Energetics of Cracked Bodies

Sanford: Chapter 6
Anderson: 2.1, 2.3, 2.4, 2.7
Kanninen and Popelar: pg. 32-37, 158-163

Equivalence of Stress Intensity and Energy Methods

111

Cracked body energetics (1)

• Total potential energy \( \pi \) of a conservative system

\[
\pi = U + \Omega
\]

\[
U = \int P \Delta = \int_0^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij}
\]

\[
\Omega = -P \Delta = -\int_{S_T} t_i u_i \, dS = -W_{ext}
\]

Load control test

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**Cracked body energetics (2)**

- If specimen is deformed by imposing displacement ($\Delta$) rather than weight ($P$), then $\Omega = 0$ (*use an old style screw machine in the lab!*)

\[ \pi = U + \Omega \]

**Stored elastic energy**

\[ U = \int P \, d\Delta = \int \sigma_{ij} \, d\varepsilon_{ij} \]

\[ \Omega = 0 \]

**Testing machine is so stiff relative to specimen that $\Delta$ does not change during crack advance**

**Energy release rate (1)**

- When the crack grows by amount $\Delta a$ during a test, the new traction free crack area is $\Delta A_c = \Delta a \times B$ (this is conventional usage of the “projected area”)

- The crack growth causes the potential energy of the structure to change

**Example shown here is load control during crack extension**
The potential energy change per unit area of crack extension is called the **energy release rate** ($J$ or $\mathcal{G}$) and is given by:

$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} \frac{\pi_2 - \pi_1}{B \Delta a}$$

A leading (-) is included in the definition so that the release rate has a positive numerical value.

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**$J$ for Displacement Controlled Loading**

- The key concept is what happens with the forces (tractions) across the crack area $\Delta a \times B$ in configuration 1 that are zero in configuration 2 after the crack grows.
- Imagine that we have simple springs connecting the crack faces over $\Delta a \times B$.
- In 1, these springs are stretched due to specimen loading and they have some stored elastic energy.
- In 2, these springs are broken (to create crack growth by $\Delta a \times B$) and no longer have the stored energy.
### J for Displacement Controlled Loading

- Something happens to the energy balance in the cracked structure since it is closed and conservative – this changes the potential energy.
- We also know that for fixed displacement loading, $\Omega = 0$ in 1 and in 2, and that the compliance (flexibility) increases in 2 due to the increased crack length.

![Diagram showing energy balance in cracked structures](image)

**1.** Some energy stored in springs ($U_{sp}$)  
**2.** No energy stored in springs

- Crack advance by $\Delta a$

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### Compute $J$ : $\Delta$ controlled loading (1)

- Measure reaction force ($R$) on load cell.
- Crack extension takes place at constant axial displacement imposed on specimen.

![Diagram showing displacement-controlled loading](image)

**1.** $\pi_1 = U_1$  
$\Omega_1 = 0$  
$\Delta_1$

**2.** $\pi_2 = U_2$  
$\Omega_2 = 0$  
$\Delta_2 = \Delta_1$

- $a_2 = a_1 + \Delta a$
- $U_1$ is area under red line
- $U_2$ is area under green line

---

Specimen compliance is inverse of stiffness. Compliance increases during crack advance.
Compute $J : \Delta$ controlled loading

(2)

$U_1$ is area under red line and includes $U_{sp}$

$U_2$ is area under green line: $U_{sp} = 0$

\[
\pi_2 - \pi_1 = U_2 - U_1
\]

\[
J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} \frac{\pi_2 - \pi_1}{B \Delta a} = -\frac{U_2 - U_1}{B \Delta a}
\]

\[
U_1 = \frac{1}{2} R_1 \Delta; \quad U_2 = \frac{1}{2} R_2 \Delta
\]

\[
J = \lim_{\Delta a \to 0} -\frac{\Delta R_2 - R_1}{2B} \frac{1}{\Delta a} = -\frac{\Delta}{2B} \left( \frac{\partial R}{\partial a} \right)_{\Delta = fixed}
\]

$J_{\Delta}$ has units of $F \cdot L / L^2$

\[
J = \lim_{a \to 0} \frac{R_1}{2} = \frac{U_1}{2}
\]

$J = \lim_{a \to 0} \frac{R_2}{2} = \frac{U_2}{2}

\]

Compute $J : \Delta$ controlled loading

(3)

- Write $R$ and $\Delta$ in terms of compliance $C$: $C = \Delta / R$
- Then the derivative can be re-written to give

\[
J = -\frac{\Delta}{2B} \left( \frac{\partial R}{\partial a} \right)_{\Delta = fixed} = -\frac{\Delta}{2B} \left( \frac{\partial (\Delta / C)}{\partial a} \right)_{\Delta = fixed}
\]

\[
J = -\frac{\Delta}{2B} \left( -\frac{C}{C^2} \frac{\partial C}{\partial a} \right)_{\Delta = fixed} = \frac{1}{2B} \frac{\Delta \Delta}{C^2} \left( \frac{\partial C}{\partial a} \right)_{\Delta = fixed} = \frac{R^2}{2B} \left( \frac{\partial C}{\partial a} \right)_{\Delta = fixed}
\]
Compute $J : \Delta$ controlled loading (3)

How to Use?

- Suppose we have a closed-form solution for the compliance
- Energy release rate follows directly by applying the above derivative (usually very simple to compute)
- Can also compute approximate compliance change in a finite element analysis for small changes in crack length

$J = \frac{R_1^2}{2B} \left( \frac{\partial C}{\partial a} \right)_{\Delta \text{ fixed}}$

$\text{Compliance, } C = \frac{a}{R}$

$J \times B \Delta a$

$R_1, R_2, a_1, a_2, \Delta$

$J$ interpretation

$U_{sp}$ is energy stored in springs

$U_1$ is area under red line: includes $U_{sp}$

$U_2$ is area under green line: $U_{sp} = 0$

- Since the system is closed, conservative, & fixed displacement during growth, the energy change between 1 and 2 must be only from the energy stored in the broken springs

$U_2 + U_{sp} = U_1$

$U_2 - U_1 = -U_{sp}$

$\pi_2 - \pi_1 = U_2 - U_1 = -U_{sp}$

$J = \frac{\partial \pi}{\partial A_c} = \lim_{\Delta \alpha \to 0} \frac{\pi_2 - \pi_1}{B \Delta a} = \frac{U_{sp}}{B \Delta a} = \frac{dU_{sp}}{dA_c}$

$J$ has units of $F \cdot L / L^2$
What does this mean?

• When $K_I$ approaches $K_{Ic}$, the cracked body in configuration 1 has just reached sufficient potential energy (in this case, stored elastic energy) available (and can give up) to break the springs (cohesive tractions) holding the crack closed over the area $B \times \Delta a$ (i.e. the energy to be released during crack advance)

• When $K_I$ from the applied displacement is $< K_{Ic}$:
  - The cohesive tractions (i.e. spring forces or strength) available from the metallurgical features exceed the applied tractions imposed by $K_I$ stress field (no break)
  - The energy available from the background (elastic) material during a crack growth $B \times \Delta a$ < energy stored in the springs at their breaking point

What does this mean? (2)

• The energy criterion was the original concept of fracture mechanics: $K_I = K_{Ic}$ came many years later

• In a few slides we show that energy and stress-intensity arguments are equivalent for linear-elastic systems

• Energy arguments are especially powerful in finite element analysis: $U$ and $\Omega$ are the most accurate quantities computed

• **Key point:** fracture conditions described w/o any mention of the metallurgical mechanisms! (caused much confusion for decades – still does)

\[
J_{\text{applied}} = J_{\text{critical}}
\]
\[ J \text{ for load control (1)} \]

1. Crack extension takes place at constant applied load

\[ \begin{align*}
\pi_1 &= U_1 + \Omega_1 \\
\Omega_1 &= -P\Delta_1 \\
U_1 &= \frac{1}{2}P\Delta_1
\end{align*} \]

\[ \begin{align*}
\pi_2 &= U_2 + \Omega_2 \\
\Omega_2 &= -P\Delta_2 \\
U_2 &= \frac{1}{2}P\Delta_2
\end{align*} \]

\[ \text{U}_1 \text{ is area under red line} \]
\[ \text{U}_2 \text{ is area under green line} \]

\[ \Delta_1 \]
\[ \Delta_2 \]

\[ \Delta_1 > \Delta_2 \]

\[ \begin{align*}
\Delta_1 &= a_1 + da_1 \\
\Delta_2 &= a_1 + da_2
\end{align*} \]

\[ \begin{align*}
\pi_1 &= U_1 + \Omega_1 \\
\Omega_1 &= -P\Delta_1 \\
U_1 &= \frac{1}{2}P\Delta_1
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\pi_2 &= U_2 + \Omega_2 \\
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U_2 &= \frac{1}{2}P\Delta_2
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\[ \text{U}_1 \text{ is area under red line} \]
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\[ \Delta_1 \]
\[ \Delta_2 \]

\[ \Delta_1 > \Delta_2 \]

\[ \begin{align*}
\Delta_1 &= a_1 + da_1 \\
\Delta_2 &= a_1 + da_2
\end{align*} \]

\[ \begin{align*}
\Omega_1 &= -P\Delta_1 \\
U_1 &= \frac{1}{2}P\Delta_1 \\
\Omega_2 &= -P\Delta_2 \\
U_2 &= \frac{1}{2}P\Delta_2
\end{align*} \]

\[ \text{Compliance, } C = \frac{\Delta}{P} \]

\[ J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} \left( \frac{\pi_2 - \pi_1}{B\Delta a} \right) = \frac{1}{2} \frac{P}{B} \left( \frac{\partial \Delta}{\partial A} \right)_{P=\text{fixed}} \]
**J for load control (3)**

- Write $P$ and $\Delta$ in terms of compliance $C$: $C = \Delta / P$
- Then the derivative can be re-written to give

$$J = \frac{P^2}{2B} \left( \frac{\partial C}{\partial a} \right)_{P=\text{fixed}}$$

**Equivalence of load-displacement control (1)**

- The crack grows when the reaction from the imposed displacement is the same as the load applied by a weight, i.e., $R = P$ (and the internal stresses are identical)
- The compliance, $C$, of a linear-elastic specimen is not a function of the load $P$ or the displacement $\Delta$. It is a function only of the crack length (and other dimensions of the specimen) and the material elastic properties
Equivalence of load-displacement control (2)

Displacement Control

\[ J = \frac{R^2}{2B} \left( \frac{\partial C}{\partial a} \right)_{\Delta=fixed} \]

Load Control

\[ J = \frac{P^2}{2B} \left( \frac{\partial C}{\partial a} \right)_{P=fixed} \]

• Consequently,

\[ R = P \]

\[ \left( \frac{\partial C}{\partial a} \right)_{\Delta=fixed} = \left( \frac{\partial C}{\partial a} \right)_{P=fixed} \]

✓ \( J \) for Load & Displacement Control are Identical

General cohesive model (1)

• More realistic models treat the “springs” as cohesive elements uniformly distributed over the crack plane

• The cohesive stress (\( t_{coh} \)) normal to the crack plane varies with the opening displacement (\( \delta \)) between the crack faces

• The cohesive law can be linear, nonlinear or damaging
General cohesive model (2)

- Compute the energy stored in cohesive forces (i.e., $U_{sp}$)

This calculation is for traction across the crack area & the total displacement across the faces.

$$U_{sp} = U_{coh} = \int_0^{\delta_c} t(\delta) B \, d\delta = B \int_0^{\delta_c} t(\delta) \, d\delta$$

$J$-K Relationship (1)

- Consider Mode I plane-strain conditions
  - Opening mode stresses acting ahead of crack in configuration 1 ($\alpha = a_1$) are given by $K_I$
  - Let crack area grow by $B \times \Delta a$ to configuration 2
  - The crack stresses relax to zero over new crack area $B \times \Delta a$ gradually as the crack opens from $a_1$ to $a_2$
  - The opening displacements must follow the Mode I solution
  - Perform “cohesive” integration to find $J$ (work done by the tractions as they relax to zero during displacement increase)

$$J = \frac{U_{coh}}{B da} = \int_0^{\delta_c} t(\delta) \, d\delta = \Gamma$$
**J-K Relationship (2)**

Plane-stress

Plane-strain

Remember: these tractions and displacements are the final values (thus a ½ is needed for energy)

The outside 2 is needed because the displacement (v) is just the upper & lower value about symmetry plane

Differential force

\[ U_{coh} = 2 \int_0^{\Delta a} \frac{1}{2} v(\bar{r}) t(\bar{r}) B \, d\bar{r} = \frac{4K_I^2}{2\pi E^p} B \int_0^{\Delta a} \frac{\Delta a - \bar{r}}{\bar{r}} \, d\bar{r} \]

\[ J = \frac{U_{coh}}{B \Delta a} = \frac{K_I^2}{E^p} \]

\[ E' = E \quad \text{Plane-stress} \]

\[ E'' = \frac{E}{1-\nu^2} \quad \text{Plane-strain} \]

**J-K Relationship (3)**

- The **stress-intensity factor** approach and the **energy-based** approach for fracture under linear-elastic conditions are identical!!
- Given \( K_I \) we can compute \( J \)
- Given \( J \) we can compute \( K_I \)
- Use the more convenient computational approach for the problem needing solution
- Energy release rates (\( J \)) are very accurate even for relatively crude finite element models that employ displacement-based element formulations
Applications of Energy Approach

Sanford: Chapter 6
Anderson: 2.3,4,7,10
Kanninen and Popelar: pg. 32-37, 158-163

Equivalence of Stress Intensity and Energy Methods

Example J computation

(1)

Use simple beam theory to compute the displacement in the arms of the specimen assuming cantilever behavior

\[
\frac{\Delta}{2} = \frac{PL^3}{3EI} = \frac{Pa^3}{3EI}, \quad I = \frac{1}{12} Bh^3
\]

Compliance, \( C = \frac{\Delta}{P} = \frac{2a^3}{3EI} \)

Double Cantilever Beam
Example $J$ computation (2)

\[
\frac{\Delta}{2} = \frac{PL}{3EI} = \frac{Pa}{3EI}; \quad I = \frac{1}{12} Bh^3
\]

Compliance, \( C = \frac{\Delta}{P} = \frac{2a^3}{3EI} \)

\[
J = \frac{P^2}{2B} \frac{\partial C}{\partial a} = \frac{P^2}{2B} \frac{\partial}{\partial a} \left( \frac{2a^3}{3EI} \right) = \frac{P^2a^2}{B} \frac{2a}{3EI} = \frac{12P^2a^2}{B^2h^3E}
\]

\[
J = \frac{12P^2a^2}{B^2h^3E}
\]

Example $J$ computation (3)

- Neglects shear deformation in the arms
- Assumes arms fixed at crack tip
- Neglects energy in remainder of specimen
- At constant load, \( J \) increases as the square of the crack length
- Verify correct physical units (must be \( F \cdot L/L^2 \))

\[
K_I = \sqrt{EJ} = \frac{2\sqrt{3}Pa}{Bh^{3/2}} \quad \text{for plane stress}
\]

This is a very good approximation, especially for long, slender arms.
Crack stability (1)

As the crack grows in length, how does the energy release rate ($J$) change at fixed load?

$$J = \frac{12P^2a^2}{B^2h^3E}$$

$$\frac{\partial J}{\partial a} = \frac{24P^2a}{B^2h^3E} = \frac{2J}{a} > 0$$

The rate of change of energy release rate is positive.

Crack stability (2)

Change to displacement control loading

$$\Delta = \frac{2Pa^3}{3EI} \Rightarrow P = \frac{3EI\Delta}{2a^3}$$

Now compute rate of change of energy release rate at fixed, imposed displacement

$$J = \frac{9EI\Delta^2}{4a^4B}$$
Crack stability (3)

- Crack growth begins when $J = J_{\text{critical}}$ for both cases
- Suppose the $J_{\text{critical}}$ is a material constant
- Then crack growth continues in load control since $J$ will be above $J_{\text{critical}}$
- In displacement control, the crack stops since $J$ decreases with growth. Additional imposed displacement is required to resume crack growth
- This also means that $K_I$ increases in load control and decreases in displacement control during crack extension

Load control

$$\frac{\partial J}{\partial a} \bigg|_P = \frac{2J}{a} > 0$$

Displacement control

$$\frac{\partial J}{\partial a} \bigg|_\Delta = \frac{J}{a} < 0$$

Use of FEA to compute $J-K_I$

- The finite element method is ideally suited to compute the energy release rate, $J$, and then $K_I$ from the $J-K_I$ relationship
- Illustrate using displacement control loading of the FE model
- Real crack length is $a$, located at edge of some shape of hole
- Mode I conditions, model only upper $\frac{1}{2}$ of specimen
- Use collapsed, 8-node 2-D elements at crack tip to best represent the strain-stress $r^{-1/2}$ singularity

$$\pi = U + \Omega = U = \frac{1}{2} (u_s)^T [K_s] \{u_s\}$$
Use of FEA to compute $J-K_I$

- Run 2 analyses with slightly longer and shorter crack lengths: just perturb the horizontal position of the 4 singularity elements
- Compute the strain energy, $U$, in each case. Most FEA codes print $U$ if asked.

$$\pi = U + \Omega = U = \frac{1}{2} \{u_s\}^T [K_s] \{u_s\}$$

$$J_{symm} = -\frac{\partial \pi}{\partial A_c} = -\frac{\partial U}{\partial A_c} = -\frac{1}{B} \frac{\partial U}{\partial a}$$

$$J_{symm} = -\frac{1}{B} \left[ U(a + \Delta a) - U(a - \Delta a) \right]$$

$$J = J_{total} = 2 \times J_{symm}$$

$$K_I = \sqrt{EJ}$$

Elastic-plastic fracture mechanics

- Anderson
  - Chp. 3 (section 3.1)
  - Chp. 7 (section 7.5)
- Sanford
  - Chp. 11 (sections 11.1, 11.2)
Elastic-plastic fracture mechanics

- Overview
- CTOD Methodology
- CTOD Estimation
- \( J \) Methodology
- \( J \) Integral
- HRR Solutions
- Features of HRR Solutions
- Asymptotic Dominance

Overview (1)

- Late 1950s-mid 1960s, nearly all research focused on Linear Elastic Fracture Mechanics (LEFM)
- Structures of most interest at the time were made of high-strength materials: Ti, Al, high-strength steels all with \( \sigma_{ys} > 100 \) ksi
  - Rockets, ICBMs
  - B-1 bomber
  - Low fracture toughness, plastic zones very small at unstable fracture
  - Extensive R&D work + ASTM standards development: E-399
Overview (2)

• In mid-1960s additional focus on (often driven by regulators)
  - Commercial & military nuclear power production
  - Offshore platforms
  - Welded pipelines
  - Low strength, high toughness steels (A36, A572, A516, A508, A533B, ...)
  - Large plastic zones at cleavage fracture, ductile instability
  - Invalidated analysis, experimental methods and assessment procedures developed for LEFM
• Relatively large R&D effort start in U.S. and abroad
  - “correlative” methodologies analogous to $K_I$: CTOD and $J$-integral

CTOD motivation

• Anderson Sections 3.1, 3.3, 7.5, 9.8
• Developed initially in the U.K. by Wells and colleagues at The Welding Institute (TWI, near Cambridge)
• Funding driven by safety of platforms in North Sea and associated pipelines
• Very physically appealing approach based on critical elastic-plastic stretching at crack tip
  - CTOD is an indirect measure of severe stretching at crack tip (no abstract energy concepts, math!)
CTOD developments

- Somewhat less popular in the U.S. where the J-integral dominated research, thinking and funding
- UK researchers (and now Edison Welding Institute in U.S.) pushed development of extensive fracture toughness testing and defect assessment procedures
- Petro-chemical industry worldwide uses CTOD for weld quality assurance, steel purchasing specs and assessment of existing defects: American Petroleum Institute, Fitness-for-Service standards

CTOD methodology

- CTOD intended as a single value of fracture toughness at instability
- Not meant to describe R-curve behavior
- For fracture after max load in a test, CTOD becomes quite ambiguous
- CTOA (crack tip opening angle) concept is an extension of CTOD for R-curve behavior (new ASTM E2472-06 standard for CTOA – much R&D activity)
- CTOD can be measured during a test by infiltration techniques, optically with high-resolution cameras, inferred from measured crack mouth opening displacement
- Plastic component of CTOD can be measured post-test directly from broken ends of specimen
CTOD methodology

- CTOD relies on its connection with the $J$-integral for a theoretical foundation ($J$-integral discussed soon)
- Major practical issue is transferability -- is the critical CTOD value measured using a deep notch SE(B) specimen in the lab the same value in a large, tension loaded structure?
- CTOD testing and assessment procedures attempt to “sidestep” this issue:
  - Specimen testing must also use the “application” thickness
  - Deep notch CTOD values are understood to be conservative values providing an additional (but not quantified) level of conservatism in assessment procedures

$$\text{Critical CTOD from deep-notch SE(B) test generally is smaller than critical CTOD measured in tension loaded components}$$

Use of CTOD methodology

- To use a CTOD methodology we need:
  - An un-ambiguous definition of CTOD both experimentally and numerically (CTOD changes with distance behind the crack front)
  - A computational procedure to estimate CTOD in a plastically deforming cracked body (at each location along crack front)
  - A experimental procedure to measure (or infer) values of CTOD at fracture instability that is robust, reliable and repeatable (can be standardized)
  - An understanding of potential effects of specimen size, loading mode (tension vs. bending), temperature, loading rate, prior ductile tearing, etc. on the critical value
  - Engineering (transferability) models for these effects to allow critical values measured in one condition to be used (“adjusted”) in another condition
  - Procedures to handle stochastic variability of measured CTOD values (e.g., cleavage fracture in steels)
  - Welds introduce more complexities - residual stresses, inhomogeneous material flow properties (how to compute...
Use of CTOD methodology

- **Testing standards**
  - ASTM E1290-08 Standard Test Method for Crack-Tip Opening Displacement (CTOD) Fracture Toughness Measurement (also an ISO standard)
  - Must test at “application thickness”
  - Plate only – no welds: appendix for welds under development
  - Single point toughness values (no R-curve)
  - Deep notch SE(B)s required, shallow notch specimens being added to quantify constraint effects
  - No guidance on transferability, stochastic
  - Prescribes how to run the test, not how to use the measured CTOD values at fracture

- **Defect assessment procedures**
  - Failure assessment diagrams (FADs)
  - UK-European: PD6493, R-6
  - US: API-579 (American Petroleum Institute)
  - See Anderson, Sections 9.3, 9.4, 9.8, 9.9

CTOD estimation

\[ v(r) = \frac{4}{\pi} K_I \sqrt{\frac{r}{2\pi}} \]

\[ \delta = 2 \times v(r = r_y) = \frac{8}{E} K_I \sqrt{\frac{1}{\pi \frac{K_I^2}{\sigma_y E}}} \]

\[ \therefore \delta = \frac{4}{\pi} \frac{K_I^2}{\sigma_y E} \]

*CTOD increases as \( K^2 \)*

(this is a nonlinear effect!)

The actual coefficient (not 4/\(\pi\)) is found from nonlinear FE analyses, varies with material strain hardening and 3-D effects.

*Suggestion: re-derive for plane-strain conditions, verify physical units are correct.*
CTOD estimation (2)

- Compute $J$ including plasticity (see later notes)
- Compute $K_J$ from $J$ (a $K$ value computed from $J$)
- Finally get CTOD from $K_J$

\[ J = \frac{K_J^2}{E'} \quad K_J = \sqrt{E'J} \]

\[ E' = E \ (\text{plane stress}) \]
\[ E' = \frac{E}{1 - \nu^2} \ (\text{plane strain}) \]

\[ \therefore \delta = \frac{4}{\pi} \frac{K_J^2}{\sigma_0 E} \quad \therefore \delta = \frac{4}{\pi} \frac{J}{\sigma_0} \]

Detailed 2-D and 3-D finite element analyses show that a more accurate form is:

\[ \therefore \delta = \frac{J}{m \sigma_{flow}} \quad m = 1.5 \text{ for plane strain and 3-D} \]

\[ \sigma_{flow} = \frac{1}{2} (\sigma_{ys} + \sigma_u) \]

CTOD estimation (3)

- Use nonlinear finite element analysis to compute CTOD as function on applied loading
- Thermal loading, residual stresses, etc. introduce NO complications
- Just extract CTOD from displacements of nodes behind crack tip at each applied load level

More elements than this needed to resolve $\theta$ variation of fields

13 nodes at this front location- collapse to a point

Extract CTOD at each front location using 45° intercept method
CTOD: example FEA model

- Example from work of Sorem, Rolfe, Dodds (*Int. J. Fracture*, V47, 1991)
- Test & analyses of A36 steel, deep notch SE(B) specimen
  - Load, then unload specimen as shown
  - Cool to liquid N2 and fracture
  - Remaining deformation is plastic
  - Superpose deformed finite element mesh (-elastic) on deformed specimen

CTOD: Example data

- Example from work of Sorem, Rolfe, Dodds (*Int. J. Fracture*, V47, 1991)
- Illustrates major effect of constraint loss on measured fracture toughness (cleavage)
- Research still on-going to construct transferability models
**J integral for elastic-plastic fracture**

- Anderson
  - Chp. 3 (3.2-3.5, App. A3.2-A3.4)
- Sanford
  - Chp. 11

**Elastic-plastic fracture mechanics**

- What is J?
  - Energy release rate concept extended to nonlinear material behavior in contained yielding
- J Methodology
- J Integral
- HRR Solutions
- Features of HRR Solutions
- Asymptotic Dominance
J methodology (1)

- Development started in early 1970s
- R & D driven and funded by safety issues of nuclear power generation
- High toughness, low strength materials
  - $K_I > 100$ at operating temperatures
  - Yield stresses 50-80 ksi
- Large plastic zones at fracture invalidate LEFM assumptions
  - Cleavage in the DBT under large-scale plasticity
  - Initiation of stable ductile tearing
  - Extensive stable tearing terminated by tearing instability
- $J$ based fracture mechanics has the same goals as CTOD approach – perhaps more focus on ductile tearing

J methodology (2)

- More abstract than CTOD (we cannot “see” a $J$), requires advanced math skills to understand, and nonlinear solid mechanics for analysis
- $J$ approach used extensively in advanced R & D efforts – it has more “headroom” than CTOD given the fundamental mechanics basis
- Major practical issue is transferability -- is the critical $J$ value or $J$-Δ$a$ curve measured using a deep-notch SE(B) specimen in the lab the same toughness in a large, tension loaded structure?
- Residual stresses, inhomogeneous materials (welds) are more difficult to address with $J$ approach
- $J$ approach mathematically (rigorously) simplifies to $K$; “correlative” fracture mechanics for SSY and LEFM – this is a major appeal of $J$
Three key developments occurred over a short time period (1966-1970)

- James Rice (then at Brown U.) explored a particular path-independent (conservation) integral for 2-D sharp crack tip configurations
  - The scalar value of the integral has units of $F \cdot L / L^2$ (energy release rate)
  - The integral has the same value when computed on all contours enclosing crack tip
  - But there are many, similar conservation integrals with path independence

Rice and his student (Rosengren) at Brown and John Hutchinson at Harvard both solved the asymptotic, Mode I crack-tip fields for a nonlinear material, analogous to the Williams solution for linear elasticity (the now-termed HRR fields)

- The leading stress-strain term of the field is singular
- For a linear-elastic material the solution is identical to the Williams solution
- The “amplitude” of the strain-stress-displacement field is again undetermined by the asymptotic solution (just like $K_I$)

- Rice then showed that the scalar value, $J$, is the amplitude of the asymptotic field analogous to $K_I$ in the Williams solution
- He did this by evaluating the $J$-integral over a circular contour enclosing the crack tip with strains-stresses-displacements given by the HRR asymptotic field
- This remains today a remarkable breakthrough!
Conservation integrals (1)

Consider a simple 2-D region \( \mathcal{R} \) through which an incompressible fluid flows in steady-state.

The velocity vector of the fluid at each point \((x, y)\) is:

\[
\vec{V}(x, y) = V_x(x, y) \hat{i} + V_y(x, y) \hat{j}
\]

Let \( \rho \) denote the mass density of the fluid. Then the net outflow of fluid from the region is given by

\[
I_\Gamma = \rho \int_{\Gamma} \vec{V}(x, y) \cdot \hat{n} \, ds = \text{net outflow}
\]

Conservation integrals (2)

If there are no sources or sinks of fluid in the region \( \mathcal{R} \). Then conservation of mass for incompressible, steady flow requires that

\[
I_\Gamma = \rho \int_{\Gamma} \vec{V}(x, y) \cdot \hat{n} \, ds = 0
\]

Suppose now there exists a point source that adds fluid to the region \( \mathcal{R} \). The strength of the source is \( \rho F \) (has units of mass/time). Then

\[
I_\Gamma = \rho \int_{\Gamma} \vec{V}(x, y) \cdot \hat{n} \, ds = \rho F
\]
Conservation integrals (3)

Now consider some other closed contour in $\mathcal{R}$ that encloses the point source. Then mass conservation also requires that

$$I_\Gamma = \rho \int_\Gamma \vec{V}(x, y) \cdot \hat{n} \, ds = \rho F$$

Then,

$$\rho \int_\Gamma \vec{V}(x, y) \cdot \hat{n} \, ds = \rho \int_\Gamma \vec{V}(x, y) \cdot \hat{n} \, ds$$

It is clear that all such contour integrals “compute” the same strength of the point source in the fluid.

Conservation integrals (4)

If we evaluate the integral over a contour within $\mathcal{R}$ that does not contain the point source, we see that

$$I_\Gamma = \rho \int_\Gamma \vec{V}(x, y) \cdot \hat{n} \, ds = 0$$
Extension to cracks (1)

Suppose we have an elastic body in equilibrium that contains a sharp crack as shown. The crack tip represents a point singularity with an effect on the mechanical fields that radiates outward from the tip location and diminishes with distance.

We readily imagine that an integral defined over a contour enclosing the crack tip could compute the strength of the singularity.

\[ J_\Gamma = \oint_\Gamma \vec{F} \cdot \hat{n} \, ds \]

A plausible measure for the strength of the singularity is the energy release rate

\[ J = -\frac{\partial \pi}{\partial A_c} \]

Extension to cracks (2)

Based on the analogy with incompressible fluid flow, we anticipate that \( J \) computed over a different contour will have the same value under suitable restrictions (e.g., no body forces applied inside the contour, no crack face loading, etc.)

\[ J_\Gamma = \frac{\partial \pi}{\partial A_c} = \oint_\Gamma \vec{F} \cdot \hat{n} \, ds \]
\[ J_\Gamma = \frac{\partial \pi}{\partial A_c} = \oint_\Gamma \vec{F} \cdot \hat{n} \, ds \]
\[ J_\Gamma = J_\Gamma = -\frac{\partial \pi}{\partial A_c} \]
Extension to cracks (3)

If we shrink the contour to the immediate crack tip region, we can define a “crack tip” value for $J$

$$J_{\text{tip}} = -\frac{\partial \pi}{\partial A_c} = \int_{\Gamma_{\text{tip}}} F(T_x, T_y, W, \ldots) \cdot \hat{n} \, ds$$

In the limit of shrinking contour sizes, the issues of body forces, crack face loading, etc. disappears.

Under these restrictions (no body forces inertial forces, isothermal, no crack-face tractions), we have

$$J_{\text{tip}} = J_{\text{far}} = -\frac{\partial \pi}{\partial A_c}$$

Extension to cracks (4)

Why of major importance?

Value of $J_{\text{tip}}$ is clearly determined by stress-strain-displacement fields very near the crack tip (the crack-tip loading that causes material separation). Hard to compute accurate fields very near tip for nonlinear conditions!

$J_{\text{far}}$ is determined by stress-strain-displacement values remote from the crack tip -- they reflect loading, geometry, boundary conditions. Should be able to compute these values accurately !!!

$J$ relates quantitatively near-tip stress-strain-displacement fields to remote geometry, loading, boundary conditions

\[ J = 0 \] when \( \Gamma \) contains no cracks or singularities!
J-integral (3)

\[ J = \int_{\Gamma} \left( W \, dy - T_y \, \frac{\partial u_y}{\partial x} \, ds \right) = \int_{\Gamma} \left( W \, dy - T_x \, \frac{\partial u_x}{\partial x} \, ds - T_y \, \frac{\partial u_y}{\partial x} \, ds \right) \]

\[ J = \int_{\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4} \left( W \, dy - T_y \, \frac{\partial u_y}{\partial x} \, ds \right) \]

\[ J = J_1 + J_2 + J_3 + J_4 \]

\[ J_1 = J_4 = 0 \text{ because } dy = 0, T_y = T_x = 0 \]

Reverse direction of contour 2 to match 1, then \( J_1 = J_2 \)

\( J_1 \) turns out to provide the positive value so we adopt it.

J-integral = energy release rate?

\[ t_{coh} = \sigma_y \delta \text{ in the continuum} \]

Thickness: \( B \)

Cohesive tractions across crack face

From previous notes

\[ U_{sp} = U_{coh} = \int_{0}^{\delta_c} [t(\delta) \, B \, da] \, d\delta = B \, da \int_{0}^{\delta_c} t(\delta) \, d\delta \]

\[ J = \frac{U_{coh}}{B \, da} = \int_{0}^{\delta_c} t(\delta) \, d\delta = \Gamma \]

Energy release per new crack face area
**J = energy release rate (2)**

\[ J = \int_{\Gamma} \left( W \frac{\partial u}{\partial x} - T_y \frac{\partial u}{\partial x} ds - T_x \frac{\partial v}{\partial x} ds \right) = -\int_{\Gamma^{-} + \Gamma^{+}} T_y \frac{\partial v}{\partial x} ds \]

\[ dx = -mds \]
\[ dy = lds \]
\[ T_y = T_x^{+} + \sigma_{yy} m = t(+1) \]
\[ \hat{n} = \hat{i} + mj \]
\[ dy = 0 \text{ and } T_x = 0 \]
\[ \text{on } \Gamma^{-} \text{ and on } \Gamma^{+} \]

**Thickness:** \( B \)

**Cohesive tractions across crack face**

**J = energy release rate (3)**

\[ J = -\int_{\Gamma^{-} + \Delta a} T_y \left( \frac{\partial u}{\partial x} \right) ds - \int_{\Gamma^{+} + \Delta a} T_y \left( \frac{\partial u}{\partial x} \right) ds \]

The only non-zero stress acting on material points along \( \Gamma \) is \( \sigma_{yy} = t(\hat{n}) \) and it is positive (tension) algebraically.

- **on** \( \Gamma^{-} : \)
  - \( \hat{n} = \hat{i} + mj = (0) i + (-1) j \)
  - \( T_x = \sigma_{xx} l + \tau_{xy} m = 0 \)
  - \( T_y = \tau_{xy} l + \sigma_{yy} m = t(-1) = -t \)
  - \( dx = -mds \Rightarrow ds = dx \)
  - for \( \Gamma^{-} = 0 \Rightarrow x = a \)
  - for \( \Gamma^{-} = \Delta a \Rightarrow x = a + \Delta a \)

- **on** \( \Gamma^{+} : \)
  - \( \hat{n} = \hat{i} + mj = (0) i + (1) j \)
  - \( T_x = \sigma_{xx} l + \tau_{xy} m = 0 \)
  - \( T_y = \tau_{xy} l + \sigma_{yy} m = t(+1) = t \)
  - \( dx = -mds \Rightarrow ds = -dx \)
  - for \( \Gamma^{+} = 0 \Rightarrow x = a + \Delta a \)
  - for \( \Gamma^{+} = \Delta a \Rightarrow x = a \)
\[ J = \text{energy release rate (4)} \]

\[ J = - \int_{a}^{a+\Delta a} -t(\delta) \left( \frac{\partial \sigma^-}{\partial x} \right) dx - \int_{a+\Delta a}^{a} t(\delta) \left( \frac{\partial \sigma^+}{\partial x} \right) (-dx) \]

\[ J = - \int_{a}^{a+\Delta a} t(\delta) \left( \frac{\partial \sigma^+}{\partial x} - \frac{\partial \sigma^-}{\partial x} \right) dx \]

\[ \Delta a \to 0 \]

\[ \delta(a) = \delta_c \text{ and } \delta(a + \Delta a) = 0 \text{ (not yet opening)} \]

\[ \frac{\partial \delta}{\partial x} dx = d\delta \]

\[ \Rightarrow J = - \int_{0}^{\delta_c} t(\delta) \ d\delta = \int_{0}^{\delta_c} t(\delta) \ d\delta \]

\[ J = \text{energy release rate (5)} \]

Same result as before for energy release rate!
Integral limitations

- Note that $J$ is a 2-D concept (the integration is around a planar contour $\Gamma$).
- There are fairly strict requirements to insure that $J$ remains independent of the integration contour $\Gamma$:
  - The material must be elastic (linear or nonlinear)
  - The material properties ($E, \nu$) can vary in $Y$ but not $X$ direction
  - The contour contains only 1 crack tip
  - No body forces inside the contour (including inertia loading)
  - No thermal or other initial strains or stresses inside the contour
  - No loading on crack faces that lie inside the contour
- Additional terms can be added to the definition of $J$ to “remove” contributions from these other effects that destroy the path independence
- When this is done, the connection of $J$ to the energy release rate is maintained but connection to the HRR field is not guaranteed