Investigating the utility of Schwarz-Christoffel mapping theory for electric machine design and analysis

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Overview

• Machine design overview
• Schwarz-Christoffel (SC) mapping
• Application to motor design
• Examples
• Comparison to Finite Element Analysis (FEA)
• Conclusions
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1) Shape
   • Used to steer the flux

2) Materials
   • Affect efficiency, weight, acoustic properties, manufacturability, cost

3) Sources
   • Characteristics and placements of currents
   • Types and placement of permanent magnets
Machine design overview

Standard Methods

• Equivalent circuit models
  • use lumped parameters
  • derived empirically
  • may ignore certain higher order effects

• Magnetic circuit models
  • usually assume the flux direction
  • fringing is empirically modelled
  • force derived from coenergy formulations
Finite Element Analysis (FEA)

- Mature, widely available
- Can be extended to 3D
- Great for analyzing existing design
- Harder to use for design
- Accuracy depends on number, type of elements
- Solution is interpolated between nodes
- Optimization is time-consuming
- Force calculation tricky to program
• Machine design overview

• **Schwarz-Christoffel (SC) mapping**
  • Application to motor design
  • Examples
  • Comparison to FEA
  • Conclusions
Def’n: A **Schwarz-Christoffel map** is a function $f$ of the complex variable $z$ that conformally maps a canonical domain in the $z$-plane (a half-plane, unit disk, rectangle, infinite strip) to a “closed” polygon in the $w$-plane.
Def’n: A conformal transformation is a complex transformation that preserves angles locally. In other words, if $\Gamma_1$ and $\Gamma_2$ are two curves that intersect at an angle $\theta_z$ in the $z$-plane at point $p$, then the images $f(\Gamma_1)$ and $f(\Gamma_2)$ intersect at an angle $\theta_w = \theta_z$ at $q = f(p)$.

- All analytic, one-to-one mappings are conformal.
**Def’n:** Let \( f(z) = f(x+iy) = g(x,y) + i h(x,y) \) be an analytic function of \( z \). Then \( f \) satisfies Laplace’s equation and \( g \) and \( h \) satisfy the Cauchy-Reimann equations and are *conjugate functions*. Thus, if one of \( g \) or \( h \) describes a scalar potential function, then the other will describe the corresponding field lines.

<table>
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<th>( g(x,y) )</th>
<th>( h(x,y) = c1 )</th>
<th>( h(x,y) = c2 )</th>
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Thm: Fundamental Theorem of Schwarz-Christoffel Mapping

Let $D$ be the interior of a polygon $P$ having vertices $w_1, \ldots, w_n$ and interior angles $\alpha_1 \pi, \ldots, \alpha_n \pi$ in counterclockwise order. Let $f$ be any conformal map from the unit disk $E$ to $D$. Then

$$f(z) = f(z_0) + C \int_{z_0}^{z} \prod_{k=1}^{n} \left(1 - \frac{\zeta}{z_k}\right)^{\alpha_k - 1} \, d\zeta$$

for some complex constants $f(z_0)$ and $C$, where $f(z_k) = w_k$ for $k = 1, \ldots, n$. 

$z$-plane \hspace{5cm} \Rightarrow \hspace{5cm} w$-plane

$E$ \hspace{5cm} $f(z_0)$ \hspace{5cm} $w_1, w_2, w_n$ \hspace{5cm} $P$
**SC parameter problem:** how do we determine the correct location of the prevertices $z_k$?

\[
f(z) = f(z_0) + C \int_{z_0}^{z} \prod_{k=1}^{n} \left(1 - \frac{\zeta}{z_k}\right)^{\alpha_k - 1} \, d\zeta
\]
SC mapping

• Most problems have no analytic solution for the prevertices
  • For $n>3$ vertices, unless lots of symmetry, no analytic solution

• Numerical solution required for
  1. Solving the parameter problem
  2. Calculating the SC integral
  3. Inverting the map
Historical milestones – machine design with SC mapping

- **1820’s**: Gauss – idea of conformal mapping
- **1867-90**: Schwarz and Christoffel discover SC formula and variants
- **1900-01**: F.W. Carter uses SC mapping for field between poles
  - “I by no means recommend that one should go to the trouble of using these somewhat difficult formulae in average practical cases…” – Carter
- **1980**: Trefethen – SCPACK FORTRAN program
- **1996**: Driscoll – SC Toolbox for Matlab®
- **1998**: Driscoll and Vavavis – CRDT algorithm for multiply elongated regions
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Design goal: Calculate the electromagnetic fields and corresponding rotor torques/forces for a given geometry and set of materials and sources
Application to motor design

Assumptions:

1. 2D developed machine cross-section
2. Air gap is a polygon (no curves) with \( n \) vertices
3. Linear magnetics
4. Periodic boundary condition (BC) at polygon edges
5. Finite, discrete currents as sources

Periodic BC

\[ w_1 \rightarrow w_n \]

Periodic BC

\[ P \]
Application to motor design

$w = u + iv$

$z = x + iy$

$z' = \exp(-iw')$

$w' = i \log(z')$
Application to motor design

\[ w = f(z) \]

\[ z = f^{-1}(w) \]

- \( H_{cc} \) known in infinite series form due to Hague circa 1930.
- Periodic BC automatically enforced

\[ H_{mot}(w) = \frac{H_{cc}(f^{-1}(w))}{\left| f'(f^{-1}(w)) \right|^*} \]
Application to motor design

SC Toolbox for MATLAB®

• Released in 1996
• Solves parameter problem for half-plane, disk, strip, rectangle, and exterior maps
• Cross-ratio formulation of the parameter problem for multiply elongated regions (CRDT)
• Computes forward and inverse maps
• Computes derivative of maps (easier)
• Graphical and object-oriented user interfaces
Application to motor design

- Three prevertices can be placed arbitrarily
- Motor air gap polygon can have multiple elongations.
  - Leads to *crowding* phenomenon
  - Multiple prevertices indistinguishable in machine precision
    - Inaccurate SC integral
CRDT algorithm

• Eliminates crowding problem
• Driscoll and Vavasis 1998
• Incorporated in the SC Toolbox
• Very well suited for multiply-elongated regions
• Tends to be $O(n^3)$
Application to motor design

Force/torque calculation

1. **Coulomb Virtual Work (CVW) method**
   - Standard for FEA analyses
   - Coenergy method
   - Eases path dependencies inherent in FEA mesh

2. **Maxwell Stress Tensor (MST) method**
   - Integrate the MST around a closed path
   - Highly path- and element-dependent for FEA
   - Ideal for SC solution
     • No path dependence
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Examples

Infinite vertices are mapped to the circle; crowding occurs.
Examples

- Infinite strip map used to plot field lines
- Constant potential surfaces
- Periodic BC not enforced here
Examples

• 40-vertex air gap polygon
• 2 coils
Solve for $H_{cc}$ using Hague’s analytic solution.

\[ \mu_r = 100 \mu_0 \]

\[ \mu_s = 100 \mu_0 \]
Examples
MST Force Calculation

Integrate tangential and normal force densities around closed path:

\[
F_t = \frac{B_n B_t}{\mu_0} \quad \quad \quad F_n = \frac{1}{2\mu_0} \left( B_n^2 - B_t^2 \right)
\]
Examples
Normalized force density vector plot

x(cm.)
y(cm.)
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Comparison to FEA

**FEA**

- Solution at mesh points with interpolation
- Solved in a finite algorithm $Ax = b$ (time-stepping algorithms may use iterative schemes)
- 3D capabilities
- Accuracy depends on type and number of elements used
- Force calculation is highly path dependent due to interpolation

**SC Mapping**

- Solution at every point, with same accuracy
- Solved iteratively by numerical integration
- 2D only
- With CRDT, accuracy depends on stopping criteria
- Force calculation is path independent
Comparison to FEA

**FEA**

- Balloon boundaries and other conditions must be enforced to simulate infinity.
- Usually solve for \( A \), then differentiate to find \( B \), introducing truncation errors.
- Geometric complexity scales with \( n^2 \).

**SC Mapping**

- Infinite vertices are naturally incorporated in the theory.
- Can solve for \( H \) directly in many cases, eliminating finite difference approximation for derivative of potential.
- Geometric complexity scales with \( n \).
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• Design benefits
  • accurate field and force calculation
  • may be possible to design in conc. cylinder domain

• Iterative map solution still hides some of the variable dependencies from the designer

• Solution scaling
  • problem complexity grows with $n$, but CRDT is $O(n^3)$. May be OK since parameter problem solved only once
  • FEA problem complexity grows with $n^2$
Conclusions

- Can be useful when fields near sharp corners of poles and teeth are needed to high accuracy

- SC mapping is a promising technique due to its accuracy