Documenting students' faulty schema and misconceptions about combinations and permutations

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***Abstract*— STEM educators have devoted increasing attention to discrete mathematics in recent years due, in part, to its strong connections with subjects like computer science, probability and statistics, and business management. Combinatorics problems, in particular, while useful for modeling concrete situations, are often considered to be tricky for students. To develop a better understanding of students' conceptions regarding problems involving permutations and combinations, a secondary data analysis using a grounded theory approach was performed on transcripts of student interviews obtained during an earlier study. Participants had recently completed a college-level discrete mathematics course with a passing grade. Analysis focused on answering two research questions: 1) What patterns of responses do students generate while producing solutions to combinatorics word problems? 2) What underlying conceptual ideas lead to these patterns?**

Keywords—combinatorics, permutations, combinations, student conceptions/misconceptions

I. INTRODUCTION

Education researchers in STEM fields have devoted increasing attention to discrete mathematics in recent years due, in part, to its strong connections with subjects like computer science, probability and statistics, and business management. The National Council of Teachers of Mathematics (NCTM) devoted their 1991 yearbook to the topic [1]. The importance of combinatorics, in particular, has been increasingly emphasized in the mathematics education literature [2-5] as well as K-12 standards documents [6]. However, while combinatorics problems are known for modeling concrete problems, they are often considered to be tricky for students. In the next section we present a brief review of the literature on college students' difficulties with introductory combinatorics. Subsequent sections are devoted to our research questions, study methods, and results/discussion.

II. LITERATURE REVIEW

We restrict our focus to the post-secondary literature. See [7] and [8] for detailed reviews of the K-12 literature. There are only a handful of peer-reviewed papers published at the post-secondary level examining students' understanding of combinatorics. Several of these articles focus on more advanced topics than we will be addressing here [9-11] or use combinatorics problems as a context in which to study other issues such as cooperative problem solving [12], metacognition [13, 14], and semiotics [15].

Recently, four unpublished thesis dissertations have examined the learning of combinatorics at the undergraduate level. In the first of these [16], Smith employed a multiple case study approach to study how students, with varying levels of experience, approached solving basic counting problems of the four basic types appearing in Table 1. Note that a *permutation* is a particular ordering of k objects selected from among n distinguishable objects ($k \leq n$) whereas a *combination* is a selection of a particular subset of k of those n objects without regard to their order.

TABLE I. FOUR BASIC TYPES OF COUNTING PROBLEMS

	Permutations (ordered)	Combinations (unordered)
No repetition allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
Repetition allowed	n^k	$\binom{n+k-1}{k}$

Smith found that the permutation problems (both with and without repetition) were easy for students, whereas combination problems were difficult. Furthermore, students approached the permutation problems from a more conceptual perspective based on the use of the multiplication rule, whereas formula recall was the strategy of choice for combination problems. Smith warned, however, that students' conceptions of permutations were rigid and resistant to generalizability beyond the simplest of counting problems. Additionally, Smith found that students' understanding of the notion of ordered/unordered sets was tenuous. Students had no problem connecting ordered sets with permutations and unordered sets with combinations. However, when asked during interviews to elaborate on the distinction, most participants appeared to

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believe that the distinction had to do with whether the problem allowed repetition.

In the second dissertation [17], Kavousian presented college-level discrete-mathematics students with an unfamiliar (and somewhat complicated) definition and a corresponding set of tasks chosen to help reveal their concept image of the new mathematical object. The study revealed that, when attempting to understand the new concept, most students did not generate their own examples (rather, they expected examples to be provided for them by the teacher), and, while many students could eventually find a formula for counting these new objects after being exposed to related tasks, they did not make many of the anticipated connections to prior knowledge (e.g., knowledge of combinations and of the binomial theorem). Furthermore, students that generated pictorial representations of their new understanding did not consider these significant in their understanding. On the contrary, an algebraic representation was considered by students to be a necessary (and often sufficient) form of understanding.

In the third dissertation [18], Lockwood conducted *think aloud* interviews with advanced undergraduate and graduate students who were asked to solve difficult counting problems. The interview data was used to develop a model of combinatorial thinking in which students tend to relate algebraic *formulas/expressions* (like those contained in Table I) to *sets of outcomes* (the collection of objects being counted) via *counting processes* (the active mental and/or physical processes of enumeration). Lockwood also noted, however, that students frequently ignore the set of outcomes in counting problems, unless specifically prompted to address it. She later used the model as an analytical tool for interpreting her data as well as other researchers' data [19]. In a separate study [20], Lockwood also examined some of the spontaneous connections that students make between different combinatorics problems. Lockwood's is (to our knowledge) the only of the four dissertations to have resulted in peer-reviewed publications.

In the final dissertation [21], Halani studied the ways in which four Arizona State students, who had no prior experience with counting problems, conceived of the sets of outcomes in combinatorics problems. Halani identified three broad categories of ways that students think about solution sets which she labeled *subsets*, *odometer*, and *problem posing*. The subsets category involves envisioning the solution set as a union of subsets, while the odometer category involves holding an item (such as a digit in a number or character in a string) constant and varying the other items systematically. The problem posing category involves the spontaneous generation of a new problem whose solution set is somehow related to the solution set of the original problem.

III. RESEARCH QUESTIONS

Evidently, there are many gaps in the post-secondary literature on combinatorics education. In particular, among the studies we reviewed, several do not focus specifically on combinatorial concepts [12-15]. And those that do tend to focus on difficult problems [9, 10, 17] or examine populations of students with a great amount of mathematical experience [11, 18-20]. The exceptions are the studies by Smith [16] and Halani [21]. However, Smith focused only on the most basic counting problems (those of the form listed in Table I), and Halani's study involved students who had had no prior instruction in combinatorics.

As a result, little has been done at the college level with problems that are just a bit more difficult than those represented in Table I. For example, little is known about how students in possession of basic knowledge of permutations and combinations (but not much more than that) approach a problem that naturally lends itself to the use of both a permutation and a combination or a problem requiring a simple combination but with an added restraint. Such problems would not lend themselves to the "order matters implies permutations" and "order does not matter implies combinations" schemas. In light of Smith's [16] claim that the notion of ordered/unordered sets is problematic for students as well as his claim that students' models of permutations are inflexible and resistant to generalizability, one would expect these intermediate level problems to be a rich site for study. With these considerations in mind, we pose the following questions:

1) What patterns of responses do students generate while producing solutions to combinatorics word problems that are slightly more complex than those in Table 1?
2) What underlying conceptual ideas lead to these patterns?

IV. METHOD

In this study, we performed a secondary data analysis of data originally gathered by different researchers in the University of Illinois at Urbana-Champaign's Computer Science Department. The data was part of a larger project to construct assessment tools to measure students' understanding of discrete mathematics concepts used in theoretical computing. The only portion of the project that will be discussed here is the part directly relevant to the current research on students' conceptions regarding combinatorics problems. The first subsection below will describe the data collection process that was carried out by a team that did not include any of the individual researchers involved in the present study. The second subsection will describe the data analysis process that was carried out by the researchers involved in the present study.

A. Data collection

Here we detail the data collection process, including details about the participants, the interview process, and the interview questions (by which we mean the combinatorics problems to be solved during the interviews).

1) Participants

In the spring of 2009, eighteen undergraduate volunteers from the University of Illinois at Urbana-Champaign enrolled in the study by responding to an email sent to recent "CS173: Discrete Mathematical Structures" students whose course performance merited a grade of B or C. Of these, there were five women and thirteen men, all but two of whom were majoring in the mathematical sciences or engineering. The other two were undeclared. Fourteen were domestic students. Ten of the eighteen were computer science or computer engineering majors who are required to take a course on discrete mathematics. Students were paid $15 for their participation. Students with grades B or C were targeted as they were more likely to generate error-prone solutions. It should be noted, however, that Illinois has a highly competitive engineering program, and the average ACT score of the students who participated was 32.6 (the 99^{th} percentile). Of the eighteen participants, only eleven of them attempted to solve one or more of the interview questions specifically involving combinatorics, and, therefore, the present study included only those eleven students. Since the data had been anonymized prior to having been given to the current research team, there is no way to know which of the original eighteen students are included in the present study.

2) Interview process

Prior to being interviewed, students were briefed about the goal of the study, which was to understand how they think through various topics in discrete mathematics. They were warned that they would frequently be asked to expand on what they were writing or saying as they solved problems and were told not to expect feedback about the correctness of solutions. Participants were then interviewed for one hour about their understanding of a variety of discrete mathematics problems. They were instructed to "think aloud," vocalizing their thoughts as they solved problems and answered questions [22]. Students' work was recorded using a document camera. Audio was recorded with a microphone. The interviewer was an advanced-undergraduate computer science major. Participants were provided with a "cheat sheet" that contained basic notation and definitions that participants were free to consult during the interview. The portion of the "cheat sheet" relevant to the counting problems contained reminders of the notation for permutations and combinations as well as the formula for combinations with repetitions (the bottom-right formula from Table 1), though none of the problems actually required the use of this formula. All interviews were transcribed verbatim, and notes regarding what students wrote and when were included in the transcripts. All student work was scanned and stored electronically for future analysis.

3) Interview questions

The interview questions were a collection of fifteen discrete mathematics problems, four of which were basic counting problems involving permutations and combinations that will be discussed in the present study. These problems resembled those that the participants were likely to have previously encountered in CS173. Not all students answered all of the questions due to time constraints. Students were allowed to pick and choose which problems to attempt. The four counting problems are listed in Table II along with possible solutions. They are labelled Problem 3 through Problem 6 since these were the original labels of the combinatorics problems among the complete set of 18 discrete math problems.

Notice that, while Problem 3 and Problem 4 each have a potential "permutation only" solution (represented by the expression on the right of the equality sign in the solution), the contexts of these two word problems lend themselves strongly to a mixed approach that involves using a combination and a permutation (represented by the expression on the left of the equality). Also notice that, while the form of the solution is the same for both problems, the questions being asked appear quite different. In Problem 3 there are eight crayons being arranged in a box, some of which happen to have the same color. Problem 4, by contrast, involves a total of twelve crayons, none of which have the same color and from which only eight are being selected for later arrangement into the box. So, while a mathematician would tend to view these two problems as isomorphic, a novice might not.

Smith's (2007) study suggests that combinations alone are difficult for students, so Problem 5 is a simple combinations without replacement problem (as in Table I) in a context in which it is clear that order does not matter. Problem 6, on the other hand, is a combinations without replacement with an additional constraint—the three red crayons cannot all be next to one another. Furthermore, the context of Problem 6 is a situation in which the order of the crayons in the final arrangement clearly matters even though the most natural solutions involve the use of only combinations. This apparent contradiction is resolved by noting that the unordered selection of several slots in which to place crayons of one color can be done in exactly the same number of ways as there are to arrange the eight crayons. It is worth noting that this problem is most easily solved indirectly by counting *all* arrangements of the crayons, then counting the six "disallowed" arrangements, and, finally, subtracting to find the number of "allowable" arrangements. It is possible to solve this problem by focusing only on the allowable arrangements (see the expression on the right side of the equality in Table II), but this approach makes the solution process much trickier, and we would expect a lower rate of success for students who choose this approach.

TABLE II. INTERVIEW QUESTIONS AND SOLUTIONS

Problem Statement	Solution
Problem 3: Suppose we have 8 crayons, three of which are the exact same color of red, and the others of which are different. Now how many ways are there of arranging the crayons in the box?	$\binom{8}{3} 5! = \frac{8!}{3!}$
Problem 4: Now suppose we have 12 different colored crayons, and a box that holds only 8. How many different arrangements of 8 crayons, selected from the 12, can be put in the box?	$\binom{12}{8} 8! = \frac{12!}{4!}$
Problem 5: A bundle of crayons is a collection of colors in no particular order. How many different bundles of 5 crayons can be made from a set of 8 uniquely colored crayons?	$\binom{8}{5} = \frac{8!}{3! \cdot 5!}$
Problem 6: Suppose you have a box of eight crayons, three of which are red, and five of which are blue. How many ways are there to arrange the crayons in the box so that the three red crayons are not all next to each other?	$\binom{8}{3} - 6 = 2\binom{6}{2} + \binom{6}{3}$

B. *Data analysis*

Three researchers at the University of Illinois participated in the data analysis: a graduate student studying math education (the author), a professor, Geoffrey, specializing in engineering education, and an advanced undergraduate, Tom, majoring in computer science. A grounded theory approach [23] was used to analyze the interview transcripts and students' written work (hereafter referred to simply as "interviews"). First, each researcher individually coded the interviews, line by line, without any predetermined coding scheme. "Codes" were essentially annotated comments that referred to a specific line (or lines) or piece of written work. This approach allows theory to arise from the data, rather than force-fitting data to a predetermined theory. Second, the researchers met to discuss one another's coding, negotiate potential differences or disagreements, and eventually produce a single document containing all agreed upon codes. (Agreement on the application of the codes was essentially universal.) Next, Tom and the author reexamined all of the coded interviews to identify and come to agreement on any themes that were appearing in the data. A list of these themes were produced and then later discussed with Geoffrey present to ensure agreement. Finally, the author reexamined the interviews in an attempt to synthesize the thematic elements and develop theories that might help explain what contributes to students' difficulties in solving counting problems.

V. RESULTS AND DISCUSSION

One issue that became apparent while studying the transcripts was that the interviewer allowed his teaching instincts to interfere with the interview protocol (especially in earlier interviews). During uncomfortable moments when a student was struggling, the interviewer frequently asked leading questions, confirmed (or disconfirmed) students' work, or simply had the student move on to the next problem. At other moments, when students indicated that they were finished with a problem, they were allowed to move on without their solution being challenged or more deeply explored. This interference complicated the analysis of the transcripts as it was no longer possible to determine what the student would have done "naturally" (without the interference). As a result, we report only on findings that we believe hold up to scrutiny *in spite of* the interviewer interference.

Each subsection below, with the exception of the first, will focus on a theme that was identified in the data and was considered noteworthy of discussion. Segments of interview transcripts are used to provide vignettes that demonstrate each theme. Participants are referred to as Student 1 through Student 11. The five themes are summarized in Table III while the results are summarized in Table IV (located the end of this section) which indicates the presence (or lack thereof) of each theme for each student/problem pair.

A. *A note on overall succes rates*

Problems 3, 4, 5, and 6 were attempted by 11, 9, 6, and 8 of the students, respectively, for a total of 34 attempts. Of these attempts, 15 (or 44%) were found to be successful. By problem, there were 5, 4, 3, and 3 successful attempts, respectively, giving success rates between 37.5% and 50%. These rates are likely upper bounds as students were allowed to skip problems and likely skipped problems that they felt they would be unable to solve. It is also worth noting that 11 of the 15 successful attempts were made by just three of the students (Students 1, 7, and 8) who succeeded on every attempt they made. These numbers indicate (in agreement with the literature) that counting problems involving combinations are quite difficult for many students.

TABLE III. LIST OF THEMES

Problem assumptions: Student questions inherent problem assumptions. Four types of assumptions that were questioned included: the shape/capacity of the crayon box, the distinguishability of crayons, the importance of order, and the number of different crayon colors were implicated in a problem statement.
Dichotomous solutions: In the context of Problems 3 and 4, the student indicates that the solution should involve a permutation or a combination only, but not both.
Allowable arrangements: For Problem 6, the student employs a solution strategy that involves focusing on the enumeration of the allowable arrangements of crayons, rather than the simpler approach of counting all possible arrangements and then subtracting the number of disallowed arrangements.
Related constraints: For Problem 6, the student fails to notice that the arrangement of the crayons of one color within the box will necessarily constrain the number of possible arrangements of the crayons of the other color.
Use of diagrams: Simply indicates whether a student used a diagram in their solution attempt. Note that a string of characters indicating a particular arrangement of crayons (e.g., RBBRRBBB) as well as any visual representation of "slots" to be filled (e.g., use of underscores to represent slots) were all counted as diagrams.

B. *Understanding the problem assumptions*

One of the most common patterns we noticed was that novice counters tended to question problem assumptions more than "experts" (students who were most successful at solving the interview problems[1]). Four types of assumptions were questioned: the shape/capacity of the crayon box, the indistinguishability of crayons of the same color, the importance of order, and the number of different colors of crayon contained in a problem. The first and last will not be elaborated upon any further as they seemed to reflect ambiguities in the wording of the interview problems more than conceptual difficulties regarding counting.

1) *Distinguishability of crayons*

Several students questioned whether crayons of the same color were distinguishable from one another as demonstrated by the following exchange that occurred during an attempt to solve Problem 3:

> Student 4: […] you have two ways of deciding whether or not the, whether or not the crayon's arrangements are different. For example, if you have

[1] We are careful to note, therefore, that "expert" behavior used in our sense is not necessarily meant to refer to behaviors characteristic of an instructor or professor.

> a box of eight, um, if these three are the exact same shade of red, you could count them as essentially the same crayon, and then not care where those are placed in the box. And that would lead you to one answer. If you count them as distinct crayons, then it would be pretty simple. […]

> Interviewer: Well we give you that they're the exact same color. So…

> Student 4: Yeah, but they're *different* crayons.

Student 4 is caught up on the distinguishability of the three red crayons and notes that, while they are all red, they are still "different" crayons. In the "real world" different red crayons are, ultimately, distinguishable, but the more expert problem solvers in this study tended to more readily "buy into" the intended problem assumption that the red crayons are indistinguishable.

2) *Importance of order*

The most common assumption that students' questioned or clearly misunderstood based upon their given solution was whether "order matters." Technically, this condition refers to the order in which distinguishable objects are arranged. However, students often ask, rather vaguely, "whether order matters" without any real indication of precisely what set of objects has its "orderability" being called into questioned. In a particular problem it may be the case that order both matters and does not matter depending upon what you are referring to. For example, in Problem 3 and Problem 6 the order of the overall arrangement of crayons matters while the order of the reds or blues alone does not matter.

Issues with order came up for each of the four problems. In the following example, Student 3, having just finished reading the statement of Problem 3, vaguely questions the issue of order using non-normative language:

> Student 3: Would this be um, according to sequence, or just combinations?

Student 9, solving the same problem, seems to have a conflict between "order mattering" and "distinguishability":

> Student 9: And the order matters, so I don't see why it points out that there are three [crayons] of the same color, well kind of.

In contrast, Student 6, while solving Problem 5, and in direct violation of the problem statement, counts the number of "bundles" as though order mattered:

> Student 6: This is the same as the last one, isn't it? [Writes down 8 choose 5 times 5!] I mean it's the same idea from the last one, you're choosing 5 crayons from a set of 8, and then put that set of 5 crayons in a specific order.

More examples of order issues will be apparent in the discussion of the closely related "dichotomous solutions" theme to be discussed below.

Consideration of these implicit problem assumptions allows us to begin to grasp the cognitive complexity that combinatorics problems require of a novice. We found that most instances of questioned assumptions occurred during a failed solution attempt. If, for each student, we count each questioned assumption only once per problem, then there were 16 such instances. Of these 16 instances, 13 occurred during a failed solution attempt by one of six different students. In particular, all but one of the seven instances in which the "order matters" assumption was questioned occurred during a failed attempt.

Students with incomplete conceptions regarding distinguishability and order were prone to developing unstable interpretations of problems that impeded progress toward a solution. This observation suggests that students may need more time than is typically devoted to discussion of the meanings of various assumptions about properties like order and distinguishability and how they relate to various problems. In the author's experience, students are typically presented with a table of basic counting formulas like the one in Table I very early on in combinatorics instruction and focus quickly moves on to more "interesting" problems in which these formulas are taken as "given" knowledge. More time is likely needed for students to develop mental models for the various counting processes represented by these formulas and how these processes are related to various types of problem assumptions regarding order and distinguishability.

C. *Dichotomous solutions*

We previously alluded to a schema that students often employ that dictates that permutations are to be used when "order matters" while combinations are to be used when order "does not matter." By *dichotomous solutions* we refer to an extension of this schema which further dictates that the solution to a counting problem must use either a permutation or a combination, but not both. Or, said differently, for a given counting problem either order matters or order does not matter (in some vague overall sense), but not both. We found evidence for the existence of this schema in solution attempts for Problem 3 and Problem 4—the two problems most likely to solicit solutions that involve both a permutation and a combination.

Four of the eleven students (Students 4, 6, 9, and 10) demonstrated this schema, and one demonstrated it on both problems (Student 10), giving a total of five instances. Note that none of these students were any of the three highly successful students (Students 1, 7, and 8). Consider first the following which occurred after Student 4 had abandoned a first solution attempt of Problem 3 and was starting on a fresh solution:

> Student 4: OK. [Reading from the cheat sheet.] Number of ways to choose elements from a set of size N… ok. So yeah, there's…yeah. The three that are the same and five that are distinct. Um…so I guess for each, each individual slot, you are choosing, for eight slots you're choosing on of…six possibilities it seems like, because these [referring to the 3 red crayons] are all counted as one possibility, um, so it might be eight choose six…umm, we have to choose R elements, oh nevermind, that doesn't take into account order. So it would be, because you care about order, it would be a permutation of…out of eight, six. [Writes "P(8,6)"] Yeah. Cause there's six distinct elements. And you want to choose all the possible ways of arranging them.

Notice that this student begins by fixating on the notation for combinations contained on the cheat sheet and continues by using choosing language ("you are *choosing*, for eight slots you're *choosing* one of…" and "so it might be eight *choose* six…umm, we have to *choose* R elements"). But the student abruptly abandons the use of combinations because they do not "take into account order" and decides to go with a permutation, which is this student's final answer.

As another example, here is Student 6 struggling with Problem 4:

> Interviewer: [Student writes 12 choose 8.] And that counts what?
>
> Student 6: How many ways to choose 8 from the 12. And then, that's pretty much it, isn't it?
>
> Interviewer: Well, what do you think?
>
> Student 6: [Repeats problem statement aloud.] Well this will count how many ways you can pick 8. Then, how many different arrangements of eight crayons is 8 factorial. So would it be times 8 factorial? No. No, that's right, because you're still counting... This is gonna count all the arrangements you can get from 12, how many different ways you can get 8 from the 12, That should be fine, shouldn't it? Just 12 choose 8?
>
> Interviewer: Ok, that's it for you?
>
> Student 6: Well, I guess, we have 12 different colored crayons, wait, no no no no. 12 different colored crayons in a box that holds only eight. Do I still use 12 choose 8? Cause it's different crayons. So if its 12 choose 8, that means, that wouldn't, that's unordered. And then we have to choose 8 that *are* ordered. Which is 8 factorial.
>
> Interviewer: Do you mean that there are two different "choosing processes" happening?
>
> Student 6: I'm, I think so. Kind of, because it's first we're choosing 8 random crayons from the 12, and then, which is kind of, unordered I guess, because you don't care which 8 you choose. Well, do you? No, yeah. I can't use 12 choose 8, because that means, ah damn. I'm confusing myself. So, they're all different colored crayons, so if you have 8 crayons, they're different colors, so there's gotta be 8 factorial in there. Errg.

The interesting thing about this attempt is that Student 6's instincts are ultimately leading to a correct solution, namely, $C(12,8)\cdot 8!$. However, the dichotomous solutions schema is preventing the student from accepting it as it mixes permutations with combinations. The student is sure at first that C(12,8) must be part of the solution, but later the student realizes that 8! must be part of the solution since order matters and reneges on the combination. This student goes on to offer $C(12,8)\cdot 8!$ as a final answer but does so with reluctance and admits that he or she is unsure if it gives the correct number or not. (Unfortunately, the interviewer immediately confirmed the answer rather than probing further.)

As illustrated by Student 6, the dichotomous solution schema tends to result in unstable interpretation of the problem assumptions. As students seesaw back and forth between wanting to use permutations and wanting to use combinations, they must also toggle between the assumptions that order matters and that order does not matter. This toggling, in turn, creates dissonance with students' intuitions (rooted in the physical context of the problem) that *both* assumptions appear to be true.

We speculate that the dichotomous solution schema is likely induced (at least partially) by instruction. Language including statements like "use permutations when order matters and use combinations when order does not matter" is probably common in instruction. It is also implicit in tables like Table I that are contained in most introductory combinatorics and discrete math textbooks. Consequently, it may be a good idea to encourage students to think of Table 1 as a collection of counting *techniques* rather than a collection of solutions to counting problems (though they are that too). Perhaps there would then be less of a tendency for students to believe that they are mutually exclusive in the solutions to counting problems. Also, as students build more robust notions regarding the concept of "order mattering"—in particular as they develop the understanding that it is not a global feature of a counting problem but, rather, a local feature that can differ for various aspects of the problem—they probably become more likely to discard the dichotomous solution schema.

D. Focus on allowable arrangements

Recall that Problem 6 has a solution (the one on the right side of the equality in Table II) that is obtained by focusing on the enumeration of "allowable" arrangements of the crayons, rather than enumerating all possible arrangements and then subtracting the number of "disallowed" arrangements (the six in which all three red crayons are adjacent to one another). The former strategy leads to a more difficult solution process as it is easier to describe and count the disallowed arrangements than it is to describe and count the allowed ones. Nevertheless, this strategy was tempting for students. Perhaps this result is not surprising as the strategy involves inspecting the actual solution set (that is, the set of different arrangements whose cardinality is the answer to the counting problem) as opposed to two sets which are not the solution set but that happen to be closely related to it. Four students (Students 2, 4, 6, and 7, making up half of the students who attempted Problem 6) tried to enumerate allowable arrangements directly. Only Student 7, who solved all problems correctly, was successful in using this approach.

The following vignette is notable in that Student 4 continues to focus on an allowable arrangements strategy, even *after* having computed the number of disallowed arrangements and after stating that the strategy is not "very slick" (emphasis ours):

> Student 4: So, um, there's one two three four five six, six total possibilities, where it's not allowed, essentially. And there's a total of, the permutations of, oh no, that's not right. Let's see. Can't do a simple permutation cause they're not distinct elements. The six is, the six was where there's not...for each, for each place it...for the first place you have two choices, red or blue, and if it's red...that's not a very slick way of doing it. And I'm not...once again, lazy, so I don't wanna go and draw out all possible combinations and just count them. Uhhh...but. *How many ways are there...so that the three crayons are not all next to each other?* Alright. Now I just have the impulse to talk, so that I am not leaving a big space on the transcript, but, nothing is actually coming into my mind at the moment, so...

Having computed that there are six disallowed arrangements, Student 4 attempts to connect this information back to the original question. But, in doing so, has reverted to a permutation-centric approach that draws the student's attention back to the allowable arrangements of red crayons. This approach leads to a rather convoluted analysis that leaves the student wondering whether his/her focus should instead be placed on the blue crayons—at which point the interviewer intervenes in an attempt to refocus the discussion on the disallowed arrangements (emphasis ours):

Student 4: How would I count that...I would put down the well I'd put down the red one's first. Um, so, I'm starting with one there's, you know there's eight choices on the first one. And then there's seven choices on the second one. So the first one, put down here [first spot], second one, just go ahead and go straight across. The third one, there's not this choice [red], but there's all the rest of the choices, so there are five choices for the remainder. And I think that's going to always be true. It's gonna be eight choices for the first one, if you're putting in red first, seven for the second one, and then five, well I guess if you put a red one in the middle...red one in the middle, one two three four, five six seven eight, and then there, a red here [to the right of the middle red crayon], and then there's, if that's in the middle then those two are both invalid [the spaces to the side of the two middle crayons], so it's possible that it could be eight, seven, and then four. *So I guess the other way to say that the three red crayons are not all next to each other is that, there is, um, there is some gap in the blue ones. That not all five blue ones are continuous.* So it could be blue, blue, I guess, yeah.

Interviewer: OK, so what did you count before? When you were doing, you know, the written count of the...

Student 4: *The original count was the, was all the possible illegal placements of red.*

Interviewer: OK. So you counted that. So what are you currently trying to count?

Student 4: I'm trying to count the total number of placements[2], using three and, five separate elements […]

Thus, some students more readily focus attention upon the set of arrangements evident in the problem statement at the expense of the other closely related sets (e.g., the set of all arrangements and the set of disallowed arrangements) that simplify the counting process. While this finding is not particularly surprising, it does suggest that the technique of examining related sets that are not explicitly mentioned in the statement of a counting problem should be focused on as an explicit goal of instruction rather than simply being an implicit aspect of the solution process of particular problems.

E. *Related constraints*

Another theme that was highlighted in the context of Problem 6 was students' failure to recognize "related constraints" on the arrangements of crayons. In the vignette above Student 4 recognized the constraint on the red crayons would result in a related constraint on the blue crayons (but mischaracterized it). Student 2, in contrast, never perceives this related constraint. Frustrated with examining allowable arrangements of red crayons, this student shifts focus to the blue ones in the following excerpt (emphasis ours):

Student 2: Yeah, um, well I don't think this is a good strategy, cause I can't really, cause these are different...I could split these up into containers, red red red red...ok. Little stuck here. So I can have, this [Draws new arrangement of red crayons], I can have like...this, [Draws another arrangement], I can have...[Draws another arrangement]...ok, well, I'm gonna think about it a different way now, *let's just ignore the reds for now*, and I know that I have five other spaces that I can choose. Blues. *So I have no restriction on where I put the blues in.* Or anyway, I can say, for the blues...order doesn't matter, so I could just do, so there's five places [Student spends some time doing calculations.] alright no, I'm stuck. *I'm trying to figure out how I can arrange just the blues in, and not worry about the reds*, so...

In the next vignette, Student 3 does not seem to realize that, once the reds have been placed, the placement of the blues is determined and continues counting the number of ways to arrange the blues (and, further, counts permutations of the eight crayons as though they were all distinguishable):

Interviewer: So, what were you just counting there?

Student 4: I was just counting um, how many positions it is possible for three of them to be together, and it's six. So I know to eliminate that. In the...

Interviewer: Eliminate it from what?

Student 4: From the thing I'm trying to calculate. So from that...it's...just...eight...[Writes, "8C3*8C5"... then tacks on, "8C3*8C5*8! - 6"]

Interviewer: So, where did you get the eight choose three times eight choose five times eight factorial?

Student 4: So eight choose three would be basically choosing three of the crayons, the three reds, and multiplying it with the five...

Interviewer: So it's, so the eight choose three counts the number of ways to place the three red crayons?

Student 4: Right right. And...

Interviewer: And, and what is the, the eight choose five counts the ways to place the blue crayons?

Student 4: Right. Um, and then this factorial is for the arrangement of all of those. And minus six is to...

Interviewer: To, yeah, take away the invalid cases.

Student 4: Right.

F. *Use of diagrams*

An unexpected finding was that the use of diagrams was associated with unsuccessful solution attempts. A diagram was produced during 19 solution attempts (representing all four problems), only four of which were successful attempts. Almost all diagrams were some sort of a representation of the slots in the crayon box (e.g. a string of horizontal dashes or squares or tick marks) or a representation of an arrangement of crayons (e.g., a string of R's and B's). So one reason for this association could be that students who are unsure of what to do may be more likely to draw specific arrangements of crayons/colors. It may also be the case that students with more counting experience develop a gestalt-like connection between the counting process and the algebraic formulas and, therefore, no longer need to use diagrams. Future research that examines

[2] Here the student is referring to the total number of arrangements allowed subject to the given restriction on the reds.

the use of diagrams in a more nuanced fashion [3] could potentially provide valuable insight into the problem solving process for counting problems.

TABLE IV. SUMMARY OF RESULTS

Understanding Assumptions / Other Themes				
	Problem 3	Problem 4	Problem 5	Problem 6
Student 1	B/-	-/-		-/D
Student 2	O/D	C/D	-/D	-/DER
Student 3	O/D	-/D		OI*/DR
Student 4	BI/D2		-/-	B/DE
Student 5	B/DE			
Student 6	IO/-	-/2	O/-	-/DE
Student 7	-/-	C/-	-/-	-/DE
Student 8	-/-	O/-	-/-	-/-
Student 9	-/D	-/2		
Student 10	-/2	-/D2	O/D	-/D
Student 11	-/D	-/D		

Table IV Key: Understanding Assumptions - **B**: questioned shape of box, **I**: questioned indistinguishability of crayons (**I*** - though not explicitly mentioned out loud, this student's solution implicitly implied distinguishable, same-colored crayons), **O**: questioned importance of order, **C**: questioned the number of crayon colors. Other Themes - **2**: dichotomous solution schema present, **E**: enumeration of allowable arrangements, **R**: failure to perceive related constraints, **D**: drew a diagram. Note: '-' indicates absence of questioned assumptions/other themes. Green/Red indicates successful/unsuccessful solution attempts. White indicates the problem was not attempted.

VI. CONCLUSION

The five themes discussed above (understanding problem assumptions, dichotomous solutions, focus on allowable arrangements, perception of related constraints, and use of diagrams) highlight various difficulties that students experience when solving basic combinatorics problems involving permutations and combinations. Many of these difficulties are attributable to overlooked subtleties regarding the problem assumptions in combinatorics problems. We have tried to highlight some of these thorny issues via the vignettes presented in the previous section and, in some cases, have offered suggestions about how some of the difficulties might be mitigated (or at least not encouraged) by instruction. We have specifically called attention to students' difficulties with understanding problem assumptions and dichotomous solutions because instructors are more likely to be blind to these difficulties as their expertise blinds them to what assumptions they are making during their own problem solving [24].

[3] More nuanced than just a binary code for whether a diagram was used or not as was done for this study. Different types of diagrams, for example, could be distinguished and details regarding how they are produced and used during the problem solving process could be studied.

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