Junior GEAR Retreat
Complex hyperbolic geometry and quasi-Fuchsian groups

John R. Parker

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1 Problem Session

The following problems shall help you assimilate the material covered in this mini course. The problems which are more challenging are marked with a $*$.

1. Let $A \in \mathrm{PU}(2,1)$. If A does not fix ∞, show that A can be decomposed as $A = T_1 DURT_2$ where T_1 and T_2 are Heisenberg translations, D is a Heisenberg dilation fixing o and ∞, U is a Heisenberg rotation fixing o and ∞ and R is a given (holomorphic) inversion swapping o and ∞.

2. The Heisenberg group $\mathfrak{N}$ is $\mathbb{C} \times \mathbb{R}$ with the group law

$$(\zeta_1, v_1) \cdot (\zeta_2, v_2) = \big(\zeta_1 + \zeta_2, v_1 + v_2 + 2\mathrm{Im}(\zeta_1 \overline{\zeta}_2)\big).$$

   Show that $(\zeta, v)^{-1} = (-\zeta, -v)$.

   The Cygan metric is defined by

$$d\big((\zeta_1, v_1), (\zeta_2, v_2)\big) = \Big| |\zeta_1 - \zeta_2|^2 - iv_1 + iv_2 - 2i\mathrm{Im}(\zeta_1\overline{\zeta}_2) \Big|^{1/2}.$$

   Show that this metric satisfies the triangle inequality.

3. Let $A \in \mathrm{SU}(2,1)$ where the Hermitian form is $\langle \cdot \, , \cdot \rangle$. Suppose that λ, μ are eigenvalues of A with eigenvectors $\mathbf{u}$, $\mathbf{v}$ respectively.

   (i) Show that $\overline{\lambda}^{-1}$ is also an eigenvalue of A.

   (ii) If $|\lambda| \neq 1$ show that $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

   (iii) If $|\lambda| \neq 1$ show that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ unless $\mu = \overline{\lambda}^{-1}$.

4. (i) Show that any element A of $\mathrm{PU}(2,1)$ can be written as the product of (anti-holomorphic) reflections two in Lagrangian planes.

   (ii) Characterise which elements A of $\mathrm{PU}(2,1)$ can be written as the product of (holomorphic) reflections (of order two) in two complex lines. Relate the distance/angle between these complex lines in terms of the trace of the lift of A to $\mathrm{SU}(2,1)$.

   (iii) Find an element A of $\mathrm{SU}(2,1)$ with real trace that cannot be written as the product of (holomorphic) reflections (of order two) in two complex lines.

5. Let A be an element of $\mathrm{SU}(2,1)$ with eigenvalue λ and trace τ.

   (i) Show that the characteristic polynomial of A is

$$\chi_A(x) = x^3 - \tau x^2 + \overline{\tau} x - 1.$$

   (ii) Using the Cayley-Hamilton theorem, show that

$$A^2 = \tau A - \overline{\tau} I + A^{-1}.$$

   (iii) Using $\chi_A(\lambda) = 0$, show that

$$A(\lambda A + A^{-1} - \tau\lambda I + \lambda^2 I) = \lambda(\lambda A + A^{-1} - \tau\lambda I + \lambda^2 I).$$

   (iv) Deduce that for any non-zero $\mathbf{z}$ the vector $(\lambda A + A^{-1} - \tau\lambda I + \lambda^2 I)\mathbf{z}$ is a λ-eigenvector of A.

6. Let A be any element of $\mathrm{SU}(2,1)$ and let $\tau = \mathrm{tr}(A)$. Let $f(\tau)$ be the function

$$f(\tau) = |\tau|^4 - 8\mathrm{Re}(\tau^3) + 18|\tau|^2 - 27.$$

   (i) Show that $\mathrm{tr}(A^2) = \tau^2 - 2\overline{\tau}$ and

$$f(\tau^2 - 2\overline{\tau}) = \left(|\tau|^2 - 1\right)^2 f(\tau).$$

   (ii) Show that $\mathrm{tr}(A^3) = \tau^3 - 3|\tau|^2 + 3$ and

$$f\left(\tau^3 - 3|\tau|^2 + 3\right) = \left(|\tau|^4 - \tau^3 - \overline{\tau}^3\right)^2 f(\tau).$$

   Interpret the above formulae in terms of eigenvalues.

7. Let A, B and C be elements of $\mathrm{SU}(2,1)$ with $ABC = I$. Show that

$$\begin{aligned}\mathrm{tr}(A^{-1})\mathrm{tr}(B) - \mathrm{tr}(A^{-1}B) &= \mathrm{tr}(B^{-1})\mathrm{tr}(C) - \mathrm{tr}(B^{-1}C) \\ &= \mathrm{tr}(C^{-1})\mathrm{tr}(A) - \mathrm{tr}(C^{-1}A).\end{aligned}$$

8. * Let A and B be loxodromic elements of $\mathrm{SU}(2,1)$ with repulsive and attractive eigenvectors $\mathbf{r}_A$, $\mathbf{a}_A$ and $\mathbf{r}_B$, $\mathbf{a}_B$ respectively. Define the cross-ratios

$$\mathbb{X}_1 = \frac{\langle \mathbf{r}_A, \mathbf{a}_B\rangle\langle \mathbf{r}_B, \mathbf{a}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_B\rangle\langle \mathbf{r}_A, \mathbf{a}_A\rangle}, \quad \mathbb{X}_2 = \frac{\langle \mathbf{a}_A, \mathbf{a}_B\rangle\langle \mathbf{r}_B, \mathbf{r}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_B\rangle\langle \mathbf{a}_A, \mathbf{r}_A\rangle}.$$

(i) Express the traces of AB and $A^{-1}B$ in terms of the eigenvalues of A, B and the cross-ratios $\mathbb{X}_1$, $\mathbb{X}_2$.

(ii) Express $\mathbb{X}_1$ and $\mathbb{X}_2$ in terms of the eigenvalues of A, B and the traces of AB and $A^{-1}B$.

Define

$$\mathbb{X}_3 = \frac{\langle \mathbf{a}_B, \mathbf{a}_A\rangle\langle \mathbf{r}_B, \mathbf{r}_A\rangle}{\langle \mathbf{r}_B, \mathbf{a}_A\rangle\langle \mathbf{a}_B, \mathbf{r}_A\rangle}.$$

(iii) Show that the sign of $\mathrm{Im}(\mathbb{X}_3)$ is determined by the sign of $\mathrm{tr}[A,B]$.

9. * Let A and B be loxodromic elements of $\mathrm{SU}(2,1)$ and suppose that $\left|\mathrm{tr}(A)\right|$ and $\left|\mathrm{tr}(B)\right|$ are large. Show that if the distance between the axes of A and B is large enough then $\langle A, B\rangle$ is discrete and free.

Express the condition on the distance between the complex axes of A and B in terms of cross-ratios.

Quantify the notion of large in the above construction. That is, give constants N and M so that if $\left|\mathrm{tr}(A)\right| > N$, $\left|\mathrm{tr}(B)\right| > N$ and the distance between the axes of A and B is greater than M then $\langle A, B\rangle$ is discrete and free.

10. (i) * Let $\langle A, BA^{-1}B^{-1}\rangle$ be a pair of pants group and take the HNN extension $\langle A, B\rangle$ by adjoining B to $\langle A, BA^{-1}B^{-1}\rangle$. Now perform a Fenchel-Nielsen twist by taking the HNN extension $\langle A, BK\rangle$ where K commutes with A. Show that we can determine $\mathrm{tr}(K)$ in terms of traces of $\langle A, BK\rangle$ and $\langle A, B\rangle$.

(ii) * Now consider $\langle A, B\rangle$ and $\langle A^{-1}, C\rangle$. Form $\langle A, B, C\rangle$, the free product of $\langle A, B\rangle$ and $\langle A^{-1}, C\rangle$ amalgamated along $\langle A\rangle$. Now perform a Fenchel-Nielsen twist to obtain $\langle A, B, KCK^{-1}\rangle$, the free product of $\langle A, B\rangle$ and $K\langle A^{-1}, C\rangle K^{-1}$ amalgamated along $\langle A\rangle$, where K commutes with A. Show that we can determine $\mathrm{tr}(K)$ in terms of traces in $\langle A, B, C\rangle$ and $\langle A, B, KCK^{-1}\rangle$.

2 Open Problem Session

Here are some open problems about complex hyperbolic quasi-Fuchsian groups.

1. Develop analytic tools that enable us to describe the space of CHQF groups (in the same way that the Teichmüller/Ahlfors/Bers theory of quasiconformal mappings enable us to describe classical quasi-Fuchsian space).

2. Let Σ be a closed (without boundary) surface of genus $g \geq 2$. Let Γ_1 and Γ_2 be two representations of $\pi_1(\Sigma)$ in the same component of CHQF space. Is there a quasiconformal map (in the sense of Korányi-Reimann) conjugating Γ_1 and Γ_2? (In the non-compact case this is false by a theorem of Miner.)

3. Let Σ be a closed (without boundary) surface of genus $g \geq 2$. Describe the component of CHQF space containing the $\mathbb{R}$-Fuchsian Teichmüller space of Σ. In particular, is it a ball? Describe other components. In particular, does bumping happen?

4. Let Σ be a closed (without boundary) surface of genus $g \geq 2$. Which closed curves on Σ is it possible to pinch in order to obtain points on the boundary of CHQF space?

5. Does every CHQF representation of a punctured surface arise in the boundary of a component of CHQF representations of a closed surface?

6. Which closed curves on a pair of pants is it possible to pinch and still have a discrete representation? Is it possible to pinch eight curves simultaneously? If so (when) is the resulting group discrete? What about for a more general surface (with or without boundary)?

7. Are there any geometrically infinite representations on the boundary of CHQF space?

8. Give information about how the Hausdorff dimension $d_H(\Lambda)$ of the limit set Λ varies in CHQF space. For example, if Σ is a punctured surface, investigate the behaviour of $d_H(\Lambda)$ along a path (parametrised by the Toledo invariant) joining $\mathbb{R}$-Fuchsian representations (where $d_H(\Lambda) = 1$) and a $\mathbb{C}$-Fuchsian representation (where $d_H(\Lambda) = 2$).

9. Which knot or link complements arise as the boundary of complex hyperbolic orbifolds? When are they rigid/when is it possible to deform them?

10. Take your favourite theorem for classical quasi-Fuchsian groups and see whether it is true for CHQF groups.

3 Bibliography

The following references will form the basis for the course. They contain more material than I shall be able to cover. In addition, their bibliographies contain many more references.

[1] John R Parker,
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[2] John R Parker & Ioannis D Platis,
Complex hyperbolic Fenchel-Nielsen coordinates,
Topology **47** (2008) 101-135.

[3] John R Parker & Ioannis D Platis,
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ed Frederick P Gardiner, Gabino Gonzalez-Diez, Christos Kourouniotis.
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[4] John R Parker & Pierre Will,
Complex hyperbolic free groups with many parabolic elements,
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