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On the Value of Communication Links in Voltage-VAR Control for Distribution Networks: A Game Theoretic Perspective

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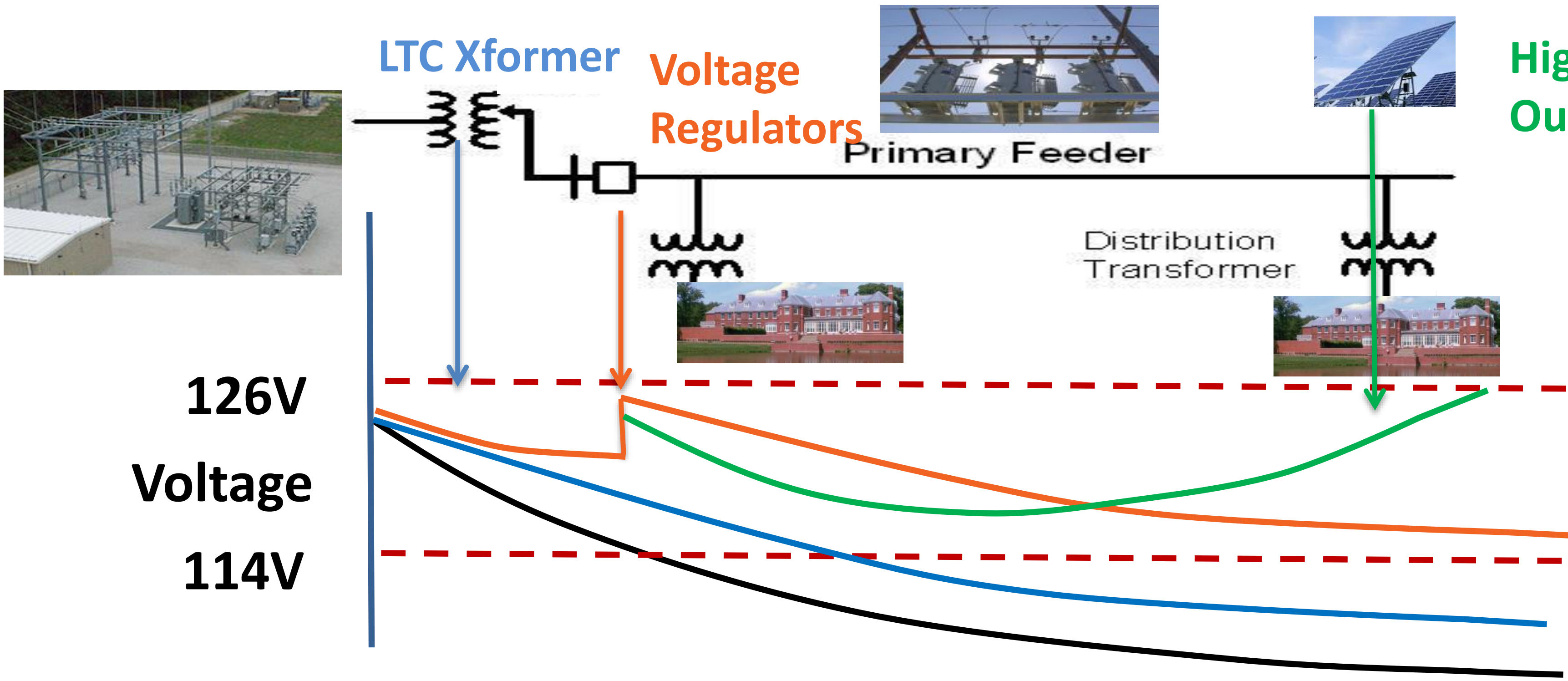
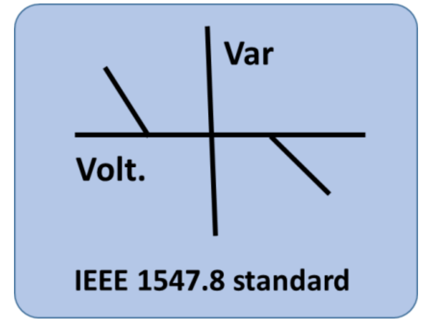
Voltage control in distribution systems



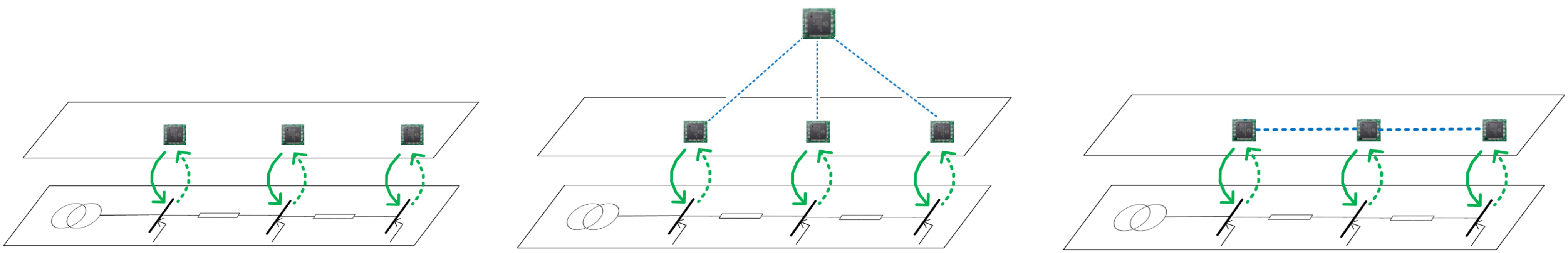
- High penetration of distributed energy resources (DERs) pose great challenges for voltage regulation
- Voltage-VAR control: DERs provide reactive power (VAR) support for voltage regulation

Motivation

- Local/droop control [Farivar ’13][Zhu ’16] suffers from system-level suboptimality with limited VAR



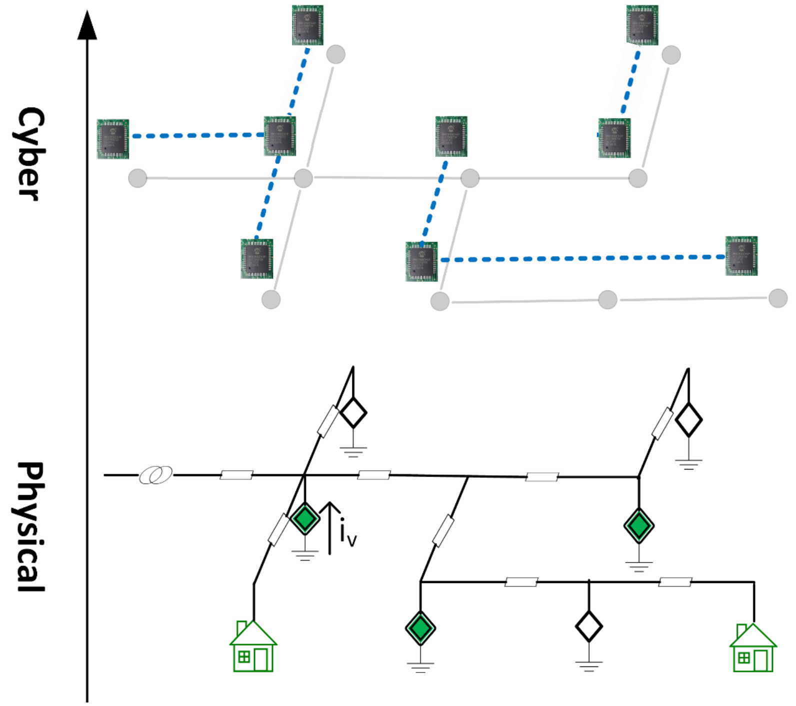
- DERs should be coordinated via communications
- Optimal power flow (OPF) based control
  – Centralized solver requires full network information
  – Distributed voltage control [Bolognani ’15][Dall’Anese-Zhu ’13][Liu-Zhu ’17] require communications with at least neighboring buses



- Communication infrastructure is limited and of low-quality in current distribution systems

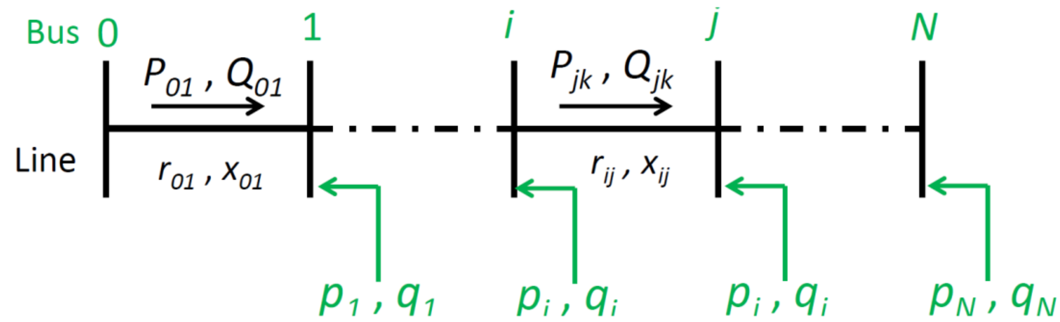
Our focus

- *Semi-local* voltage control
- Questions
  - How to characterize the equilibrium of voltage-VAR control with scarce communication links (not necessarily strongly connected)?
  - How to design an implementable algorithm to achieve this equilibrium?
  - How to deploy communication links following this communication-optimality tradeoff?



Modeling

- Consider a distribution network of tree topology $(\mathcal{N}, \mathcal{E})$



- LinDist power flow model [Baran '89], where $\mathbf{M}^o = \left[\mathbf{m}_0^T \;\; \mathbf{M}^T\right]^T$ is the graph incidence matrix , $\mathbf{D}_x := \mathrm{diag}\{x_{ij}\}$ and $\mathbf{D}_r := \mathrm{diag}\{r_{ij}\}$

$$-\mathbf{M}\mathbf{P} = -\mathbf{p}$$

$$-\mathbf{M}\mathbf{Q} = -\mathbf{q}$$

$$\mathbf{m}_0 v_0 + \mathbf{M}^T \mathbf{v} = \mathbf{D}_r \mathbf{P} + \mathbf{D}_x \mathbf{Q}$$

- The effectiveness of the LinDistFlow model has been validated in [Farivar '13][Sulc '14][Zhu '16]

Problem formulation

- Voltage-VAR relation
  - Denote $\mathbf{B} := \mathbf{M}\mathbf{D}_x^{-1}\mathbf{M}^T$ and $\mathbf{w} := \mathbf{M}\mathbf{D}_x^{-1}\mathbf{D}_r\mathbf{M}^{-1}\mathbf{p} - v_0\mathbf{M}\mathbf{D}_x^{-1}\mathbf{m}_0$

operating point

reduced weighted Laplacian matrix

$$\mathbf{B}\mathbf{v} = \mathbf{q} + \mathbf{w} \xrightarrow[\bar{\mathbf{v}}=\mathbf{X}\mathbf{w}]{\mathbf{X}=\mathbf{B}^{-1}} \mathbf{v} = \mathbf{X}\mathbf{q} + \bar{\mathbf{v}}$$

Lemma 1: Matrix $\mathbf{X}$ is positive definite (PD) with nonnegative entries.

- Voltage control problem: DC-OPF $\mathcal{P}_0$

$$\begin{aligned} \min_{\mathbf{v},\mathbf{q}} \quad & \frac{1}{2}\|\mathbf{v}-\boldsymbol{\mu}\|_2^2 + \frac{1}{2}\|\mathbf{q}\|_{\mathbf{C}}^2 \\ s.t. \quad & \mathbf{v} = \mathbf{X}\mathbf{q} + \bar{\mathbf{v}} \\ & \underline{\mathbf{q}} \le \mathbf{q} \le \bar{\mathbf{q}} \end{aligned}$$

  - Goal: minimize voltage mismatch and VAR provision cost under limited VAR resources
  - Desired voltage profile $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{C} = \mathrm{diag}\{c_1, \cdots, c_N\}$ with $c_j \geq 0$.

Game theoretic characterization

- Consider a communication graph $(\mathcal{N}, \mathcal{E}_c)$ with K connected components $\mathcal{K} = \{1, \cdots, K\}$, the buses in the k-th component constitute $\mathcal{K}_k$
- Each bus can only take care of the cost within the component
- A strategic game $\mathcal{G} = \langle \mathcal{K}, \{\mathcal{Q}_k\}_{k \in \mathcal{K}}, \{U_k\}_{k \in \mathcal{K}} \rangle$ with K players whose action sets are VAR injection

$$\mathbf{q}_k \in \mathcal{Q}_k := \left[\underline{\mathbf{q}}_k, \bar{\mathbf{q}}_k\right]$$

- Define payoff functions $U_k : \mathcal{Q} \to \mathbb{R}$

$$\begin{aligned} U_k(\mathbf{q}_k, \mathbf{q}_{-k}) &= \frac{1}{2}\|\mathbf{v}_k - \boldsymbol{\mu}_k\|_2^2 + \frac{1}{2}\|\mathbf{q}_k\|_{\mathbf{C}_k}^2 \\ &= \frac{1}{2}\|\mathbf{X}_{k,k}\mathbf{q}_k + \mathbf{X}_{k,-k}\mathbf{q}_{-k} + \bar{\boldsymbol{\mu}}_k\|_2^2 + \frac{1}{2}\|\mathbf{q}_k\|_{\mathbf{C}_k}^2 \end{aligned}$$

  - $\mathbf{q}_{-k} = [\mathbf{q}_i]_{i \neq k}$ is the VAR injection of all other connected components
  - $\mathbf{X}_{i,k}$ is the block submatrix of $\mathbf{X}$ with proper dimension, $\bar{\boldsymbol{\mu}}_k = \bar{\mathbf{v}}_k - \boldsymbol{\mu}_k$

Nash equilibrium and its existence

- Nash Equilibrium (NE) of the game $\mathcal{G}$: $\mathbf{q}^* = (\mathbf{q}_k, \mathbf{q}_{-k})^T$ satisfies $\forall k \in \mathcal{K}$

$$U_k\left(\mathbf{q}_k^*, \mathbf{q}_{-k}^*\right) \leq U_k\left(\mathbf{q}_k, \mathbf{q}_{-k}^*\right), \forall \mathbf{q}_k \in \mathcal{Q}_k$$

- Note each submatrix $\mathbf{X}_{k,k}$ is PD and each $\mathcal{Q}_k$ is convex and compact

Prop 1: The set of the Nash equilibrium of the game $\mathcal{G}$ is nonempty.

- Equilibrium conditions (EC): Karush–Kuhn–Tucker (KKT) conditions for each player's convex optimization problem

$$\forall k \in \mathcal{K}:$$

$$\begin{cases} \mathbf{X}_{k,k}^T\left(\mathbf{X}_{k,k}\mathbf{q}_k + \mathbf{X}_{k,-k}\mathbf{q}_{-k} + \bar{\boldsymbol{\mu}}_k\right) + \mathbf{C}_k\mathbf{q}_k + \bar{\boldsymbol{\lambda}}_k - \underline{\boldsymbol{\lambda}}_k = \mathbf{0} \\ \bar{\boldsymbol{\lambda}}_k^T\left(\mathbf{q}_k - \bar{\mathbf{q}}_k\right) = 0 \\ \underline{\boldsymbol{\lambda}}_k^T\left(\mathbf{q}_k - \underline{\mathbf{q}}_k\right) = 0 \\ \bar{\boldsymbol{\lambda}}_k \geq \mathbf{0}, \underline{\boldsymbol{\lambda}}_k \geq \mathbf{0} \\ \bar{\mathbf{q}}_k \leq \mathbf{q}_k \leq \underline{\mathbf{q}}_k \end{cases}$$

Equilibrium conditions

- Compact form of the EC:
  - Denote $\tilde{\mathbf{X}} = \mathrm{diag}\left\{\mathbf{X}_{k,k}\right\}_{k\in\mathcal{K}}$ a block diagonal matrix

$$\begin{cases} \tilde{\mathbf{X}}(\mathbf{X}\mathbf{q}+\bar{\boldsymbol{\mu}})+\mathbf{C}\mathbf{q}+\bar{\boldsymbol{\lambda}}-\underline{\boldsymbol{\lambda}}=\mathbf{0} \\ \bar{\boldsymbol{\lambda}}^T(\mathbf{q}-\bar{\mathbf{q}})=0 \\ \underline{\boldsymbol{\lambda}}^T(\mathbf{q}-\underline{\mathbf{q}})=0 \\ \bar{\boldsymbol{\lambda}}\geq\mathbf{0},\underline{\boldsymbol{\lambda}}\geq\mathbf{0} \\ \underline{\mathbf{q}}\leq\mathbf{q}\leq\bar{\mathbf{q}} \end{cases}$$

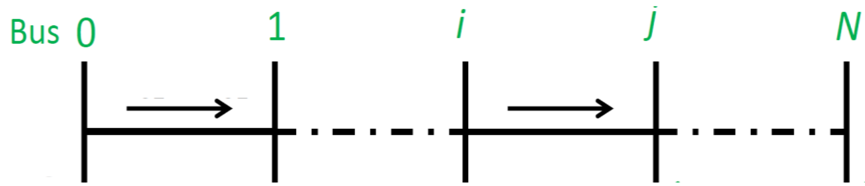
  - The property of the NE is determined by the property of $\tilde{\mathbf{X}}\mathbf{X}+\mathbf{C}$

Uniqueness of the NE

Prop 2: For any $\mathbf{C} \geq \mathbf{0}$, the NE of the game $\mathcal{G}$ is unique for any operating point captured by $\bar{\boldsymbol{\mu}}$ if and only if the matrix $\tilde{\mathbf{X}}\mathbf{X}$ is a P-matrix, i.e., every principal minor of $\tilde{\mathbf{X}}\mathbf{X}$ is positive.

- The proof stems from the theory of linear complementarity problem (LCP)
- P-property of $\tilde{\mathbf{X}}\mathbf{X}$ can be proved for some special distribution networks

Prop 3: For any communication structure captured by $\tilde{\mathbf{X}}$, the matrix $\tilde{\mathbf{X}}\mathbf{X}$ is a P-matrix if each bus on $(\mathcal{N}, \mathcal{E})$ has degree no greater than 2.



Prop 4: For any $\mathbf{C} \geq \mathbf{0}$, the NE of the game $\mathcal{G}$ is unique for any operating point captured by $\bar{\boldsymbol{\mu}}$ if the matrix $\tilde{\mathbf{X}}\mathbf{X}$ is PD.

- The most common sufficient condition for the NE's uniqueness

Uniqueness of the NE

- In general, $\tilde{\mathbf{X}}\mathbf{X}$ is not provably to be a P-matrix or PD matrix
- However, uniqueness is not a big concern in practice
  - Actual operating point $\bar{\boldsymbol{\mu}}$ does not necessarily lead to nonuniqueness
  - $\tilde{\mathbf{X}}\mathbf{X}$ is a P-matrix in most simulation cases
  - $\mathbf{C}$ usually suffices to make $\tilde{\mathbf{X}}\mathbf{X} + \mathbf{C}$ a P-matrix or even a PD matrix

Two special cases

- **Case 1**: no-communication among DERs

Theorem 1: Given the game $\mathcal{G}$, if there is no communication among buses, i.e., if $\tilde{\mathbf{X}} = \mathrm{diag}\{\mathbf{X}\}$, then the NE is equivalent to the solution to a convex optimization problem $\mathcal{P}_1$

$$\begin{aligned} \min_{\mathbf{v},\mathbf{q}} \quad & \frac{1}{2}\|\mathbf{v}-\boldsymbol{\mu}\|^2_{\mathbf{X}^{-1}} + \frac{1}{2}\|\mathbf{q}\|^2_{\tilde{\mathbf{X}}^{-1}\mathbf{C}} \\ s.t. \quad & \mathbf{v} = \mathbf{X}\mathbf{q} + \bar{\mathbf{v}} \\ & \underline{\mathbf{q}} \leq \mathbf{q} \leq \bar{\mathbf{q}} \end{aligned}$$

which is unique for any $\mathbf{C} \geq \mathbf{0}$.

- It can be proved the EC and the optimality condition of $\mathcal{P}_1$ are equivalent
- The uniqueness also follows *Prop 2* that $\tilde{\mathbf{X}}\mathbf{X}$ is a P-matrix in this case
- The value of communications is captured by comparing the solution to the original problem $\mathcal{P}_0$ and $\mathcal{P}_1$

Two special cases

- Observations
  - Buses contribute less VAR resources than optimal and become selfish
  - The objective of $\mathcal{P}_1$ happens to be the weighted potential function of $\mathcal{G}$
  - The EC is invariant up to a PD diagonal matrix scaling on $\bar{\boldsymbol{\lambda}}$ and $\underline{\boldsymbol{\lambda}}$

Corollary 4: For any $\mathbf{C} \geq \mathbf{0}$, the NE of the game $\mathcal{G}$ is unique if the matrix $\tilde{\mathbf{X}}\mathbf{X}$ is diagonally stable, i.e., there exists a diagonal $\mathbf{D} \geq \mathbf{0}$ such that $\mathbf{D}\tilde{\mathbf{X}}\mathbf{X} + \mathbf{X}\tilde{\mathbf{X}}\mathbf{D}$ is PD.

- This condition is weaker than *Prop 3* that $\tilde{\mathbf{X}}\mathbf{X}$ is PD
- Conditions for diagonal stability has been investigated in control society, see [Baker '78][Kraaij '91]

Two special cases

- **Case 2**: communication among DERs in the same situation

Theorem 2: Given the game $\mathcal{G}$, assume $\mathbf{C} = c\mathbf{I}$ for some $c \geq 0$, if the solution to $\mathcal{P}_1$ makes the bus within one communication component have the same VAR injection situation, then the solution constitutes one NE of $\mathcal{G}$.

$q_j = \bar{q}_j, q_j = \underline{q}_j$, or $\underline{q}_j < q_j < \bar{q}_j$

- It is proved by showing the solution to $\mathcal{P}_1$ always satisfies the EC
- Observation: if $\mathbf{C} = \mathbf{0}$, the NE is the same as that in **Case 1**, i.e., it gains no benefit to place links among buses that already in the same situation

Voltage regulation algorithm

- Gradient-projection-based algorithm: easy to implement
- Pseudo-gradient

$$\mathbf{F}(\mathbf{q}) := [\nabla_{\mathbf{q}_k} U_k(\mathbf{q})]_{k\in\mathcal{K}} = \tilde{\mathbf{X}}(\mathbf{X}\mathbf{q}+\bar{\boldsymbol{\mu}}) + \mathbf{C}\mathbf{q} = \tilde{\mathbf{X}}(\mathbf{v}-\boldsymbol{\mu}) + \mathbf{C}\mathbf{q}$$

- The up-to-date information on other components' VAR injection is reflected in intra-component measurements $\mathbf{v}_k(t)$
- Asynchronous update is necessary

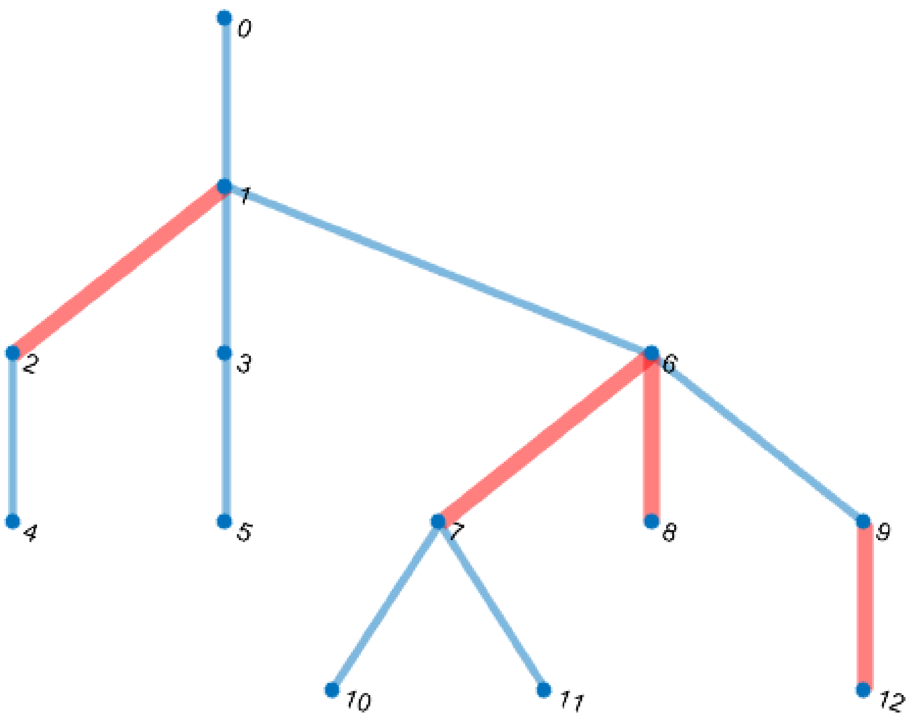
Iterations when an update is executed

$$\mathbf{q}_k(t+1) = \begin{cases} \left[\mathbf{q}_k(t) - \epsilon(\mathbf{X}_{k,k}\mathbf{v}_k(t) + \mathbf{C}_k\mathbf{q}_k(t))\right]_{\mathcal{Q}_k}, & t \in \mathcal{T}_k \\ \mathbf{q}_k(t), & t \notin \mathcal{T}_k \end{cases} \quad (*)$$

Prop 5: The asynchronous update (*) with bounded update delay converges to the unique NE of $\mathcal{G}$ with small $\epsilon > 0$ such that the spectrum radius $\rho[\mathbf{I} - \epsilon(\tilde{\mathbf{X}}\mathbf{X} + \mathbf{C})] < 1$, if the matrix $\tilde{\mathbf{X}}\mathbf{X} + \mathbf{C}$ is PD.

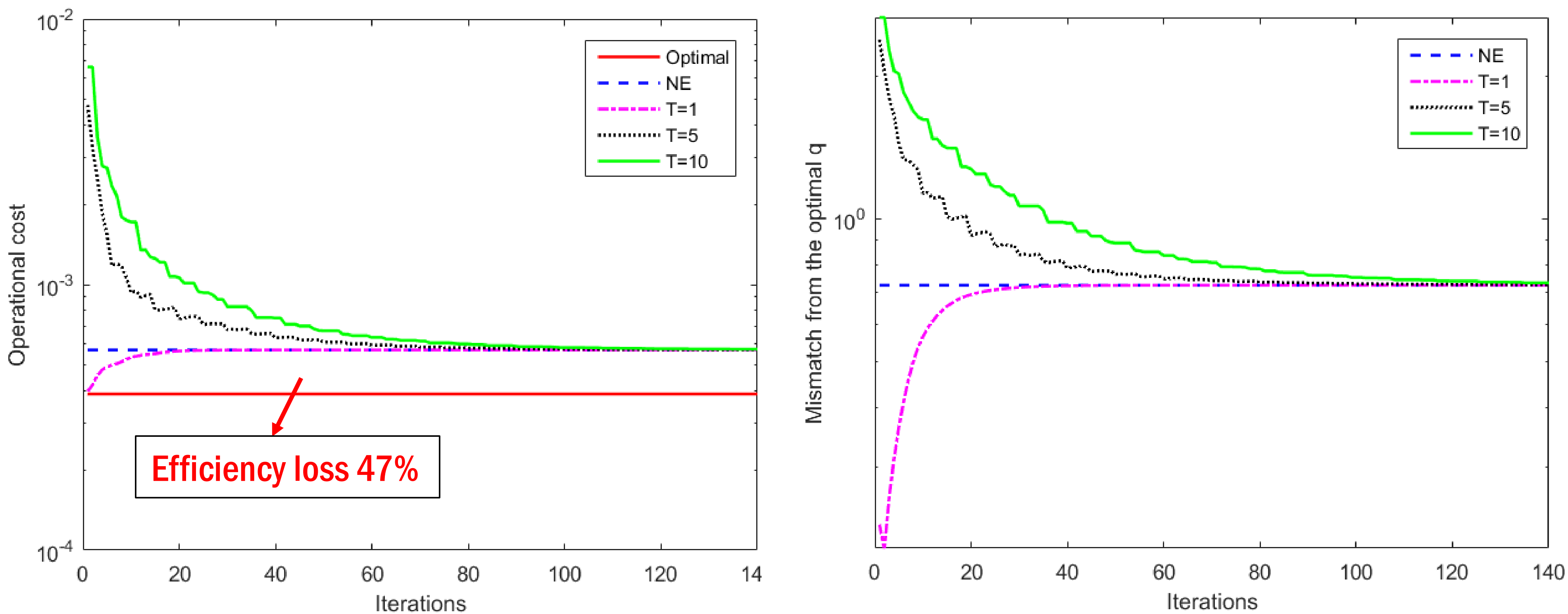
Simulations

- A radial network of $N = 13$ buses with line impedance $(.233 + j.366)\Omega$
- $K = 8$ communication components such that the smallest eigenvalue of $(\tilde{\mathbf{X}}\mathbf{X} + \mathbf{X}\tilde{\mathbf{X}})/2$ is -9.31×10^{-7}, and $\tilde{\mathbf{X}}\mathbf{X}$ is a P-matrix
- Test for both cases when $\tilde{\mathbf{X}}\mathbf{X} + \mathbf{C}$ is indefinite and PD with different $\mathbf{C} \geq \mathbf{0}$



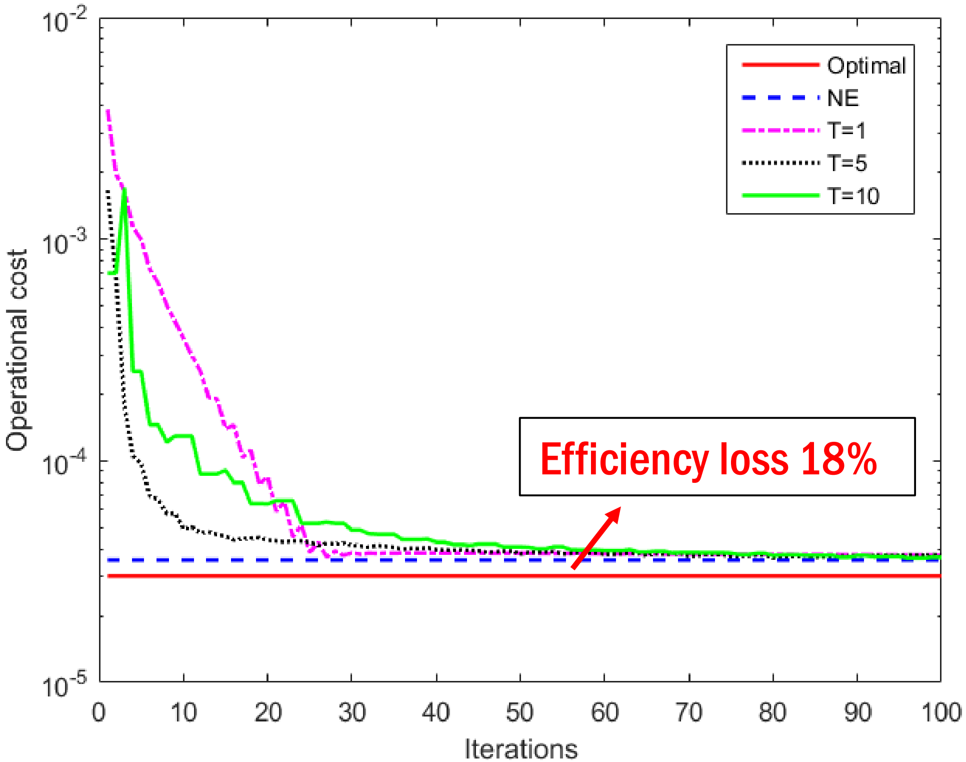
Topology of the distribution and communication networks

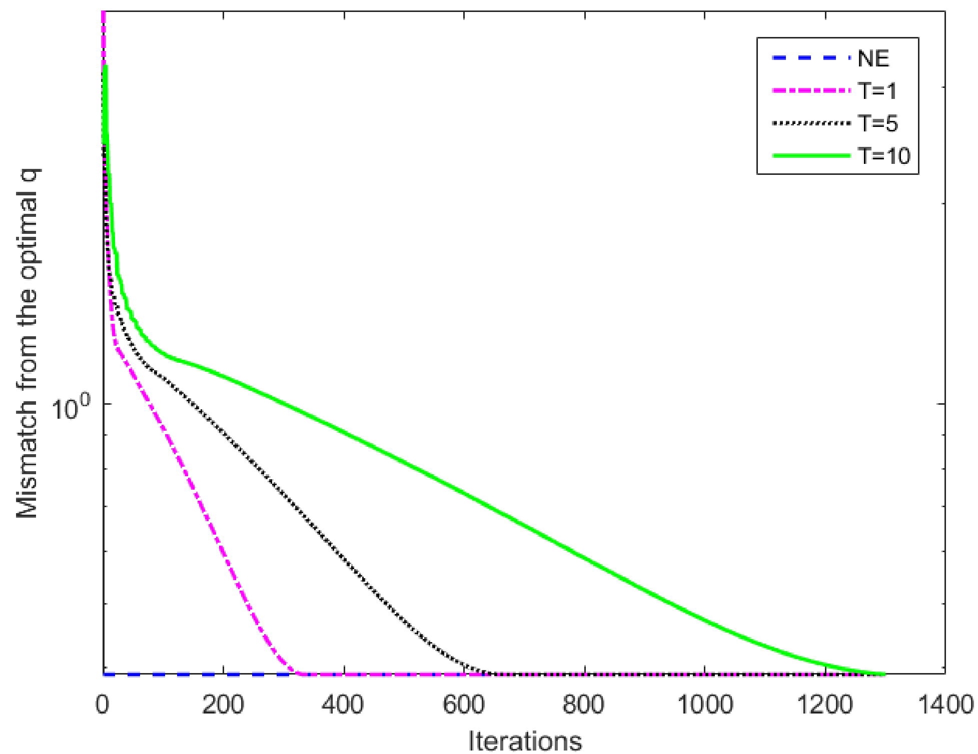
Simulations



$\tilde{\mathbf{X}}\mathbf{X} + \mathbf{C}$ is PD

Simulations





$\tilde{\mathbf{X}}\mathbf{X} + \mathbf{C}$ is indefinite but a P-matrix

Communication link deployment

- How to deploy communication links and design $\tilde{\mathbf{X}}$ in an optimal way?
- The link deployment can be formulated as a bilevel optimization problem:

$$\min_{\{\mathbf{s}_k\},\mathbf{q},\bar{\boldsymbol{\lambda}},\underline{\boldsymbol{\lambda}}} \quad \frac{1}{2}\|\mathbf{X}\mathbf{q}+\bar{\boldsymbol{\mu}}\|_2^2+\frac{1}{2}\|\mathbf{q}\|_{\mathbf{C}}^2$$

$$s.t. \quad \text{Equilibrium Conditions}$$

$$\tilde{\mathbf{X}} = \sum_{k=1}^{K} \text{diag}\{\mathbf{s}_k\}\, \mathbf{X} \text{diag}\{\mathbf{s}_k\}$$

$$\mathbf{s}_k \in \{0,1\}^N, \forall k$$

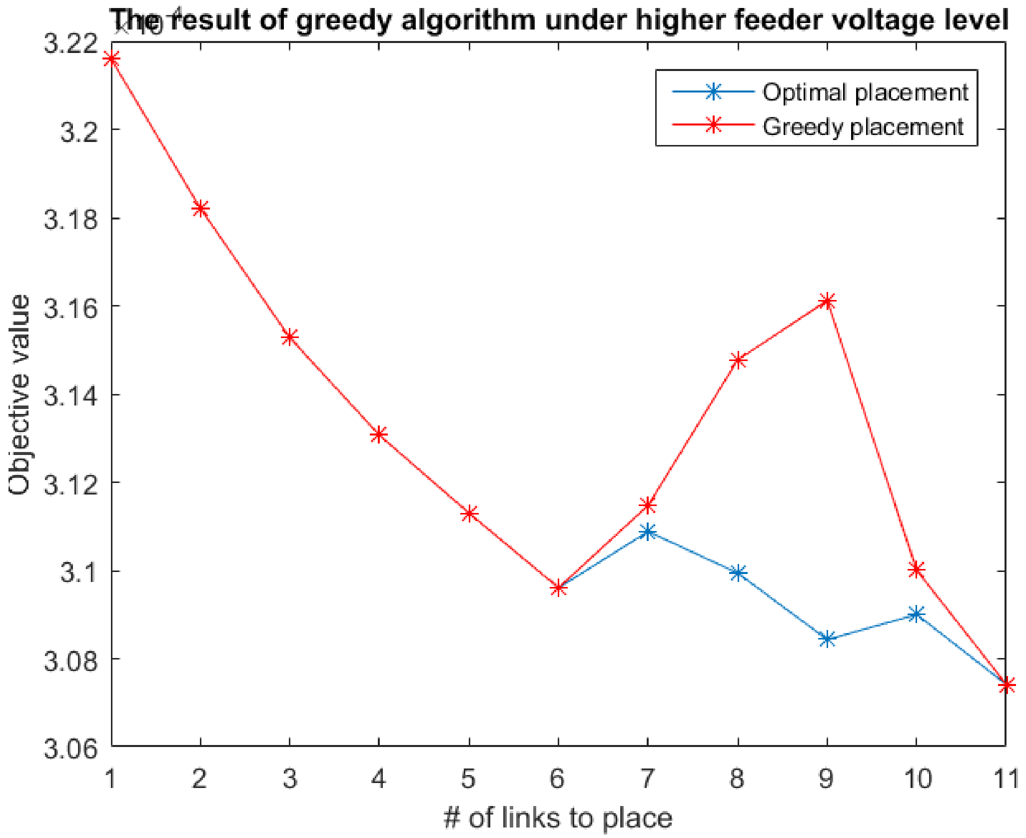
$$\sum_{k=1}^{K} \mathbf{s}_k = \mathbf{1}$$

$$1 \leq |\mathbf{s}_k|_0 \leq C_k, \forall k$$

- The bilevel problem is difficult to solve

Communication link deployment

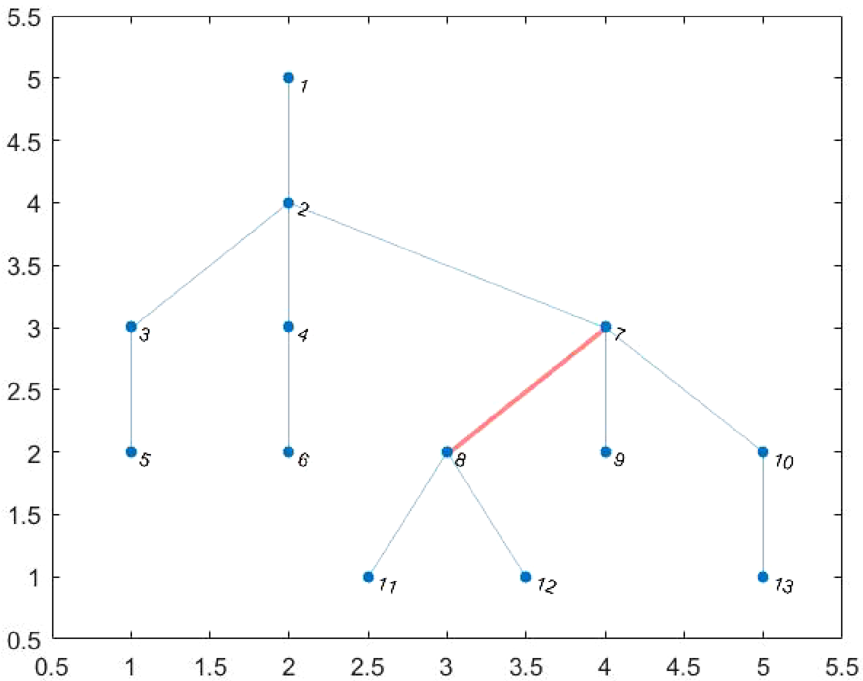
- Simulation results
  - With high feeder voltage level that some VAR injections are negative
  - Adding communication links is not always beneficial



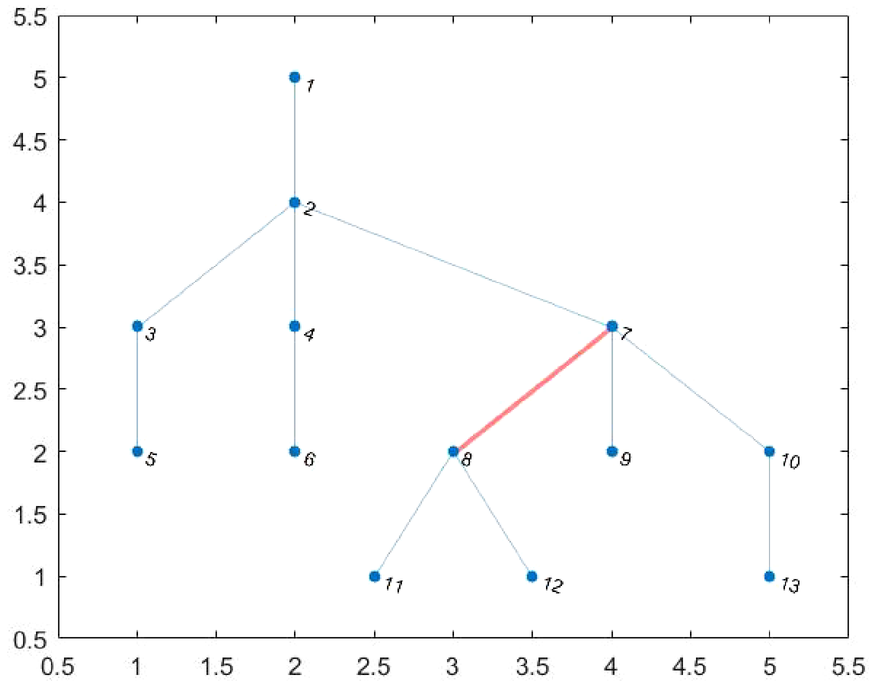
Communication link deployment

- Simulation results
  - With low feeder voltage level such that all VAR injections are positive
  - The sequence of link placement of two algorithms are different

Optimal deployment

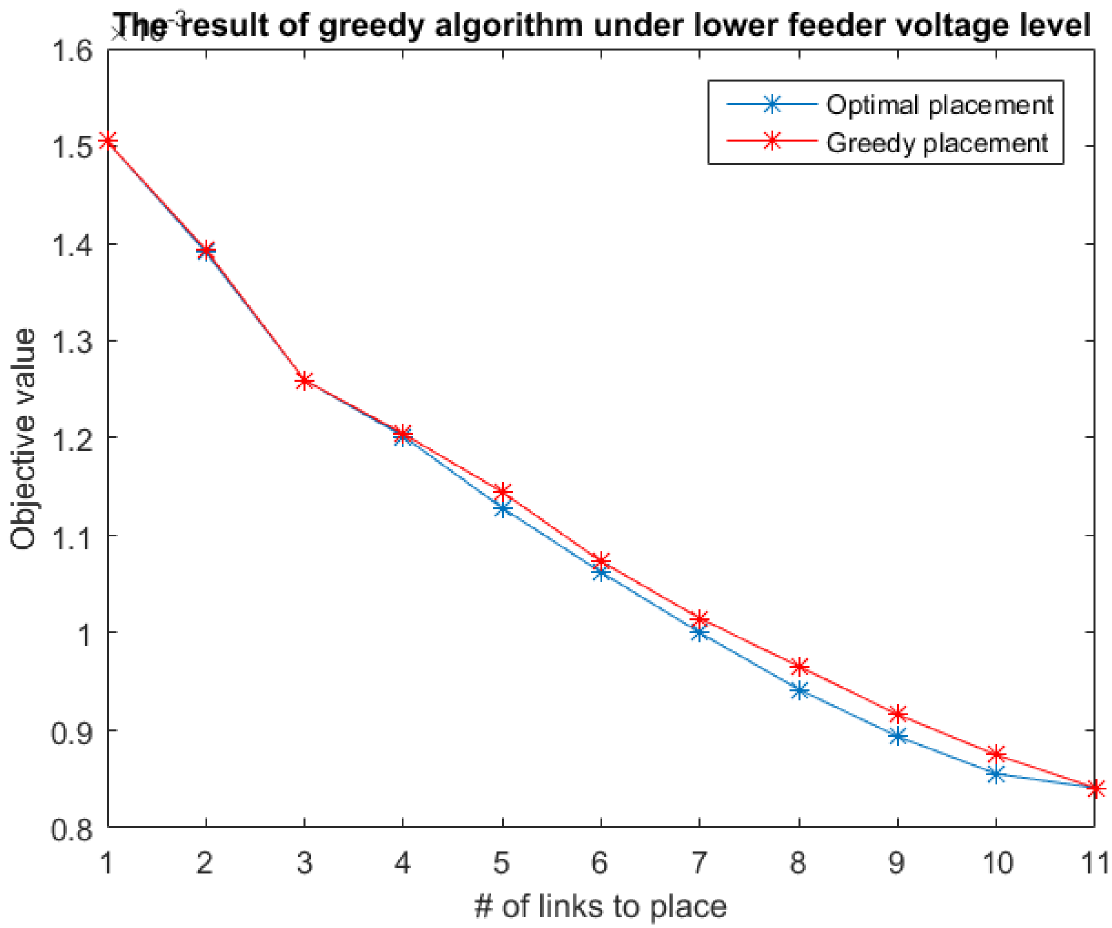


Greedy deployment



Communication link deployment

- Simulation results
  - The greedy algorithm is not so bad



Summary

- Characterize the equilibrium of the voltage-VAR control under limited communications from a game theoretic perspective
- Analyze the Nash equilibrium and investigate its general existence and uniqueness conditions
- Make connections between the NE and a convex optimization problem in two special cases
- Develop an asynchronous control algorithm that respects the communication limitations
- Propose a bilevel optimization problem for communication link deployment with preliminary simulation results
- Future work
  - Solve the bilevel problem by approximation or relaxation

Questions

Acknowledgement

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Uniqueness of the NE

- Denote $\mathbf{F}(\mathbf{q}) := \left(\nabla_{\mathbf{q}_k} U_k(\mathbf{q})\right)_{k=1,\cdots,K} = \tilde{\mathbf{X}}^T \left(\mathbf{X}\mathbf{q} + \tilde{\boldsymbol{\mu}}\right)$
- Uniqueness of NE: common sufficient conditions
  - Strict monotonicity of $\mathbf{F}(\mathbf{q})$ [Scutari '10]: $\tilde{\mathbf{X}}^T\mathbf{X} + \mathbf{X}^T\tilde{\mathbf{X}}$ needs to be PD, which does not hold for many examples of $\tilde{\mathbf{X}}$ in simulation
  - Diagonally strictly convexity [Rosen '58]: need to find a positive diagonal matrix $\mathbf{D}$ such that $\mathbf{D}\tilde{\mathbf{X}}^T\mathbf{X} + \mathbf{X}^T\tilde{\mathbf{X}}\mathbf{D}$ is PD. This requires $\tilde{\mathbf{X}}^T\mathbf{X}$ to be diagonally stable for the existence of such $\mathbf{D}$
  - Best response function constitutes a contraction mapping [Shum '07] or a standard function [Yates, '95]: no closed-form solution to a vector quadratic programming with box constraints
  - Prove the fixed point mapping is a contraction: Use the Projection Theorem [Bertsekas '89]. This is equivalent to requiring $\tilde{\mathbf{X}}^T\mathbf{X} + \mathbf{X}^T\tilde{\mathbf{X}}$ to be PD