# On Computational Thinking, Inferential Thinking and "Data Science"

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*September 24, 2016*

# What Is the Big Data Phenomenon?

- Science in confirmatory mode (e.g., particle physics)
  - *inferential issue*: massive number of nuisance variables
- Science in exploratory mode (e.g., astronomy, genomics)
  - *inferential issue*: massive number of hypotheses
- Measurement of human activity, particularly online activity, is generating massive datasets that can be used (e.g.) for personalization and for creating markets
  - *inferential issues*: many, including heterogeneity, unknown sampling frames, compound loss function

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- "*It should only improve as we collect more data; in particular it shouldn't slow down*"
- "*There are serious privacy concerns of course, and they vary across the clients*"

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- Big Data analysis requires a thorough blending of computational thinking and inferential thinking
- What I mean by computational thinking
  - abstraction, modularity, scalability, robustness, etc.
- Inferential thinking means (1) considering the real-world phenomenon behind the data, (2) considering the sampling pattern that gave rise to the data, and (3) developing procedures that will go “backwards” from the data to the underlying phenomenon

# The Challenges are Daunting

- The core theories in computer science and statistics developed separately and there is an oil and water problem
- Core statistical theory doesn't have a place for runtime and other computational resources
- Core computational theory doesn't have a place for statistical risk

# Outline

- Inference under privacy constraints
- Inference under communication constraints
- The variational perspective

# Part I: Inference and Privacy

with John Duchi and Martin Wainwright

# Privacy and Data Analysis

- Individuals are not generally willing to allow their personal data to be used without control on how it will be used and now much privacy loss they will incur
- “Privacy loss” can be quantified via differential privacy
- We want to trade privacy loss against the value we obtain from “data analysis”
- The question becomes that of quantifying such value and juxtaposing it with privacy loss

# Privacy

query

database

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query

$\downarrow$

database

$\downarrow$

$$\tilde{\theta}$$

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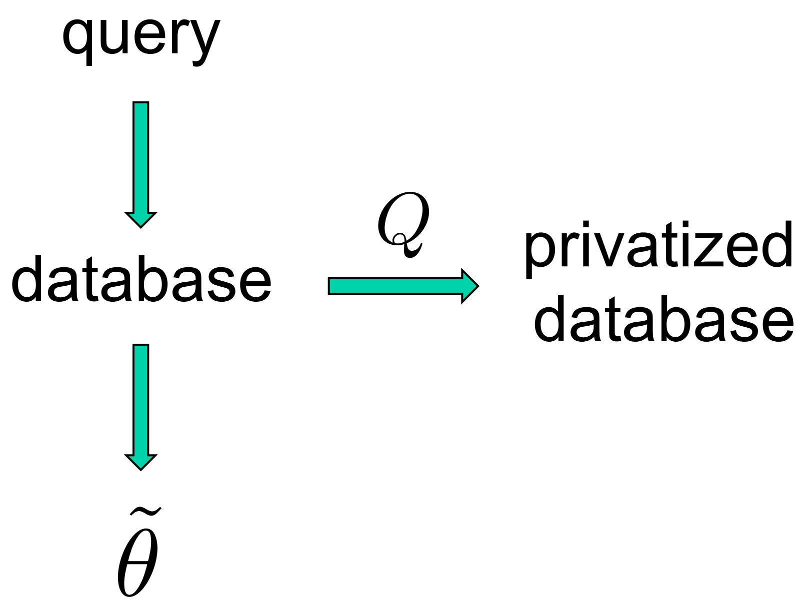

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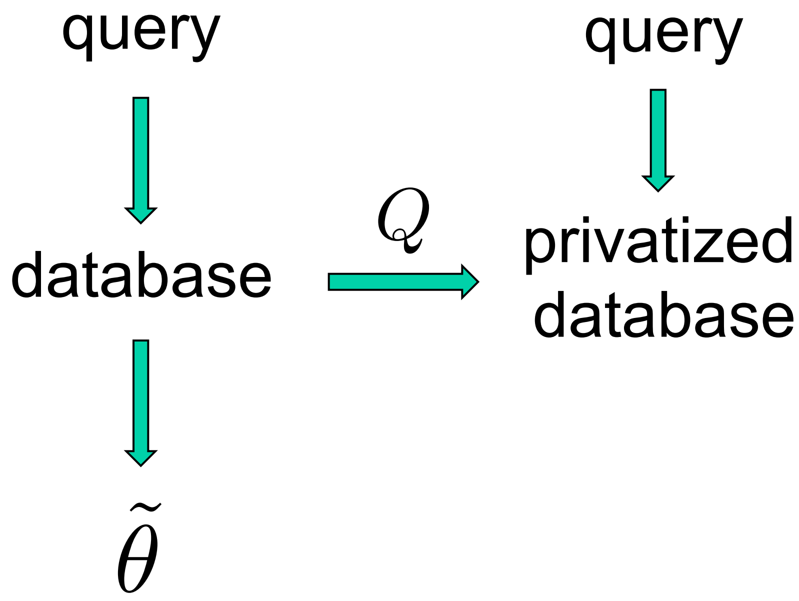

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$$\begin{tikzcd}
\text{query} \arrow[d] & & \text{query} \arrow[d] \\
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Classical problem in differential privacy: show that $\hat{\theta}$ and $\tilde{\theta}$ are close under constraints on $Q$

# Inference

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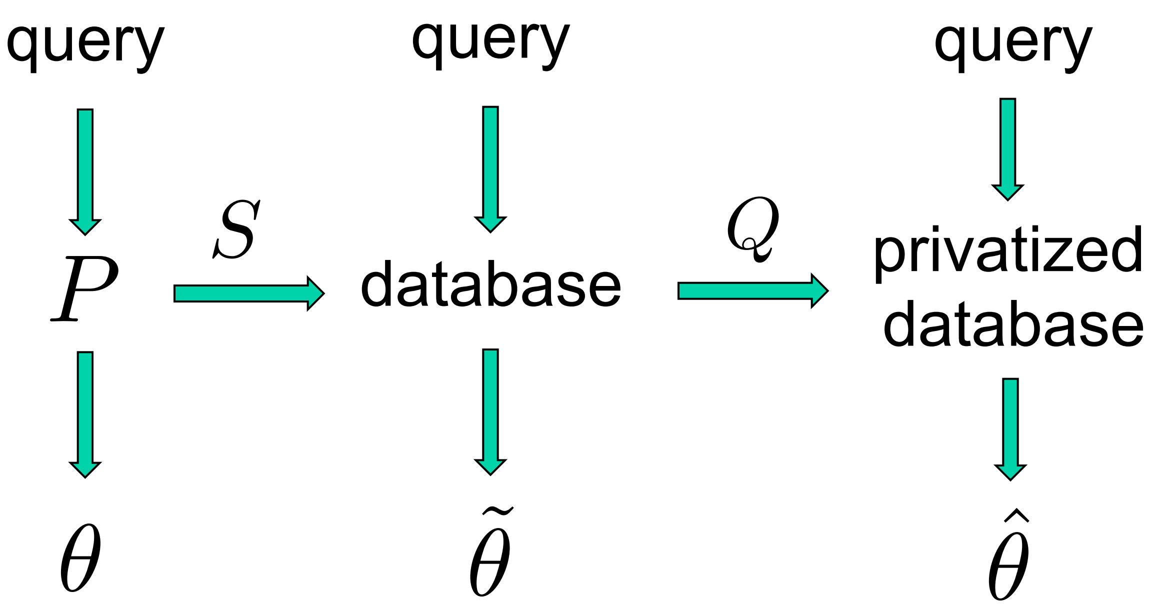
Classical problem in statistical theory: show that $\tilde{\theta}$ and $\theta$ are close under constraints on $S$

# Privacy and Inference




The privacy-meets-inference problem: show that $\theta$ and $\hat{\theta}$ are close under constraints on $Q$ and on $S$

# Background on Inference

- In the 1930's, Wald laid the foundations of statistical decision theory
- Given a family of distributions $\mathcal{P}$, a parameter $\theta(P)$ for each $P \in \mathcal{P}$, an estimator $\hat{\theta}$, and a loss $l(\hat{\theta}, \theta(P))$, define the risk:

$$R_P(\hat{\theta}) := \mathbb{E}_P \left[ l(\hat{\theta}, \theta(P)) \right]$$

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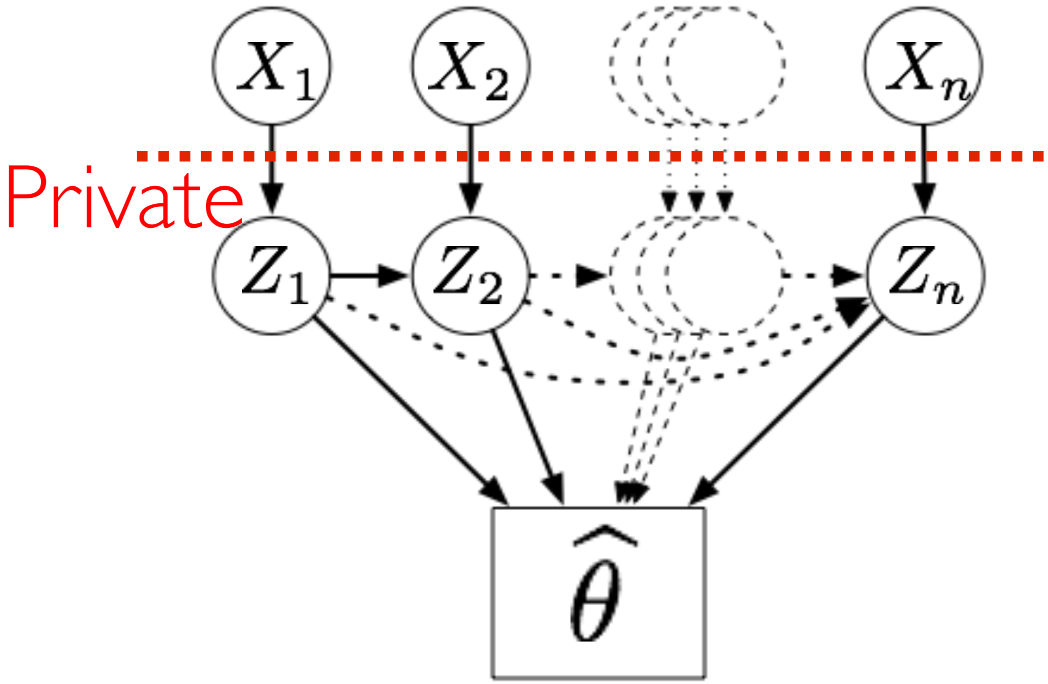
- Minimax principle [Wald, '39, '43]: choose estimator minimizing worst-case risk:

$$\sup_{P \in \mathcal{P}} \mathbb{E}_P\left[l(\hat{\theta}, \theta(P))\right]$$

# Local Privacy



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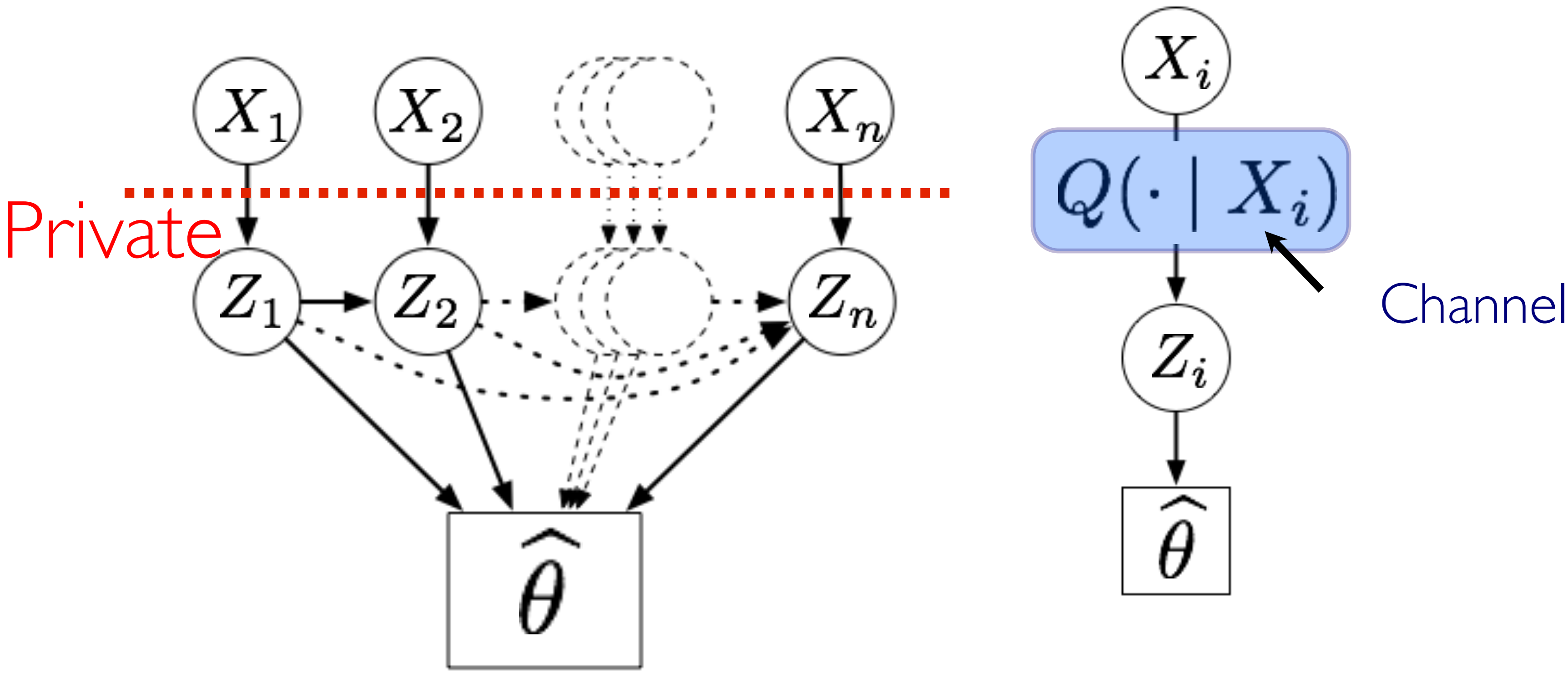



Individuals $i \in \{1, \ldots, n\}$ with private data $X_i \stackrel{\text{iid}}{\sim} P$

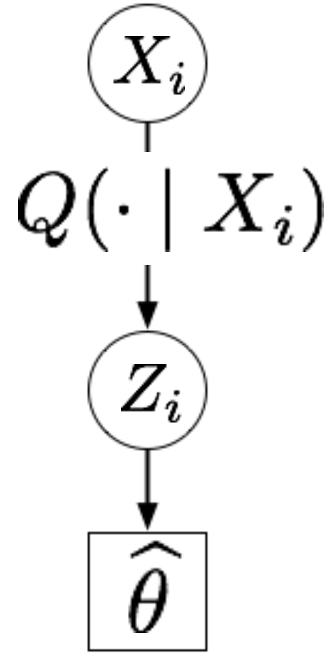
Estimator $\quad Z_1^n \mapsto \widehat{\theta}(Z_1^n)$

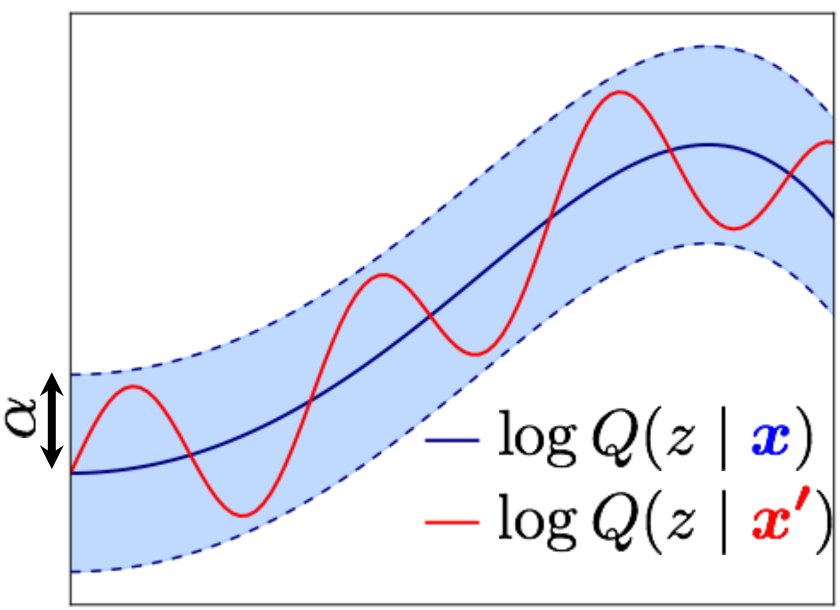
# Differential Privacy

**Definition:** channel $Q$ is $\alpha$-differentially private if

$$\sup_{S, x \in \mathcal{X}, x' \in \mathcal{X}} \frac{Q(Z \in S \mid x)}{Q(Z \in S \mid x')} \leq \exp(\alpha)$$

[Dwork, McSherry, Nissim, Smith 06]




Given $Z$, cannot reliably discriminate between $x$ and $x$'

# Private Minimax Risk

- Parameter $\theta(P)$ of distribution
- Family of distributions $\mathcal{P}$
- Loss $\ell$ measuring error
- Family $\mathcal{Q}_\alpha$ of private channels

$\alpha$ -private Minimax risk

$$\mathfrak{M}_n(\theta(\mathcal{P}), \ell, \alpha) := \inf_{Q \in \mathcal{Q}_\alpha} \inf_{\widehat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P,Q}\left[\ell(\widehat{\theta}(Z_1^n), \theta(P))\right]$$

Best $\alpha$-private channel

Minimax risk under privacy constraint

# Vignette: Private Mean Estimation

**Example:** estimate reasons for hospital visits

Patients admitted to hospital for substance abuse

Estimate prevalence of different substances

| | |
|---|---|
| 1 | Alcohol |
| 1 | Cocaine |
| 0 | Heroin |
| 0 | Cannabis |
| 0 | LSD |
| 0 | Amphetamines |

Proportions

$$\theta = \begin{array}{l} \theta_1 = .45 \\ \theta_2 = .32 \\ \theta_3 = .16 \\ \theta_4 = .20 \\ \theta_5 = .00 \\ \theta_6 = .02 \end{array}$$

# Vignette: Mean Estimation

Consider estimation of mean $\theta(P) := \mathbb{E}_P[X] \in \mathbb{R}^d$, with errors measured in $\ell_\infty$-norm, for

$$\mathcal{P}_d := \{\text{distributions } P \text{ supported on } [-1,1]^d\}$$

**Proposition**:

Minimax rate

$$\mathfrak{M}_n(\mathcal{P}_d, \|\cdot\|_\infty) \asymp \min\left\{1, \frac{\sqrt{\log d}}{\sqrt{n}}\right\}$$

(achieved by sample mean)

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Private minimax rate for $\alpha = O(1)$

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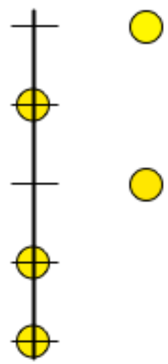
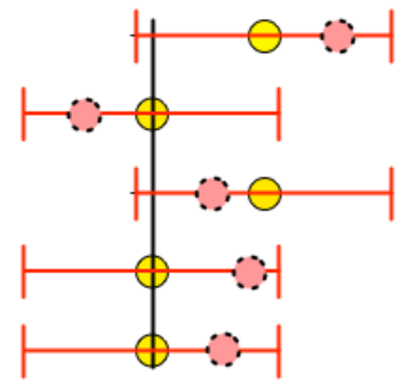
**Note:** Effective sample size $n \mapsto n\alpha^2/d$

# Optimal mechanism?

$$X = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



$$Z = X + W = \begin{bmatrix} 1 + W_1 \\ 0 + W_2 \\ 1 + W_3 \\ 0 + W_4 \\ 0 + W_5 \end{bmatrix}$$



Non-private observation

**Idea 1**: add independent noise (e.g. Laplace mechanism)

[Dwork et al. 06]

**Problem:** magnitude much too large (this is unavoidable: *provably sub-optimal*)

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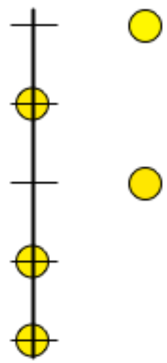

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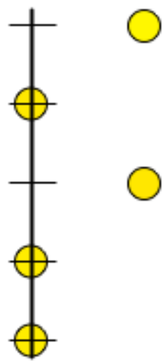
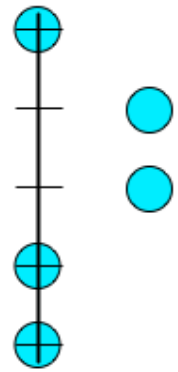


Non-private observation

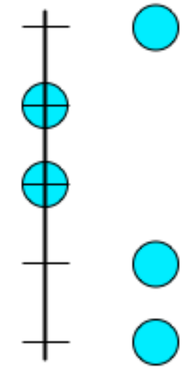
$$v = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



View 1

(closer: 3 overlap)

$$1 - v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$



View 2

(farther: 2 overlap)

- Draw $v$ uniformly in $\{0, 1\}^d$

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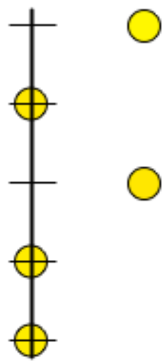

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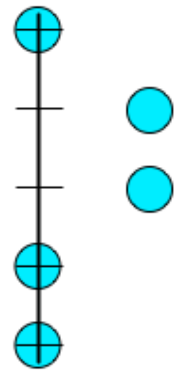

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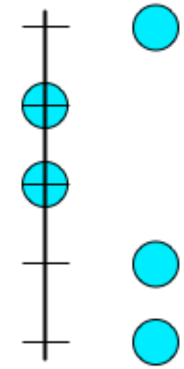


View 2

(farther: 2 overlap)

- Draw $v$ uniformly in $\{0, 1\}^d$

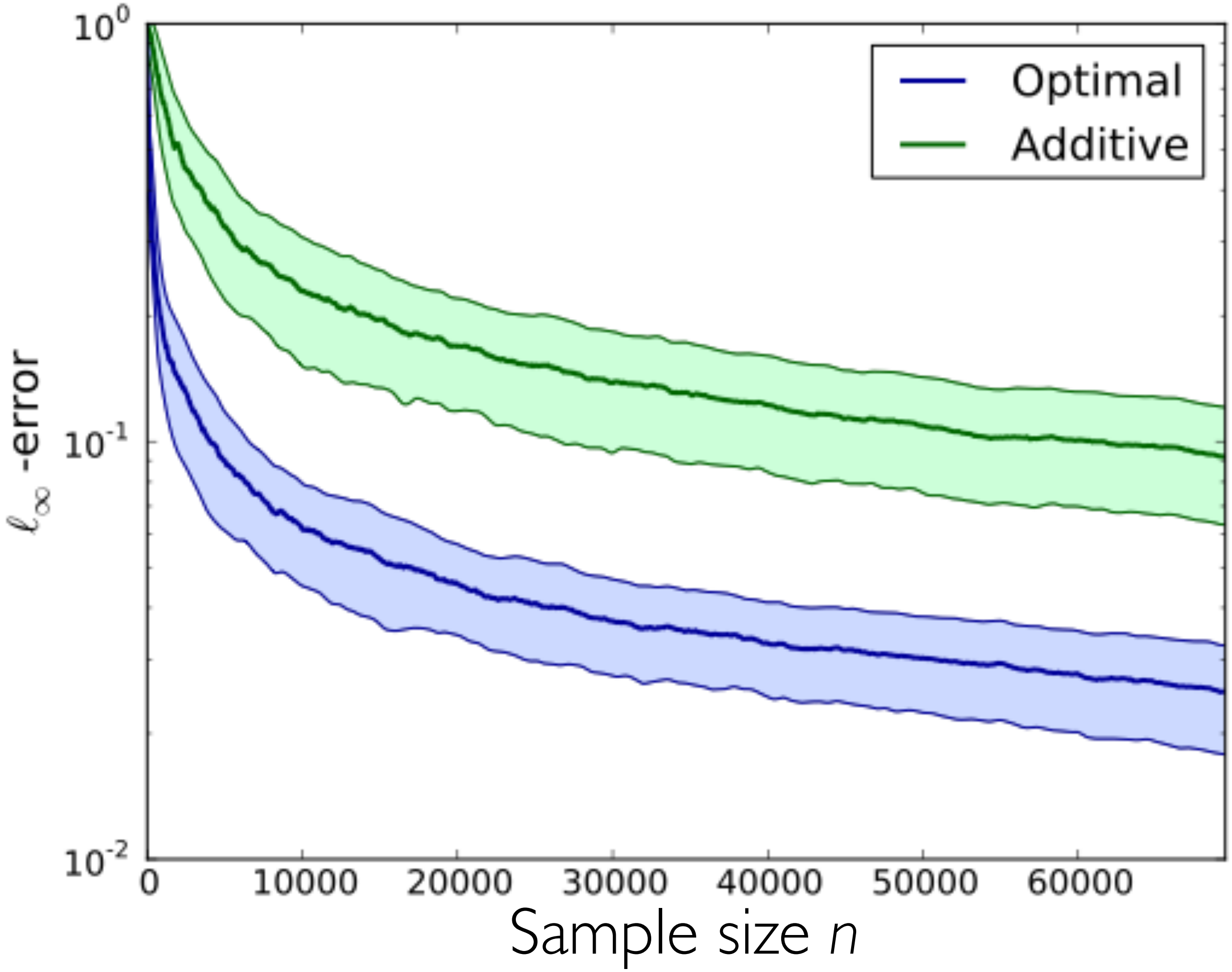
- With probability $\frac{e^\alpha}{1 + e^\alpha}$

  choose closer of $v$ and $1 - v$ to $X$

- otherwise, choose farther

At end:
Compute sample average
and
de-bias

# Empirical evidence




Data source:
Drug Abuse
Warning
Network

Estimate proportion of emergency room visits
involving different substances

# Additional Examples

- Fixed-design regression
- Convex risk minimization
- Multinomial estimation
- Nonparametric density estimation

- Almost always, the effective sample size reduction is:

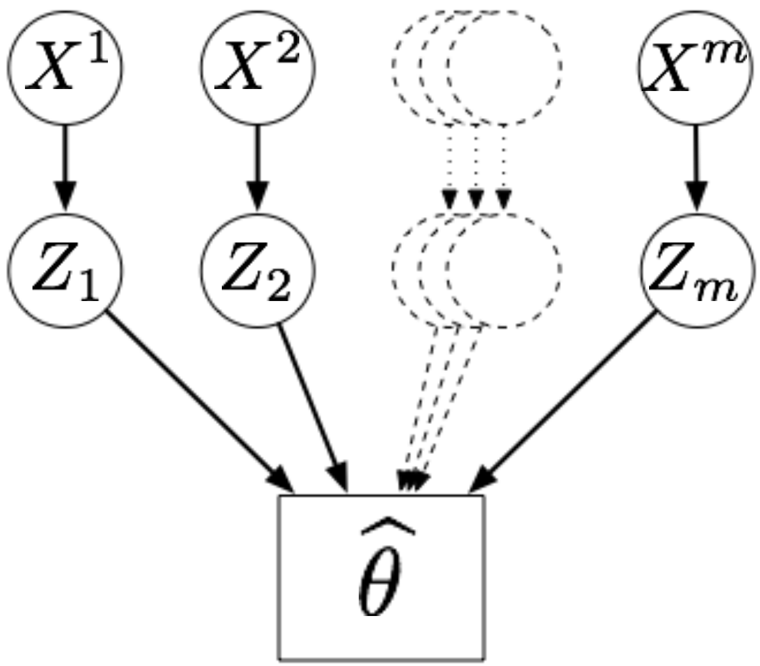
$$n \mapsto \frac{n\alpha^2}{d}$$

# Part III: Inference and Compression

with Yuchen Zhang, John Duchi and Martin Wainwright

# Communication Constraints

- Large data necessitates distributed storage
- Independent data collection (e.g., hospitals)
- Privacy




Setting: each of $m$ agents has sample of size $n$

$$X^i = (X_1^i, X_2^i, \ldots, X_n^i)$$

Messages $Z_i$ to fusion center

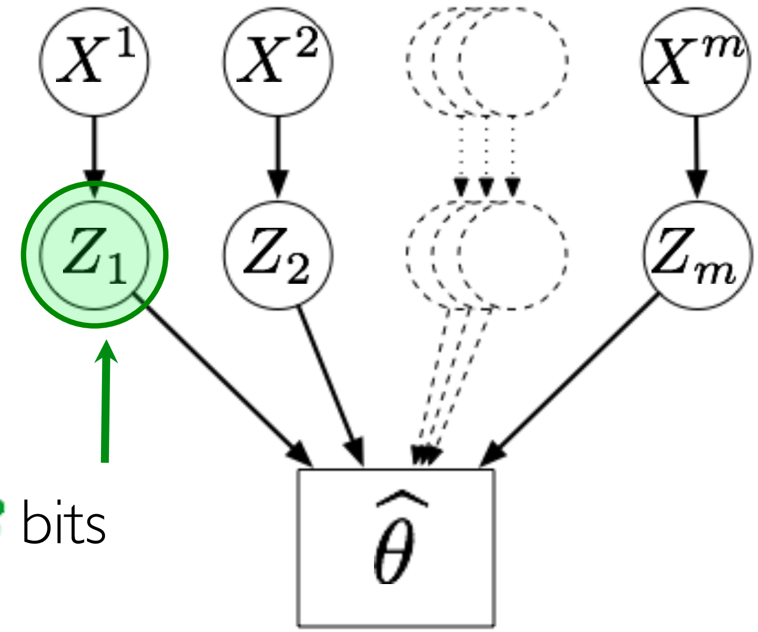
**Question:** tradeoffs between communication and statistical utility?

[Yao 79; Abelson 80; Tsitsiklis and Luo 87; Han & Amari 98; Tatikonda & Mitter 04; ...]

# Minimax Communication

**Central object of study:**

- Parameter $\theta(P)$ of distribution
- Family of distributions $\mathcal{P}$
- Loss $\|\cdot\|_2^2$

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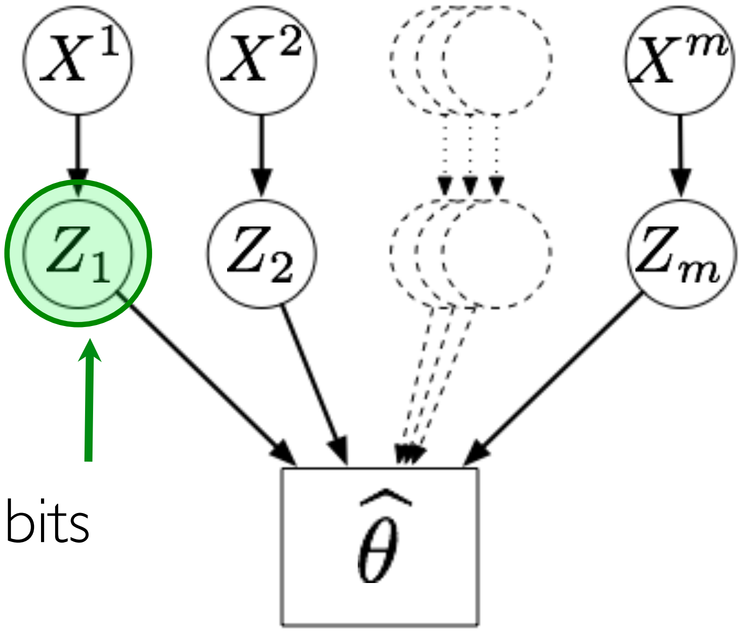



Constrained to be $\leq B$ bits

Minimax risk with $B$-bounded communication

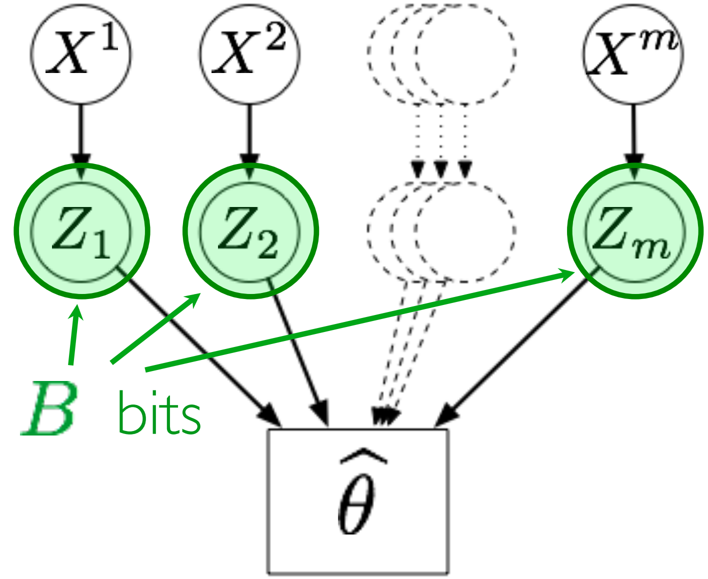
$$\mathfrak{M}_n(\theta(\mathcal{P}), B) := \inf_{\pi \in \Pi_B} \inf_{\widehat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[ \|\widehat{\theta}(Z_1^m) - \theta(P)\|_2^2 \right]$$

Best protocol $Z_i = \pi(X^i)$ with $Z_i$ smaller than $B$ bits

# Vignette: Mean Estimation

Consider estimation in normal location family, $\theta \in [-1, 1]^d$

$$X_j^i \stackrel{\text{iid}}{\sim} \mathsf{N}(\theta, \sigma^2 I_{d \times d})$$



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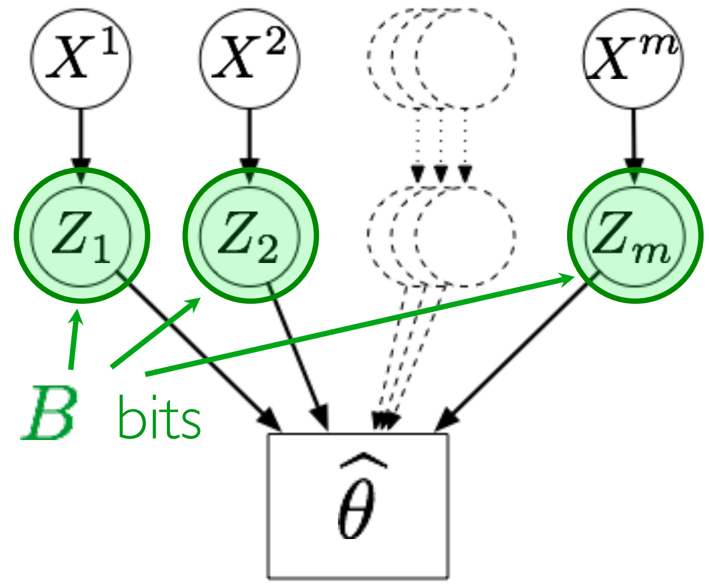


**Theorem:** when each agent has sample of size $n$

Minimax rate

$$\mathbb{E}[\|\widehat{\theta}(X^1,\ldots,X^m) - \theta\|_2^2] \asymp \frac{\sigma^2 d}{nm}$$

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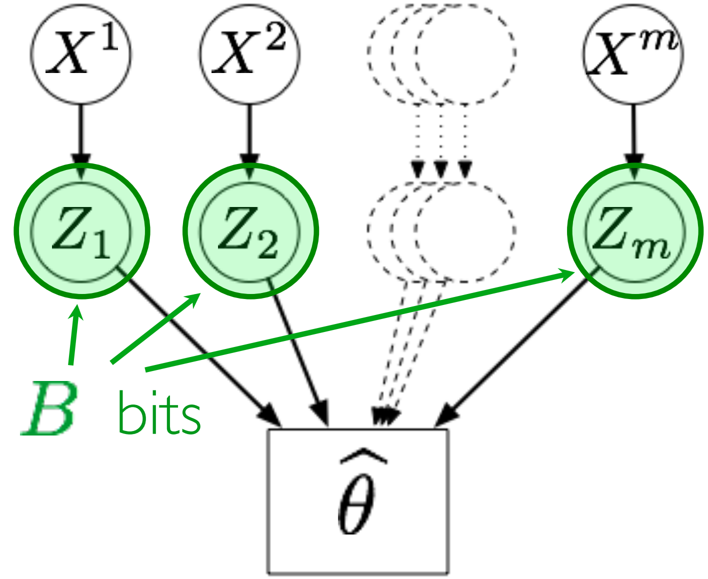


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Minimax rate with *B*-bounded communication

$$\frac{d}{B \wedge d} \frac{1}{\log m} \frac{\sigma^2 d}{nm} \lesssim \mathfrak{M}_n(\mathcal{N}_d, B) \lesssim \frac{d \log m}{B \wedge d} \frac{\sigma^2 d}{nm}$$

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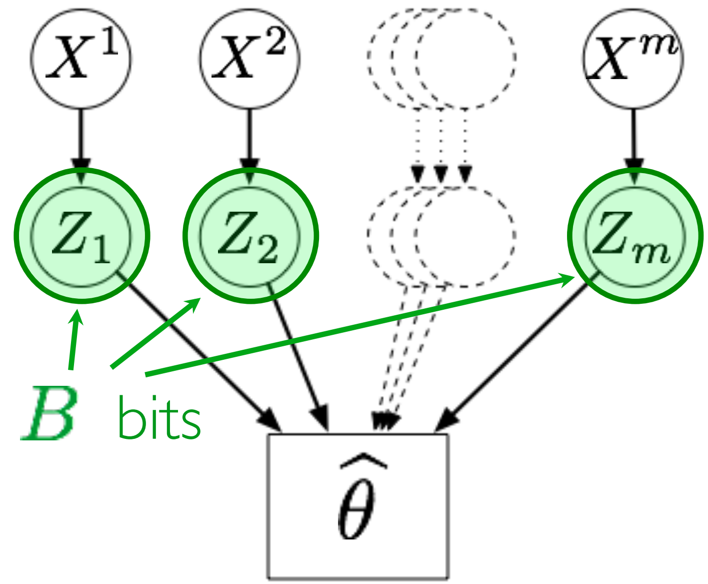


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Consequence: each sends $\approx d$ bits for optimal estimation

# Computation and Inference

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- Hard!

# Computation and Inference: Mechanisms and Bounds

- Tradeoffs via convex relaxations
  - linking runtime to convex geometry and risk to convex geometry
- Tradeoffs via concurrency control
  - optimistic concurrency control
- Bounds via optimization oracles
  - number of accesses to a gradient as a surrogate for computation
- Bounds via communication complexity
- Tradeoffs via subsampling
  - bag of little bootstraps, variational consensus Monte Carlo

# A Variational Framework for Accelerated Methods in Optimization

with Andre Wibisono and Ashia Wilson

July 12, 2016

# Accelerated gradient descent

**Setting:** Unconstrained convex optimization

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- Accelerated gradient descent:

$$\begin{aligned} y_{k+1} &= x_k - \beta \nabla f(x_k) \\ x_{k+1} &= (1 - \lambda_k) y_{k+1} + \lambda_k y_k \end{aligned}$$

obtains the (optimal) convergence rate of $O(1/k^2)$

# The acceleration phenomenon

Two classes of algorithms:

- **Gradient methods**
  - Gradient descent, mirror descent, cubic-regularized Newton's method (Nesterov and Polyak '06), etc.
  - Greedy descent methods, relatively well-understood

# The acceleration phenomenon

Two classes of algorithms:

- **Gradient methods**
  - Gradient descent, mirror descent, cubic-regularized Newton's method (Nesterov and Polyak '06), etc.
  - Greedy descent methods, relatively well-understood
- **Accelerated methods**
  - Nesterov's accelerated gradient descent, accelerated mirror descent, accelerated cubic-regularized Newton's method (Nesterov '08), etc.
  - Important for both theory (optimal rate for first-order methods) and practice (many extensions: FISTA, stochastic setting, etc.)
  - *Not* descent methods, faster than gradient methods, still mysterious

# Accelerated methods

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  - Optimized first-order method (Drori, Teboulle '14; Kim, Fessler '15)
  - Geometric shrinking (Bubeck, Lee, Singh '15)
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  - ...

  But only for strongly convex functions, or first-order methods

**Question:** What is the underlying mechanism that generates acceleration (including for higher-order methods)?

# Accelerated methods: Continuous time perspective

- Gradient descent is discretization of gradient flow

$$\dot{X}_t = -\nabla f(X_t)$$

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- Su, Boyd, Candes '14: Continuous time limit of accelerated gradient descent is a second-order ODE

$$\ddot{X}_t + \frac{3}{t}\dot{X}_t + \nabla f(X_t) = 0$$

# Accelerated methods: Continuous time perspective

- Gradient descent is discretization of gradient flow

$$\dot{X}_t = -\nabla f(X_t)$$

(and mirror descent is discretization of natural gradient flow)

- Su, Boyd, Candes '14: Continuous time limit of accelerated gradient descent is a second-order ODE

$$\ddot{X}_t + \frac{3}{t}\dot{X}_t + \nabla f(X_t) = 0$$

- These ODEs are obtained by taking continuous time limits. Is there a deeper generative mechanism?

**Our work:** A general variational approach to acceleration

A systematic discretization methodology

# Bregman Lagrangian

Define the **Bregman Lagrangian**:

$$\mathcal{L}(x, \dot{x}, t) = e^{\gamma_t + \alpha_t} \left( D_h(x + e^{-\alpha_t}\dot{x}, x) - e^{\beta_t} f(x) \right)$$

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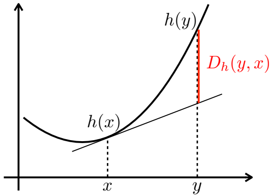
- Function of position $x$, velocity $\dot{x}$, and time $t$
- $D_h(y, x) = h(y) - h(x) - \langle \nabla h(x), y - x \rangle$ is the Bregman divergence
- $h$ is the convex distance-generating function



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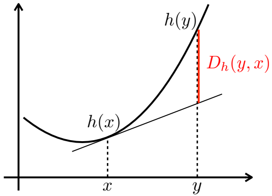

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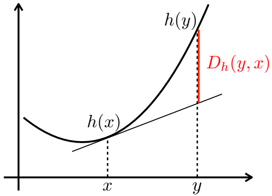

# Bregman Lagrangian

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$$\mathcal{L}(x, \dot{x}, t) = e^{\gamma_t - \alpha_t} \left( \frac{1}{2} \|\dot{x}\|^2 - e^{2\alpha_t + \beta_t} f(x) \right)$$

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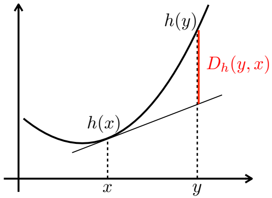

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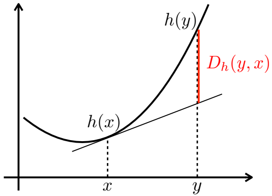



**Ideal scaling** conditions:

$$\dot{\beta}_t \leq e^{\alpha_t}$$

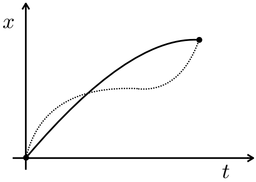
$$\dot{\gamma}_t = e^{\alpha_t}$$

# Bregman Lagrangian

$$\mathcal{L}(x, \dot{x}, t) = e^{\gamma_t + \alpha_t} \Big( D_h(x + e^{-\alpha_t} \dot{x}, x) - e^{\beta_t} f(x) \Big)$$

Variational problem over curves:

$$\min_X \int \mathcal{L}(X_t, \dot{X}_t, t) \, dt$$




Optimal curve is characterized by **Euler-Lagrange** equation:

$$\frac{d}{dt} \left\{ \frac{\partial \mathcal{L}}{\partial \dot{x}}(X_t, \dot{X}_t, t) \right\} = \frac{\partial \mathcal{L}}{\partial x}(X_t, \dot{X}_t, t)$$

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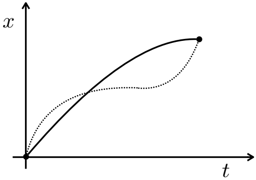


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E-L equation for Bregman Lagrangian under ideal scaling:

$$\ddot{X}_t + (e^{\alpha_t} - \dot{\alpha}_t)\dot{X}_t + e^{2\alpha_t + \beta_t} \left[ \nabla^2 h(X_t + e^{-\alpha_t} \dot{X}_t) \right]^{-1} \nabla f(X_t) = 0$$

# General convergence rate

### Theorem

*Theorem Under ideal scaling, the E-L equation has convergence rate*

$$f(X_t) - f(x^*) \leq O(e^{-\beta_t})$$

# General convergence rate

**Theorem**

*Theorem Under ideal scaling, the E-L equation has convergence rate*

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**Proof.** Exhibit a Lyapunov function for the dynamics:

$$\mathcal{E}_t = D_h\left(x^*,\, X_t + e^{-\alpha_t}\dot{X}_t\right) + e^{\beta_t}(f(X_t) - f(x^*))$$
$$\dot{\mathcal{E}}_t = -e^{\alpha_t+\beta_t}D_f(x^*, X_t) + (\dot{\beta}_t - e^{\alpha_t})e^{\beta_t}(f(X_t) - f(x^*)) \;\leq\; 0$$

$\square$

**Note:** Only requires convexity and differentiability of $f, h$

# Polynomial convergence rate

For $p > 0$, choose parameters:

$$\alpha_t = \log p - \log t$$
$$\beta_t = p \log t + \log C$$
$$\gamma_t = p \log t$$

E-L equation has $O(e^{-\beta_t}) = O(1/t^p)$ convergence rate:

$$\ddot{X}_t + \frac{p+1}{t}\dot{X}_t + Cp^2 t^{p-2}\Big[\nabla^2 h\Big(X_t + \frac{t}{p}\dot{X}_t\Big)\Big]^{-1}\nabla f(X_t) = 0$$

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For $p = 2$:

- Recover result of Krichene et al with $O(1/t^2)$ convergence rate
- In Euclidean case, recover ODE of Su et al: $\ddot{X}_t + \frac{3}{t}\dot{X}_t + \nabla f(X_t) = 0$

# Time dilation property (reparameterizing time)

($p = 2$: accelerated gradient descent)

$$O\left(\frac{1}{t^2}\right): \quad \ddot{X}_t + \frac{3}{t}\dot{X}_t + 4C\Big[\nabla^2 h\Big(X_t + \frac{t}{2}\dot{X}_t\Big)\Big]^{-1}\nabla f(X_t) = 0$$

$$\Big\downarrow \quad \text{speed up time: } Y_t = X_{t^{3/2}}$$

$$O\left(\frac{1}{t^3}\right): \quad \ddot{Y}_t + \frac{4}{t}\dot{Y}_t + 9Ct\Big[\nabla^2 h\Big(Y_t + \frac{t}{3}\dot{Y}_t\Big)\Big]^{-1}\nabla f(Y_t) = 0$$

($p = 3$: accelerated cubic-regularized Newton's method)

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($p = 3$: accelerated cubic-regularized Newton's method)

- All accelerated methods are traveling the same curve in space-time at different speeds
- Gradient methods don't have this property
  - From gradient flow to rescaled gradient flow: Replace $\frac{1}{2}\|\cdot\|^2$ by $\frac{1}{p}\|\cdot\|^p$

# Time dilation for general Bregman Lagrangian

$$O(e^{-\beta_t}): \quad \text{E-L for Lagrangian } \mathcal{L}_{\alpha,\beta,\gamma}$$

$$\Big\downarrow \ \text{speed up time: } Y_t = X_{\tau(t)}$$

$$O(e^{-\beta_{\tau(t)}}): \quad \text{E-L for Lagrangian } \mathcal{L}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$$

where

$$\begin{aligned}
\tilde{\alpha}_t &= \alpha_{\tau(t)} + \log \dot{\tau}(t) \\
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\end{aligned}$$

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**Question:** How to discretize E-L while preserving the convergence rate?

# Discretizing the dynamics (naive approach)

Write E-L as a system of first-order equations:

$$Z_t = X_t + \frac{t}{p}\dot{X}_t$$

$$\frac{d}{dt}\nabla h(Z_t) = -Cpt^{p-1}\nabla f(X_t)$$

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Euler discretization with time step $\delta > 0$ (i.e., set $x_k = X_t$, $x_{k+1} = X_{t+\delta}$):

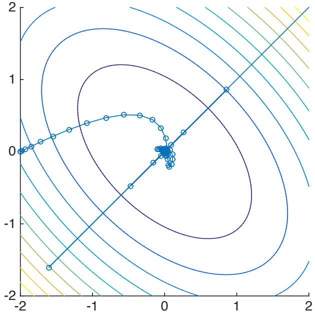
$$x_{k+1} = \frac{p}{k+p}z_k + \frac{k}{k+p}x_k$$
$$z_k = \arg\min_z \left\{ Cpk^{(p-1)}\langle \nabla f(x_k), z\rangle + \frac{1}{\epsilon}D_h(z, z_{k-1}) \right\}$$

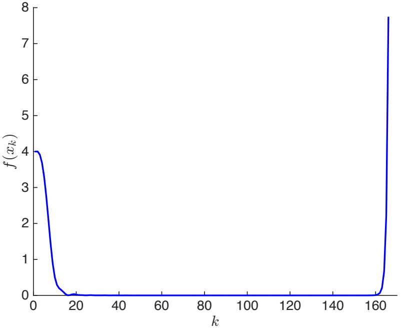
with step size $\epsilon = \delta^p$, and $k^{(p-1)} = k(k+1)\cdots(k+p-2)$ is the rising factorial

# Naive discretization doesn't work

$$x_{k+1} = \frac{p}{k+p} z_k + \frac{k}{k+p} x_k$$
$$z_k = \arg\min_z \left\{ Cpk^{(p-1)} \langle \nabla f(x_k), z \rangle + \frac{1}{\epsilon} D_h(z, z_{k-1}) \right\}$$

Cannot obtain a convergence guarantee, and empirically unstable



# Modified discretization

Introduce an auxiliary sequence $y_k$:

$$x_{k+1} = \frac{p}{k+p} z_k + \frac{k}{k+p} y_k$$
$$z_k = \arg\min_z \left\{ Cpk^{(p-1)} \langle \nabla f(y_k), z \rangle + \frac{1}{\epsilon} D_h(z, z_{k-1}) \right\}$$

Sufficient condition: $\langle \nabla f(y_k), x_k - y_k \rangle \geq M \epsilon^{\frac{1}{p-1}} \| \nabla f(y_k) \|_*^{\frac{p}{p-1}}$

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Assume $h$ is uniformly convex: $D_h(y, x) \geq \frac{1}{p} \|y - x\|^p$

## Theorem

*Theorem*

$$f(y_k) - f(x^*) \leq O\left(\frac{1}{\epsilon k^p}\right)$$

**Note:** Matching convergence rates $1/(\epsilon k^p) = 1/(\delta k)^p = 1/t^p$

Proof using generalization of Nesterov's estimate sequence technique

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# Higher-order gradient update

Higher-order Taylor approximation of $f$:

$$f_{p-1}(y; x) \;=\; f(x) + \langle \nabla f(x), y - x \rangle + \cdots + \frac{1}{(p-1)!} \nabla^{p-1} f(x)(y - x)^{p-}$$

Higher-order gradient update:

$$y_k \;=\; \arg\min_{y} \left\{ f_{p-1}(y; x_k) + \frac{2}{\epsilon p} \|y - x_k\|^p \right\}$$

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Higher-order gradient update:

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Assume $f$ is smooth of order $p - 1$:

$$\|\nabla^{p-1} f(y) - \nabla^{p-1} f(x)\|_* \leq \frac{1}{\epsilon} \|y - x\|$$

## Theorem

*Lemma*

$$\langle \nabla f(y_k), x_k - y_k \rangle \geq \frac{1}{4} \epsilon^{\frac{1}{p-1}} \|\nabla f(y_k)\|_*^{\frac{p}{p-1}}$$

Can use this to complete the modified discretization process!

# Accelerated higher-order gradient method

$$x_{k+1} = \frac{p}{k+p} z_k + \frac{k}{k+p} y_k$$

$$y_k = \arg\min_y \left\{ f_{p-1}(y; x_k) + \frac{2}{\epsilon p} \|y - x_k\|^p \right\} \quad \leftarrow O\left(\frac{1}{\epsilon k^{p-1}}\right)$$

$$z_k = \arg\min_z \left\{ Cpk^{(p-1)} \langle \nabla f(y_k), z \rangle + \frac{1}{\epsilon} D_h(z, z_{k-1}) \right\}$$

If $\nabla^{p-1} f$ is $(1/\epsilon)$-Lipschitz and $h$ is uniformly convex of order $p$, then:

$$f(y_k) - f(x^*) \leq O\left(\frac{1}{\epsilon k^p}\right) \quad \leftarrow \text{ accelerated rate}$$

$p = 2$: Accelerated gradient/mirror descent

$p = 3$: Accelerated cubic-regularized Newton's method (Nesterov '08)

$p \geq 2$: Accelerated higher-order method

# Recap: Gradient vs. accelerated methods

How to design dynamics for minimizing a convex function $f$?

Rescaled gradient flow

$$\dot{X}_t = -\nabla f(X_t) \,/\, \|\nabla f(X_t)\|_*^{\frac{p-2}{p-1}}$$

$$O\left(\frac{1}{t^{p-1}}\right)$$

Higher-order gradient method

$O\left(\frac{1}{\epsilon k^{p-1}}\right)$ when $\nabla^{p-1} f$ is $\frac{1}{\epsilon}$-Lipschitz

matching rate with $\epsilon = \delta^{p-1} \Leftrightarrow \delta = \epsilon^{\frac{1}{p-1}}$

# Recap: Gradient vs. accelerated methods

How to design dynamics for minimizing a convex function $f$?

| Rescaled gradient flow | Polynomial Euler-Lagrange equation |
| --- | --- |
| $\dot{X}_t = -\nabla f(X_t) \,/\, \lVert \nabla f(X_t) \rVert_*^{\frac{p-2}{p-1}}$ | $\ddot{X}_t + \frac{p+1}{t}\dot{X}_t + t^{p-2}\left[\nabla^2 h\left(X_t + \frac{t}{p}\dot{X}_t\right)\right]^{-1} \nabla f(X_t) =$ |
| $O\left(\frac{1}{t^{p-1}}\right)$ | $O\left(\frac{1}{t^{p}}\right)$ |
| **Higher-order gradient method** | **Accelerated higher-order method** |
| $O\left(\frac{1}{\epsilon k^{p-1}}\right)$ when $\nabla^{p-1} f$ is $\frac{1}{\epsilon}$-Lipschitz | $O\left(\frac{1}{\epsilon k^{p}}\right)$ when $\nabla^{p-1} f$ is $\frac{1}{\epsilon}$-Lipschitz |
| matching rate with $\epsilon = \delta^{p-1} \Leftrightarrow \delta = \epsilon^{\frac{1}{p-1}}$ | matching rate with $\epsilon = \delta^{p} \Leftrightarrow \delta = \epsilon^{\frac{1}{p}}$ |

# Summary: Bregman Lagrangian

- Bregman Lagrangian family with general convergence guarantee

$$\mathcal{L}(x, \dot{x}, t) = e^{\gamma_t + \alpha_t} \left( D_h(x + e^{-\alpha_t} \dot{x}, x) - e^{\beta_t} f(x) \right)$$

- Polynomial subfamily generates accelerated higher-order methods: $O(1/t^p)$ convergence rate via higher-order smoothness

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- Polynomial subfamily generates accelerated higher-order methods: $O(1/t^p)$ convergence rate via higher-order smoothness
- Exponential subfamily: $O(e^{-ct})$ rate via uniform convexity
- Understand structure and properties of Bregman Lagrangian: Gauge invariance, symmetry, gradient flows as limit points, etc.
- Bregman Hamiltonian:

$$\mathcal{H}(x, p, t) = e^{\alpha_t + \gamma_t} \Big( D_{h^*}\left(\nabla h(x) + e^{-\gamma_t} p,\, \nabla h(x)\right) + e^{\beta_t} f(x) \Big)$$

# Discussion

- Many conceptual and mathematical challenges arising in taking seriously the problem of “Big Data”
- Facing these challenges will require a rapprochement between “computational thinking” and “inferential thinking”
  - bringing computational and inferential fields together at the level of their foundations