GEAR Junior Retreat

Homework Problems

(1) Prove directly from definition that the visual boundary $\partial_\infty F_n$, of a finitely generated free group is a Cantor set. To do this, you must prove that $\partial_\infty F_n$ is totally disconnected and perfect.

(2) Show that if G is a non-elementary hyperbolic group, then ∂_T is discrete.

(3) Show that if X is a proper, complete, CAT(0) space, then the identity map $\partial_T X \to \partial_\infty X$ is continuous. This is proved in Bridson-Haefliger but try to figure it out from definition. You should be able to form a verbal sentence to explain why this is true - and then you could write a precise proof!

(4) Let $\Gamma = F_2 \times \mathbb{Z}$. Then Γ acts geometrically on $X = T \times \mathbb{R}$ where T is the regular 4-valence tree. Prove that $\partial_\infty X$ is not locally connected in two ways. First, "compute" its boundary and observe that you know the space you get is not locally connected because you took point set topology at some point. Second, prove directly from definition that a specific point is a point of non-local connectivity.

(5) ($\star$) Find an example of a one-ended CAT(0) group that is *not* hyperbolic that has locally connected 1-dimensional boundary. And don't use $\mathbb{Z} \oplus \mathbb{Z}$ or anything quasi-isometric to it!

(6) ($\star$) Prove that any group of the form $F_2 \rtimes \mathbb{Z}$ is a CAT(0) group. This is not true for $F_n \rtimes \mathbb{Z}$ for $n \geq 3$. You can try to construct an example yourself but it might be a little tough. Gersten gave a beautiful and simple example of such things. We can talk about these in the course.

Open Problems

(1) Suppose Γ is a one-ended group acting geometrically on a CAT(0) space X with ∂_∞ locally connected. If Γ acts geometrically on another CAT(0) space Y, must Y be locally connected?

(2) Suppose Γ acts geometrically on CAT(0) spaces X and Y. Does there exist a bijection between the components of $\partial_T X$ and $\partial_T Y$?

(3) **Vague Question:** How do the topologies (Tits and visual) of a space X determine the splittings of a a group Γ acting geometrically on X?