Junior GEAR Retreat
SL_2-character varieties of 2 and 3-manifolds through examples.

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Abstract: We will study many examples of character varieties of surfaces and 3-manifolds groups. Along the way, we will review their algebraic properties as their symplectic structure (if any), ideal points, torsion form and boundary structures.

1 Problem Session

The following problems shall help you assimilate the material covered in this mini course. The problems which are more challenging are marked with a *.

1. Describe the character variety of the following groups: $\mathbb{Z}, \mathbb{Z}^2, F_2, \mathbb{Z}/p\mathbb{Z}$.

2. Compute explicitly the symplectic structure of the character variety of the torus $(S^1)^2$.

3. Show that the character variety of the figure eight knot whose group is $\langle u, v | wv = uw \rangle$ where $w = v^{-1}uvu^{-1}$ is

$$\{(x, y) \in \mathbb{C}^2 / (x^2 - y - 2)(2x^2 + y^2 - x^2y - y - 1) = 0\}$$

   where $x = \operatorname{tr} \rho(u)$ and $y = \operatorname{tr} \rho(uv)$.*

4. Describe the character variety of the Heisenberg 3-manifold $H(\mathbb{R})/H(\mathbb{Z})$ where $H(A) = \{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in A\}$.

5. Find the ideal points of the character variety of the figure eight knot and the corresponding incompressible surfaces. *

6. Compute the torsion form of the handle body of genus 2.
7. Find a relation between the symplectic form on the character variety of a surface and the Reidemeister torsion, viewed as a volume form. *
8. Describe the character variety of the complement of the torus knot $T_{p,q}$ whose fundamental group is $\langle a, b | a^p = b^q \rangle$.
9. Let M be the complement of three fibers of the Hopf fibration $p : S^3 \to S^2$. Describe its character variety and the application induced by the restriction on the boundary.*

2 Open Problem Session

The following questions or broad ideas are currently being studied in areas related to the mini course.

1. Understand the precise relation between skein modules of 3-manifolds and character varieties. Is the first the algebra of functions on the second?

2. This question is indeed a question of reductibility: is it true that the skein module of the complement of a knot in S^3 is reduced? (does not have non trivial nilpotent elements)

3. Let A be the A-polynomial of a knot $K \subset S^3$, and X_A its Hamiltionian vector field viewed as a vector field on $X(S^3 \setminus K)$. Let T be the Reidemeiser torsion viewed as a 1-form on $X(S^3 \setminus K)$: compute $\frac{L_{X_A} T}{T}$. This question has implication in topological quantum field theory.

4. Show that the skein module of a knot complement in S^3 is free over $\mathbb{C}[t, t^{-1}]$.

3 Bibliography

The following material will be useful for the course.

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4. J. Marché, Geometry of the representation spaces in SU(2). Strasbourg master class on geometry, 333370, IRMA Lect. Math. Theor. Phys., 18, Eur. Math. Soc., Zrich, 2012.

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7. P. B. Shalen. Representations of 3-manifold groups. Handbook of geometric topology, 955-1044, North-Holland, Amsterdam, 2002.