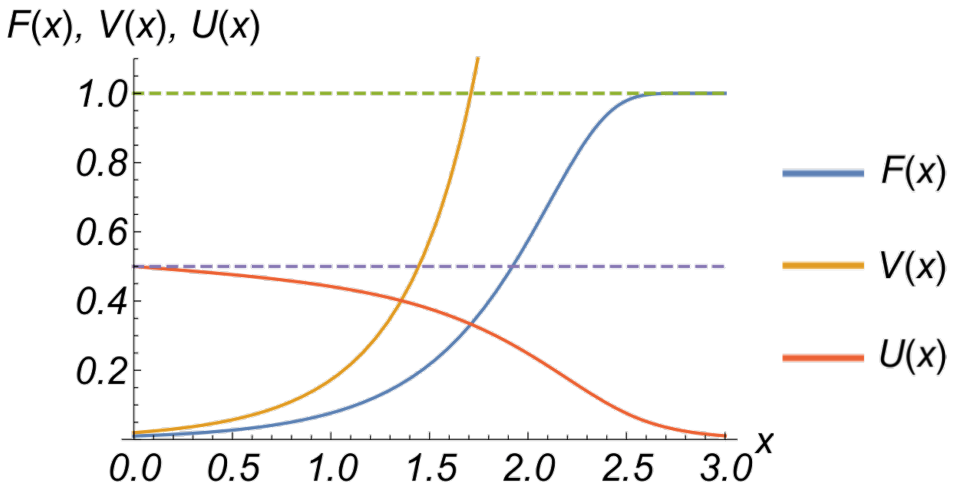
Let $U(x) = F(x)/V(x)$. Then $U(0) = 1/2$, want $U(+\infty) = 1\ldots$



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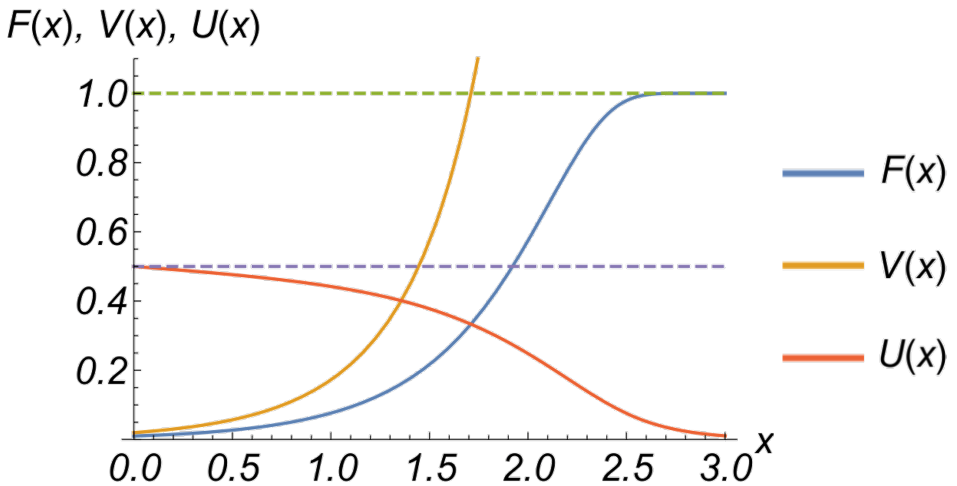

$$\dot{U} = -\lambda U(1-U) + (1-F)(1-G) \leq -\lambda U(1-U) + 1$$

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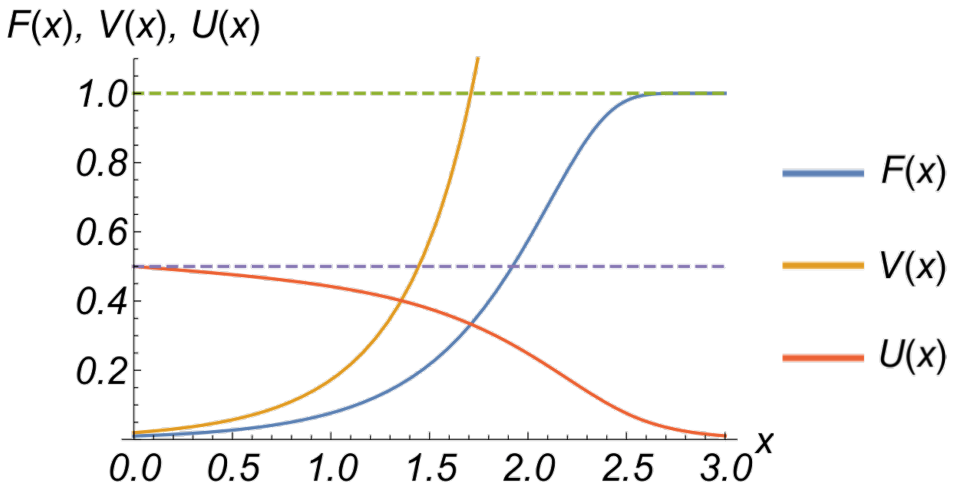

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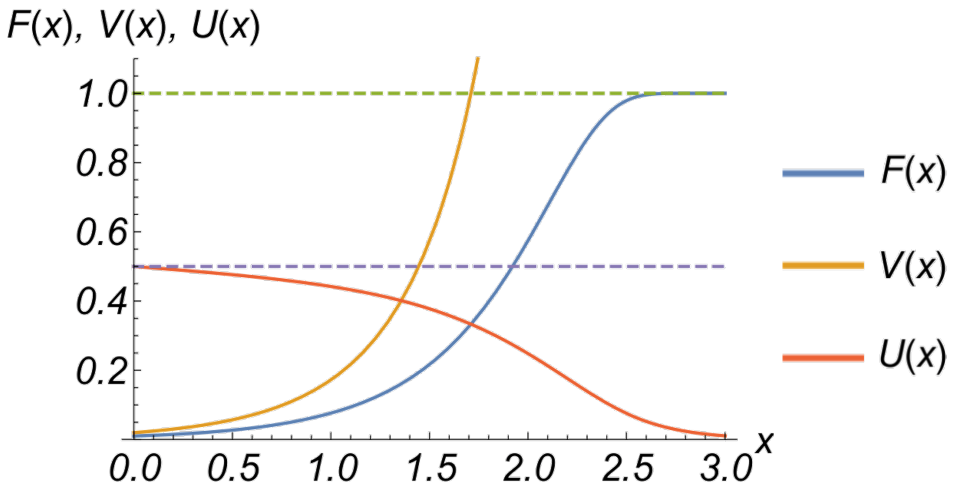

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No fixed distribution on finite values: cost of un-planted edge is $+\infty$

$\Rightarrow$ suggests almost perfect recovery

A Unique Solution when $\lambda < 4$

$(F, G, V, W) \Longleftrightarrow (U, V, W)$: three-dimensional dynamical system

$$\dot{U} = -\lambda U(1-U) + (1-UV)\,(1-(1-U)W)$$
$$\dot{V} = \lambda V(1-U)$$
$$\dot{W} = \lambda WU$$

Initial conditions: $U(0) = \frac{1}{2}, V(0) = W(0) = \epsilon$

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Lemma

If $\lambda < 4$ there is a unique $\epsilon_0 \in (0,1)$ such that

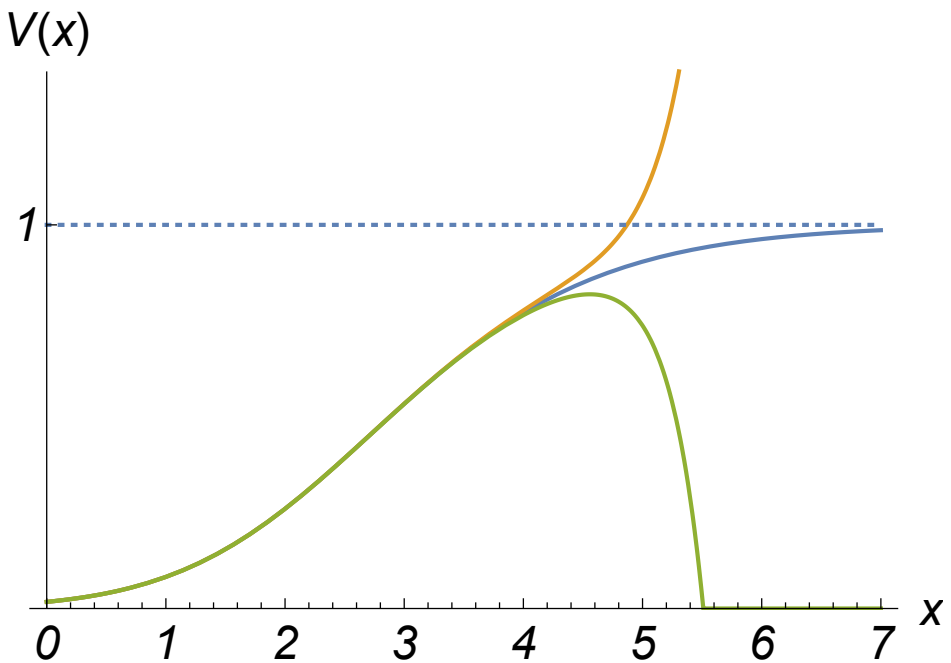
If $\epsilon \in [0, \epsilon_0)$, $U(x) \to +\infty$

If $\epsilon = \epsilon_0$, $U(x) \to 1$ and $V(x) \to 1$

If $\epsilon \in (\epsilon_0, 1]$, $V(x) \to +\infty$

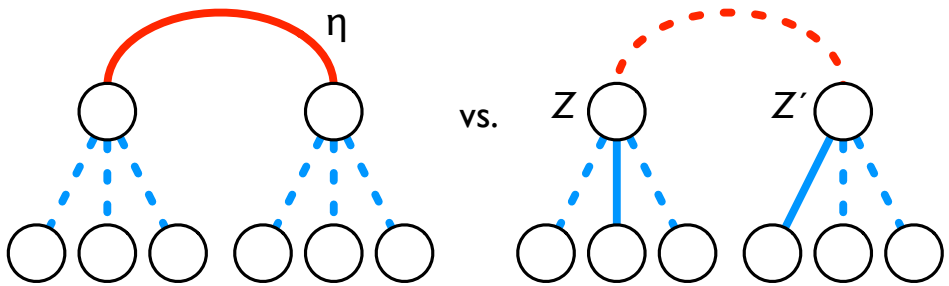
A Unique Solution when $\lambda < 4$

Geometric intepretation: $(U = 1, V = 1, W = 0)$ is a saddle point.
If $V(0) = \epsilon_0$ we approach the saddle point along its unstable manifold.



This gives cdfs $F, V \to 1$ of the unique fixed point distribution

Computing the Overlap for $\lambda < 4$



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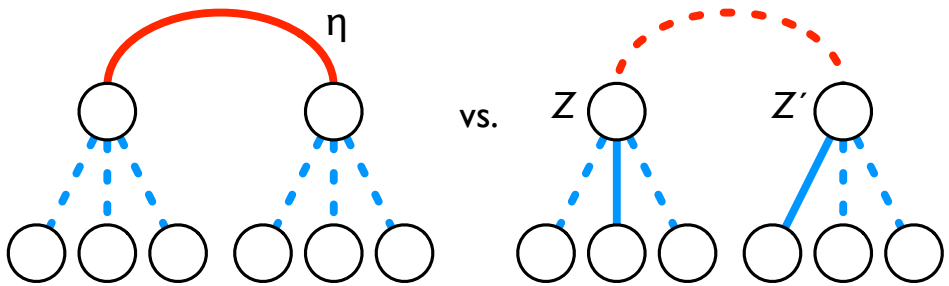

$$\alpha(\lambda) = \mathbb{P}[\eta < Z + Z'] = 1 - \mathbb{E}_\eta \int_{-\infty}^{+\infty} f(x) F(\eta - x)\, dx$$

$$= 1 - \int_{-\infty}^{+\infty} f(x)\, \mathbb{E}_\eta F(\eta - x)\, dx$$

$$= 1 - \int_{-\infty}^{+\infty} (1 - F(x))(1 - G(x)) V(x) W(x)\, dx$$

$$= 1 - 2 \int_{0}^{+\infty} (1 - F(x))(1 - G(x)) V(x) W(x)\, dx$$

Open Problems

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- Other planted structures: spanning trees, traveling salespeople?

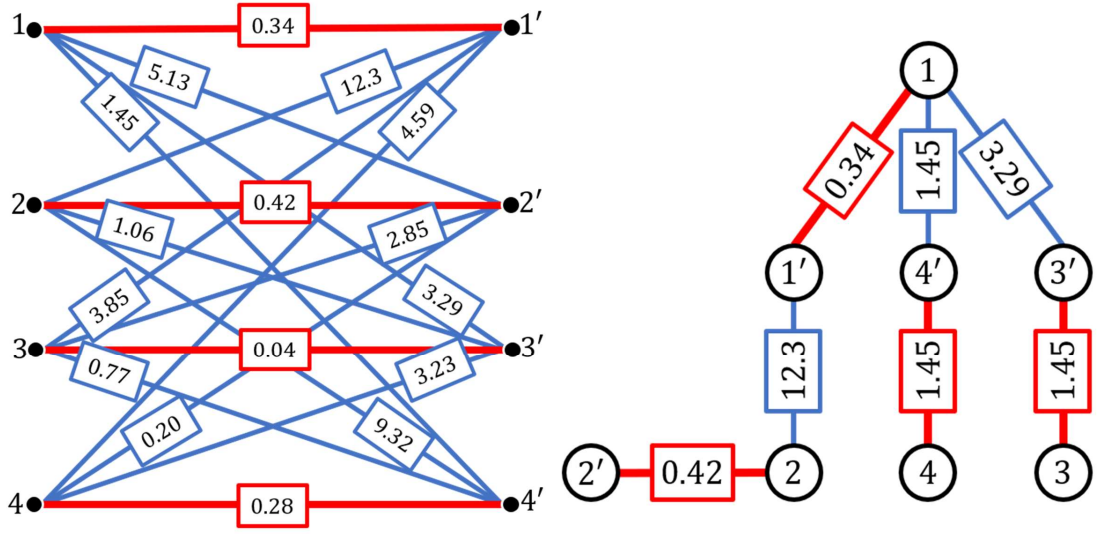
High Level Takeaway

– The Objective Method: applicable as long as there is a correlation decay, useful in combinatorial optimization problems

Thanks for your Attention

Proof Idea: Local Weak Convergence (Aldous [1, 2])

Unfolding Map



– Choose v uniformly at random and unfold the bipartite graph

Proof Idea: Local Weak Convergence (Aldous [1, 2])

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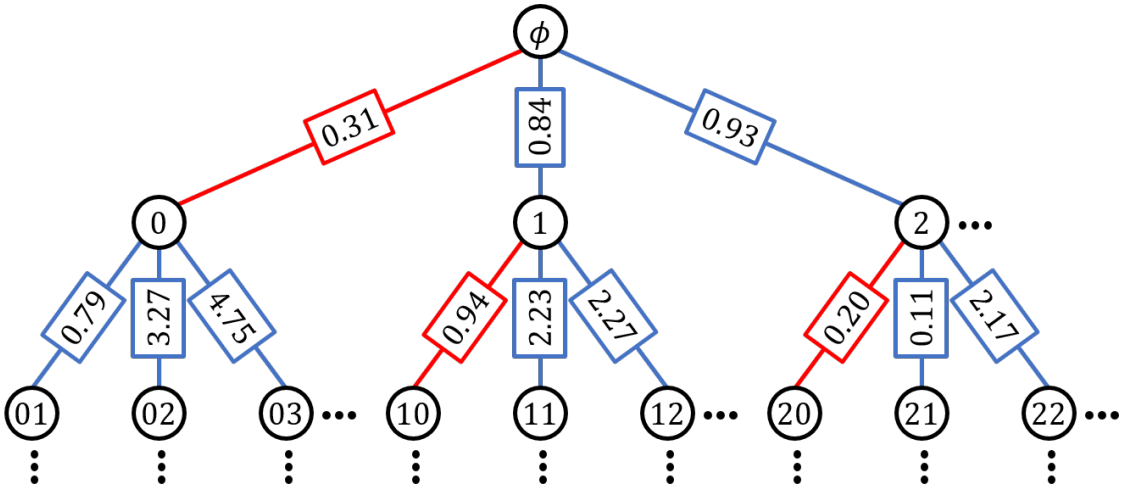

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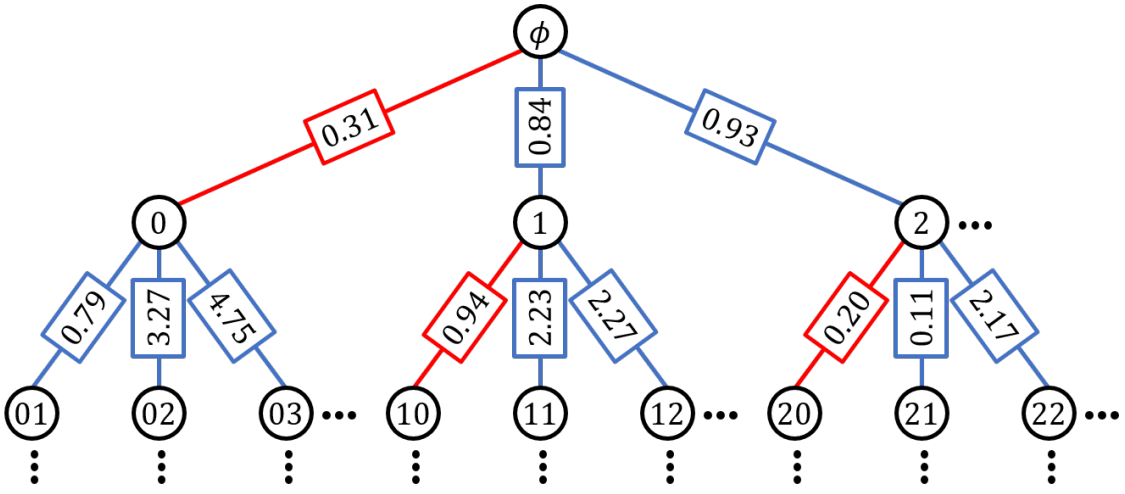

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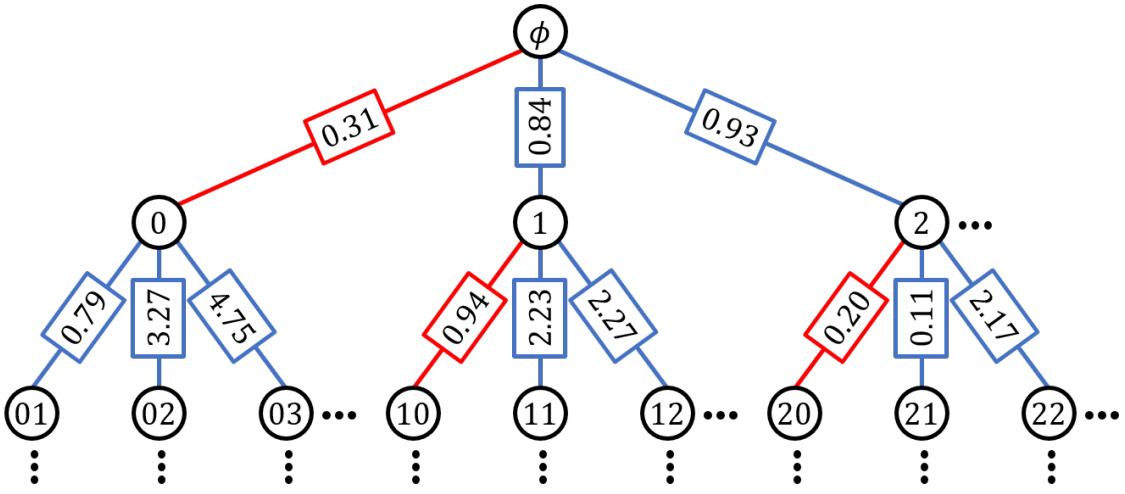

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- The corresponding matching on infinite tree satisfies this property: *involution invariant*

Proof Idea: Local Weak Convergence (Aldous [1, 2])

Minimum Involution Invariant Matching on T_∞

Lemma

If the system of recursive distributional equation has a solution, then this solution corresponds to the minimum involution invariance matching $\mathcal{M}_{opt}$ on T_∞. Moreover, this matching is unique.

Corollary

If the system of recursive distributional equation has a solution, then this solution is unique.

We have constructed the minimum involution invariant $\mathcal{M}_{\text{opt}}$ on T_∞ and computed its cost and overlap when $\lambda < 4$

Proof Idea: Local Weak Convergence (Aldous [1, 2])

From $\mathcal{M}_{\text{opt}}$ to $\widehat{M}$ and Vise-Versa

Easy Half: From $\widehat{M}$ to $\mathcal{M}_{\text{opt}}$

- Any sequence of $\{M_n\}$ of matchings on $K_{n,n}$ has a subsequence $\{M_{n_j}\}$ that converges to a (possibly random) involution invariant matching on T_∞, in the local weak sense

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- Local weak convergence, compactness, and Skorokhod's theorem

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- Any sequence of $\{M_n\}$ of matchings on $K_{n,n}$ has a subsequence $\{M_{n_j}\}$ that converges to a (possibly random) involution invariant matching on T_∞, in the local weak sense
- Local weak convergence, compactness, and Skorokhod's theorem

Hard Half: From $\mathcal{M}_{\text{opt}}$ to $\widehat{M}$

Proof Idea: Local Weak Convergence (Aldous [1, 2])

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- Almost-doubly-stochastic matrix

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- By invariance, M_∞^* and $\mathcal{M}_{\text{opt}}$ differ at the root

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Combine all the above

- By invariance, M^*_∞ and $\mathcal{M}_{\text{opt}}$ differ at the root
- M^*_∞ often chooses the wrong partner for $\varnothing$

Proof Idea: Local Weak Convergence (Aldous [1, 2])

From $\mathcal{M}_{\text{opt}}$ to $\widehat{M}$ and Vise-Versa

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- By invariance, M^*_∞ and $\mathcal{M}_{\text{opt}}$ differ at the root
- M^*_∞ often chooses the wrong partner for $\varnothing$
- Right partner given by recursion $\Rightarrow$ differential equations

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Together these imply $\lim_{n\to\infty} \text{overlap}(\widehat{M}_n) = \text{overlap}(\mathcal{M}_{\text{opt}})$

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