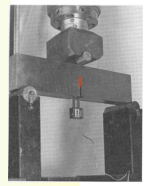
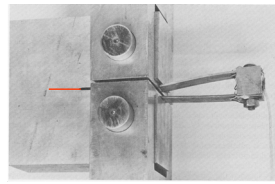
Energetics of Cracked Bodies

Sanford: Chapter 6
Anderson: 2.1, 2.3, 2.4, 2.7
Kanninen and Popelar: pg. 32-37, 158-163





Equivalence of Stress Intensity and Energy Methods

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Cracked body energetics (1)

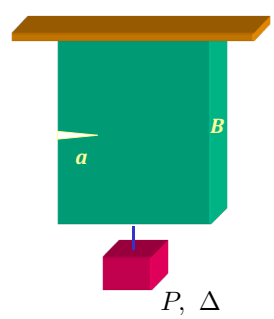
- **Total potential energy (π) of a conservative system**

$$\pi = U + \Omega$$

Stored elastic energy — ***Potential energy of applied loads***

$$U = \int P\, d\Delta = \int_V \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}$$

$$\Omega = -P\Delta = -\iint_{S_T} t_i u_i\, dS = -W_{ext}$$



Load control test

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Cracked body energetics (2)

- If specimen is deformed by imposing displacement (Δ) rather than weight (P), then Ω = 0 (*use an old style screw machine in the lab!*)

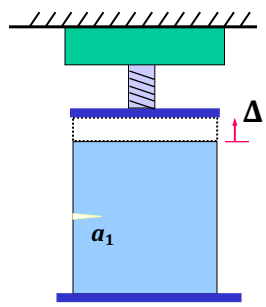
$$\pi = U + \Omega$$

Potential energy of applied loads

Stored elastic energy

$$U = \int P\, d\Delta = \int_V \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}$$

$$\underline{\Omega = 0}$$



Testing machine is so stiff relative to specimen that Δ does not change during crack advance

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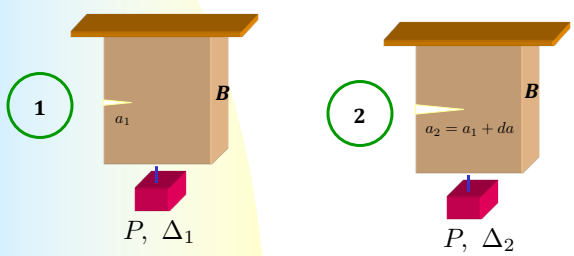
Energy release rate (1)

- When the crack grows by amount Δa during a test, the new traction free crack area is $\Delta A_c = \Delta a \times B$ (this is conventional usage of the "projected area")
- The crack growth causes the potential energy of the structure to change



Example shown here is load control during crack extension

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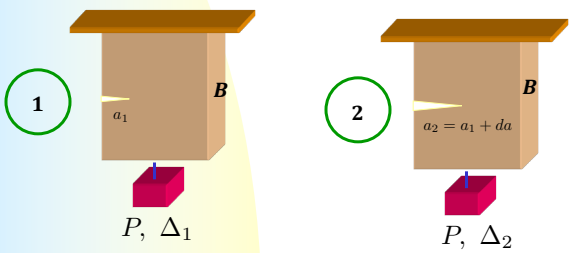
Energy release rate (2)

- The potential energy change per unit area of crack extension is called the ***energy release rate*** (J or $\mathcal{G}$) and is given by

$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} -\frac{\pi_2 - \pi_1}{B \Delta a}$$

A leading (-) is included in the definition so that the release rate has a positive numerical value



Example shown here is load control during crack extension

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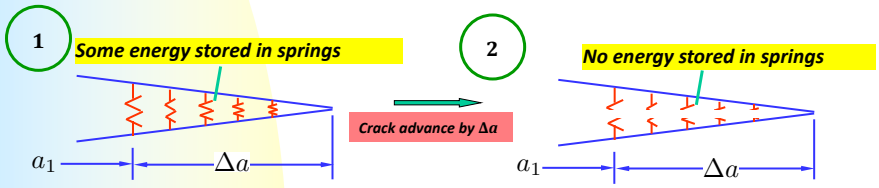
J for Displacement Controlled Loading

- The key concept is what happens with the forces (tractions) across the crack area $\Delta a \times B$ in configuration 1 that are zero in configuration 2 after the crack grows
- Imagine that we have simple springs connecting the crack faces over $\Delta a \times B$
- In 1, these springs are stretched due to specimen loading and they have some stored elastic energy
- In 2, these springs are broken (to create crack growth by $\Delta a \times B$) and no longer have the stored energy



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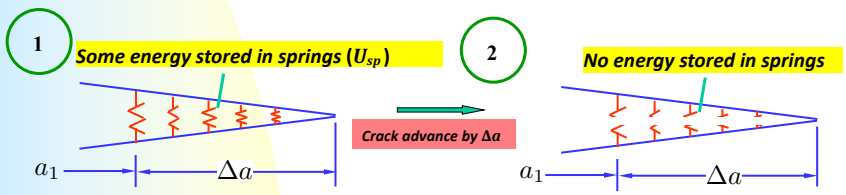
J for Displacement Controlled Loading

- Something happens to the energy balance in the cracked structure since it is closed and conservative – this changes the potential energy
- We also know that for fixed displacement loading, $\Omega = 0$ in 1 and in 2, and that the *compliance (flexibility) increases* in 2 due to the increased crack length



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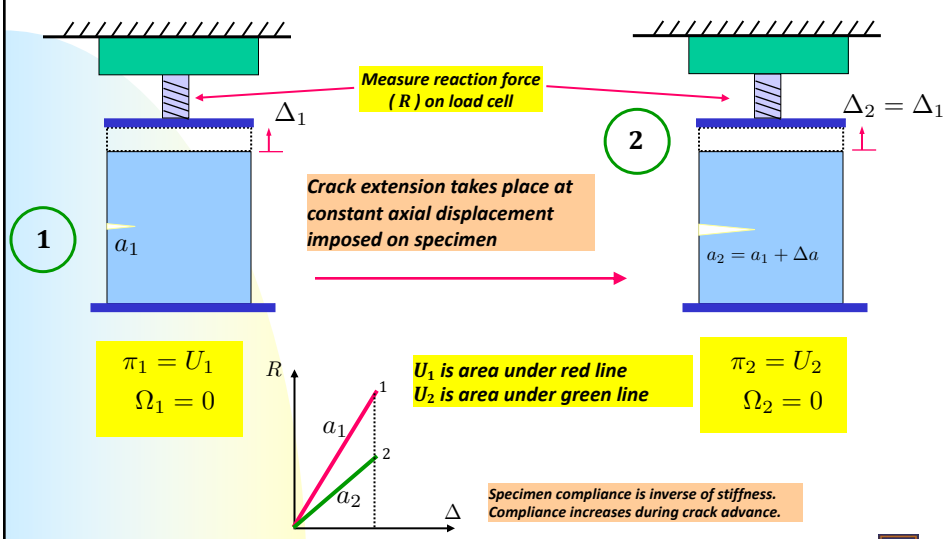
Compute *J* : Δ controlled loading (1)



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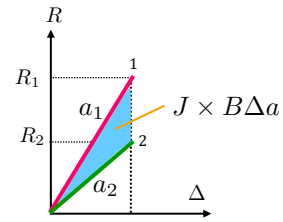
Compute J : Δ controlled loading (2)

U_1 is area under red line and includes U_{sp}
U_2 is area under green line: U_{sp} = 0



$$\pi_2 - \pi_1 = U_2 - U_1$$

$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} -\frac{\pi_2 - \pi_1}{B\Delta a} = -\frac{U_2 - U_1}{B\Delta a}$$

$$U_1 = \frac{1}{2} R_1 \Delta; \quad U_2 = \frac{1}{2} R_2 \Delta$$

$$J = \lim_{\Delta a \to 0} -\frac{\Delta}{2B} \frac{R_2 - R_1}{\Delta a} = -\frac{\Delta}{2B} \left(\frac{\partial R}{\partial a} \right)_{\Delta = fixed}$$

J has units of F-L / L²

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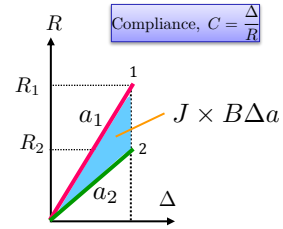
Compute J : Δ controlled loading (3)

- **Write R and Δ in terms of compliance C: C = Δ / R**
- **Then the derivative can be re-written to give**

Compliance, $C = \frac{\Delta}{R}$



$$J = -\frac{\Delta}{2B} \left(\frac{\partial R}{\partial a} \right)_{\Delta = fixed} = -\frac{\Delta}{2B} \underbrace{\left(\frac{\partial (\Delta / C)}{\partial a} \right)_{\Delta = fixed}}$$

= 0 *(fixed grips, Δ does not change with a)*

$$\left(\frac{\partial (\Delta / C)}{\partial a} \right)_{\Delta = fixed} = \frac{C \left(\cancel{\frac{\partial \Delta}{\partial a}} \right)_{\Delta = fixed} - \Delta \left(\frac{\partial C}{\partial a} \right)_{\Delta = fixed}}{C^2} = -\frac{\Delta}{C^2} \left(\frac{\partial C}{\partial a} \right)_{\Delta = fixed}$$

$$J = -\frac{\Delta}{2B} \left(-\frac{\Delta}{C^2} \right) \left(\frac{\partial C}{\partial a} \right)_{\Delta = fixed} = \frac{1}{2B} \frac{\Delta}{C} \frac{\Delta}{C} \left(\frac{\partial C}{\partial a} \right)_{\Delta = fixed} = \frac{R^2}{2B} \left(\frac{\partial C}{\partial a} \right)_{\Delta = fixed}$$

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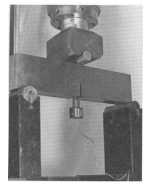
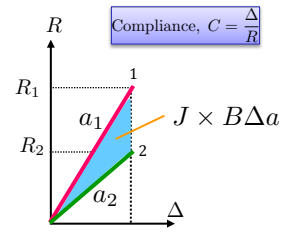
Compute J : Δ controlled loading (3)

$$J = \frac{R^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{\Delta = fixed}$$





How to Use ?

- **Suppose we have a closed-form solution for the compliance**
- **Energy release rate follows directly by applying the above derivative (usually v. simple to compute)**
- **Can also compute <u>approximate compliance change</u> in a finite element analysis for small changes in crack length**

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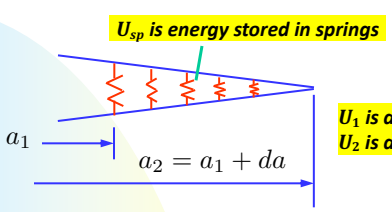
J interpretation



U_1 is area under red line: includes U_{sp}
U_2 is area under green line: U_{sp}= 0

- **Since the system is closed, conservative, & fixed displacement during growth, the energy change between 1 and 2 must be *only* from the energy stored in the broken springs**

$$\left.\begin{aligned} U_2 + U_{sp} &= U_1 \\ U_2 - U_1 &= -U_{sp} \end{aligned}\right\} \rightarrow \pi_2 - \pi_1 = U_2 - U_1 = -U_{sp}$$

$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} -\frac{\pi_2 - \pi_1}{B\Delta a} = -\frac{-U_{sp}}{B\Delta a} = \frac{U_{sp}}{dA_c}$$

J has units of F-L / L²

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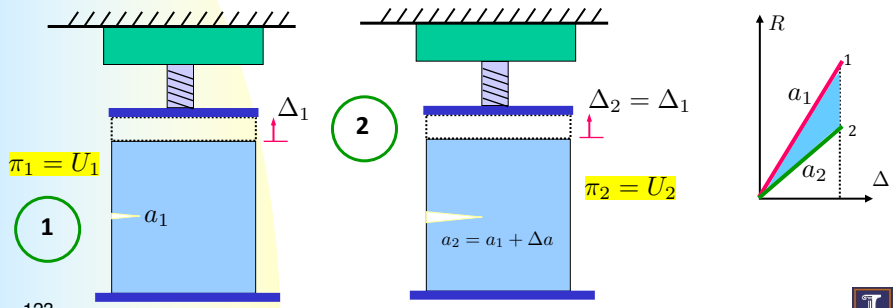
What does this mean?

- When K_I approaches K_{Ic}, the cracked body in configuration 1 has just reached sufficient potential energy (in this case, stored elastic energy) available (and can give up) to break the springs (cohesive tractions) holding the crack closed over the area $B \times \Delta a$ (*i.e.* the energy to be *released* during crack advance)
- When K_I from the applied displacement is < K_{Ic}:
  - The cohesive tractions (*i.e.* spring forces or strength) *available* from the metallurgical features exceed the applied tractions imposed by K_I stress field (no break)
  - The energy available from the background (elastic) material during a crack growth $B \times \Delta a$ < energy stored in the springs at their breaking point



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What does this mean? (2)

- The energy criterion was the original concept of fracture mechanics: $K_I = K_{Ic}$ came many years later
- In a few slides we show that energy and stress-intensity arguments are equivalent for linear-elastic systems
- Energy arguments are especially powerful in finite element analysis: U and Ω are the most accurate quantities computed
- *Key point:* fracture conditions described w/o any mention of the metallurgical mechanisms! (caused much confusion for decades – still does)

$$J_{applied} = J_{critical}$$

Original concept of fracture mechanics

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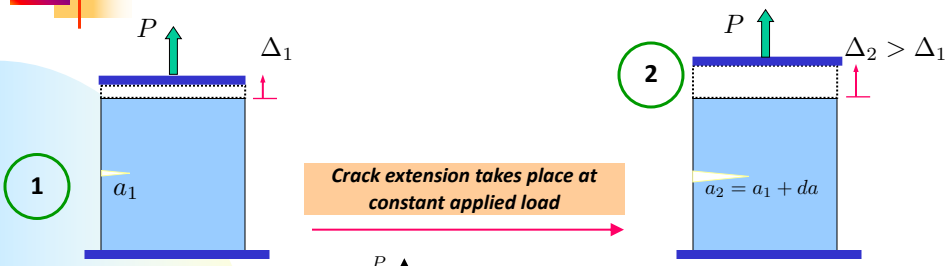
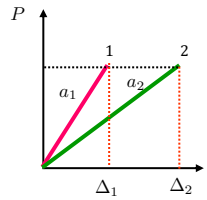
J for load control (1)



Crack extension takes place at constant applied load

$$\pi_1 = U_1 + \Omega_1$$
$$\Omega_1 = -P\Delta_1$$
$$U_1 = \frac{1}{2}P\Delta_1$$



$$\pi_2 = U_2 + \Omega_2$$
$$\Omega_2 = -P\Delta_2$$
$$U_2 = \frac{1}{2}P\Delta_2$$

U_1 is area under red line
U_2 is area under green line

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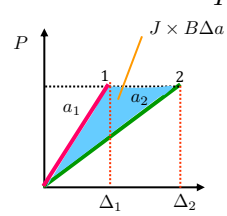
J for load control (2)

Compliance, $C = \dfrac{\Delta}{P}$

$$\left.\begin{aligned}\Omega_1 &= -P\Delta_1\\ U_1 &= \frac{1}{2}P\Delta_1\end{aligned}\right\} \quad \pi_1 = U_1 + \Omega_1 = -\frac{1}{2}P\Delta_1$$

$$\left.\begin{aligned}\Omega_2 &= -P\Delta_2\\ U_2 &= \frac{1}{2}P\Delta_2\end{aligned}\right\} \quad \pi_2 = U_2 + \Omega_2 = -\frac{1}{2}P\Delta_2$$



$$J = -\frac{\partial \pi}{\partial A_c} = \lim_{\Delta a \to 0} -\frac{\pi_2 - \pi_1}{B\Delta a} = \frac{1}{2}\frac{P\left(\Delta_2 - \Delta_1\right)}{B\,da} = \frac{P}{2B}\left(\frac{\partial \Delta}{\partial a}\right)_{P=fixed}$$

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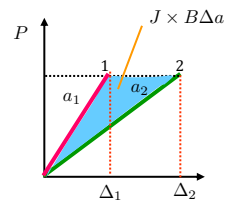
J for load control (3)

$$J = -\frac{\partial \pi}{\partial A_c} = \frac{P}{2B}\left(\frac{\partial \Delta}{\partial a}\right)_{P=fixed}$$

Compliance, $C = \dfrac{\Delta}{P}$



- **Write *P* and Δ in terms of compliance *C*: *C* = Δ / *P***
- **Then the derivative can be re-written to give**

$$J = \frac{P}{2B}\left(\frac{\partial \Delta}{\partial a}\right)_{P=fixed} = \frac{P}{2B}\underbrace{\left(\frac{\partial (PC)}{\partial a}\right)_{P=fixed}}$$

= 0 *(fixed load, P does not change with a)*

$$\left(\frac{\partial (PC)}{\partial a}\right)_{P=fixed} = C\underbrace{\left(\frac{\partial P}{\partial a}\right)_{P=fixed}}_{0} + P\left(\frac{\partial C}{\partial a}\right)_{P=fixed} = P\left(\frac{\partial C}{\partial a}\right)_{P=fixed}$$

$$J = \frac{P^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{P=fixed}$$

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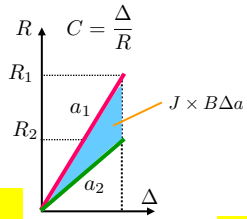
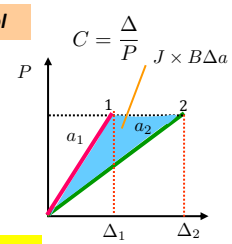
Equivalence of load-displacement control (1)

Displacement Control



$$J = \frac{R^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{\Delta=fixed}$$

Load Control



$$J = \frac{P^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{P=fixed}$$

- **The crack grows when the reaction from the imposed displacement is the same as the load applied by a weight, *i.e.*, *R* = *P* (and the internal stresses are identical)**
- **The compliance, *C*, of a linear-elastic specimen is not a function of the load *P* or the displacement Δ. It is a function only of the crack length (and other dimensions of the specimen) end the material elastic properties**

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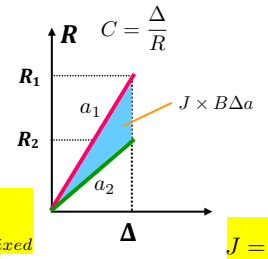
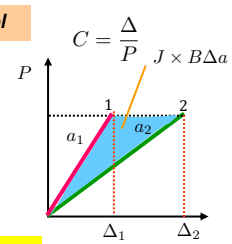
Equivalence of load-displacement control (2)

Displacement Control



$$J = \frac{R^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{\Delta = fixed}$$

Load Control



$$J = \frac{P^2}{2B}\left(\frac{\partial C}{\partial a}\right)_{P = fixed}$$

- **Consequently,**

$$R = P$$

$$\left(\frac{\partial C}{\partial a}\right)_{\Delta = fixed} = \left(\frac{\partial C}{\partial a}\right)_{P = fixed}$$

✓ ***J*** **for Load & Displacement Control are Identical**

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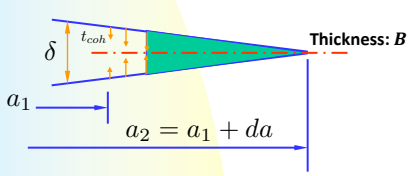
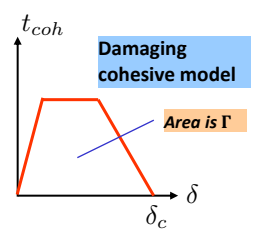
General cohesive model (1)

- **More realistic models treat the "springs" as <u>cohesive elements</u> uniformly distributed over the crack plane**
- **The cohesive stress (t_{coh}) normal to the crack plane varies with the opening displacement (δ) between the crack faces**
- **The cohesive law can be linear, nonlinear or damaging**





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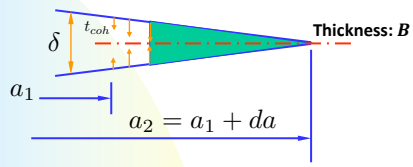
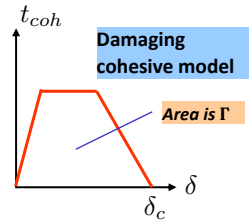
General cohesive model (2)

- Compute the energy stored in cohesive forces (i.e., U_{sp})





This calculation is for traction across the crack area & the total displacement across the faces.

$$U_{sp} = U_{coh} = \int_0^{\delta_c} \underbrace{[t(\delta)\ Bda]}_{\text{traction x area = force}} d\delta = Bda \int_0^{\delta_c} t(\delta)\ d\delta$$

$$J = \frac{U_{coh}}{Bda} = \int_0^{\delta_c} t(\delta)\ d\delta = \Gamma$$

Energy release per new crack face area

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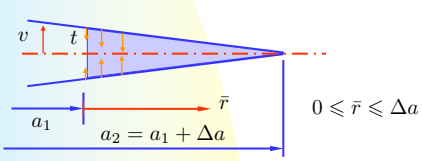
J-K Relationship (1)

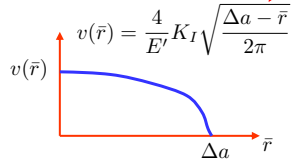
- Consider Mode I plane-strain conditions
  - Opening mode stresses acting ahead of crack in configuration 1 ($a = a_1$) are given by K_I
  - Let crack area grow by $B \times \Delta a$ to configuration 2
  - The crack stresses *relax* to zero over new crack area $B \times \Delta a$ gradually as the crack opens from a_1 to a_2
  - The opening displacements must follow the Mode I solution
  - Perform "cohesive" integration to find J (work done by the tractions as they relax to zero during displacement increase)



$0 \leqslant \bar{r} \leqslant \Delta a$

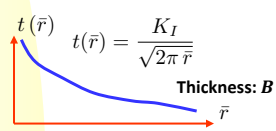
$r = \Delta a - \bar{r}$

$$v(\bar{r}) = \frac{4}{E'} K_I \sqrt{\frac{\Delta a - \bar{r}}{2\pi}}$$



$$E' = \frac{E}{1-\nu^2}$$

Plane-strain



$$t(\bar{r}) = \frac{K_I}{\sqrt{2\pi\,\bar{r}}}$$

Thickness: B

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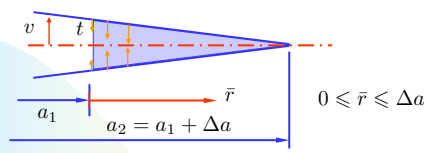
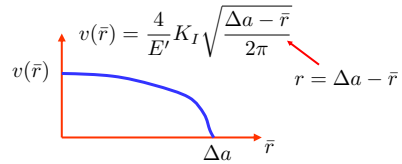
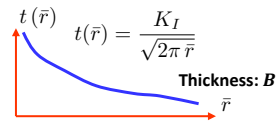
J-K Relationship (2)



$0 \leqslant \bar{r} \leqslant \Delta a$





$$U_{coh} = 2 \int_0^{\bar{r}=\Delta a} \frac{1}{2}\, v(\bar{r})\, \underbrace{t(\bar{r})\; B\, d\bar{r}}_{\text{Differential force}} = \frac{4K_I^2}{2\pi\, E'} B \underbrace{\int_0^{\Delta a} \sqrt{\frac{\Delta a - \bar{r}}{\bar{r}}}\, d\bar{r}}_{\frac{\Delta a \pi}{2}}$$

$$J = \frac{U_{coh}}{B \Delta a} = \frac{K_I^2}{E'}$$

$E' = E$ **Plane-stress**

$E' = \dfrac{E}{1-\nu^2}$ **Plane-strain**

Remember: *these tractions and displacements are the final values (thus a ½ is needed for energy)*
The outside 2 is needed because the displacement (v) is just the upper & lower value about symmetry plane

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J-K Relationship (3)

- The ***stress-intensity factor*** approach and the ***energy-based*** approach for fracture under linear-elastic conditions are identical!!
- Given K_I we can compute J
- Given J we can compute K_I
- Use the more convenient computational approach for the problem needing solution
- Energy release rates (J) are very accurate even for relatively crude finite element models that employ <u>displacement-based</u> element formulations

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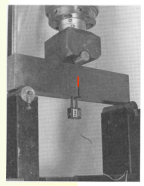
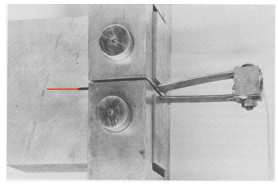
Applications of Energy Approach

Sanford: Chapter 6
Anderson: 2.3,4,7,10
Kanninen and Popelar: pg. 32-37, 158-163





Equivalence of Stress Intensity and Energy Methods

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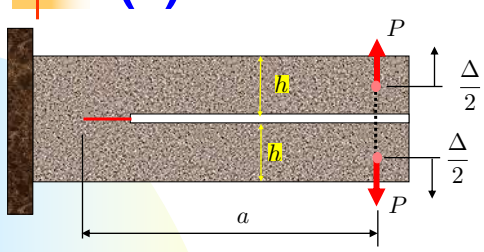
Example *J* computation (1)



Double Cantilever Beam

Thickness: B

Use simple beam theory to compute the displacement in the arms of the specimen assuming cantilever behavior

$$\frac{\Delta}{2} = \frac{PL^3}{3EI} = \frac{Pa^3}{3EI}; \quad I = \frac{1}{12}Bh^3$$

total displacement

$$Compliance, \quad C = \frac{\Delta}{P} = \frac{2a^3}{3EI}$$

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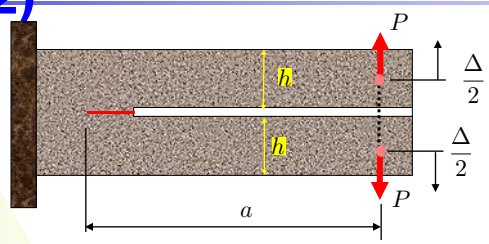
Example *J* computation (2)



Thickness: B

$$\frac{\Delta}{2} = \frac{PL^3}{3EI} = \frac{Pa^3}{3EI}; \quad I = \frac{1}{12}Bh^3$$

$$Compliance, \quad C = \frac{\Delta}{P} = \frac{2a^3}{3EI}$$

$$J = \frac{P^2}{2B}\frac{\partial C}{\partial a} = \frac{P^2}{2B}\frac{\partial}{\partial a}\left(\frac{2a^3}{3EI}\right) = \frac{P^2a^2}{BEI} = \frac{12P^2a^2}{B^2h^3E}$$

$$J = \frac{12P^2a^2}{B^2h^3E}$$

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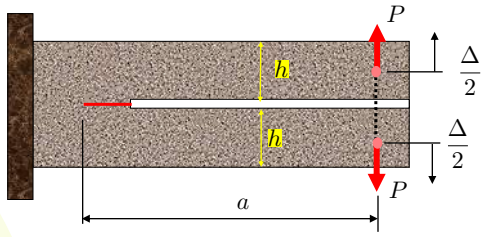
Example *J* computation (3)



Thickness: B

$$J = \frac{12P^2a^2}{B^2h^3E}$$

- Neglects shear deformation in the arms
- Assumes arms fixed at crack tip
- Neglects energy in remainder of specimen
- At constant load, *J* increases as the square of the crack length
- Verify correct physical units (must be $F\text{-}L/L^2$)

$$K_I = \sqrt{EJ} = \frac{2\sqrt{3}Pa}{Bh^{3/2}}$$

for plane stress

This is a very good approximation, especially for long, slender arms

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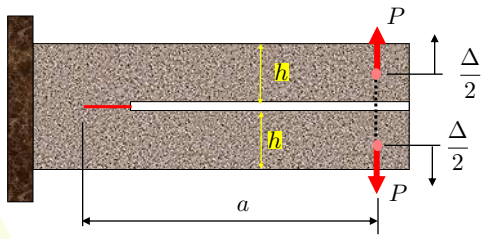
Crack stability (1)



Thickness: B

$$J = \frac{12P^2a^2}{B^2h^3E}$$

As the crack grows in length, how does the energy release rate (*J*) change at fixed load?

$$\left.\frac{\partial J}{\partial a}\right|_P = \frac{24P^2a}{B^2h^3E} = \frac{2J}{a} > 0$$

The rate of change of energy release rate is *positive*

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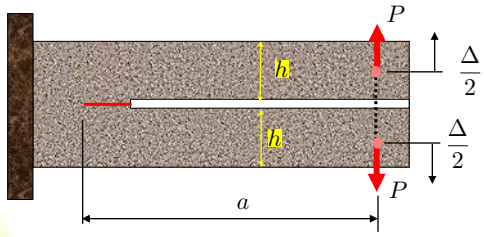
Crack stability (2)



Thickness: B

$$J = \frac{12P^2a^2}{B^2h^3E}$$

Change to displacement control loading

$$\Delta = \frac{2Pa^3}{3EI} \Rightarrow P = \frac{3EI\Delta}{2a^3}$$

$$J = \frac{9EI\Delta^2}{4a^4B}$$

Now compute rate of change of energy release rate at fixed, imposed displacement

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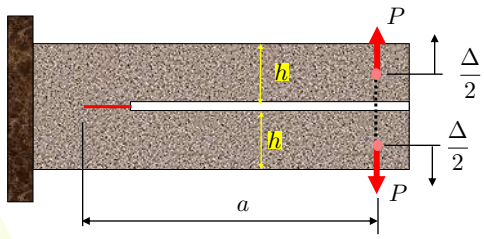
Crack stability (3)



Thickness: B

$$\left.\frac{\partial J}{\partial a}\right|_P = \frac{2J}{a} > 0$$

Load control

$$\left.\frac{\partial J}{\partial a}\right|_\Delta = -\frac{J}{a} < 0$$

Displacement control

- Crack growth begins when $J=J_{critical}$ for both cases
- Suppose the $J_{critical}$ is a material constant
- Then crack growth continues in load control since J will be above $J_{critical}$
- In displacement control, the crack stops since J decreases with growth. Additional imposed displacement is required to resume crack growth
- This also means that K_I increases in load control and decreases in displacement control during crack extension

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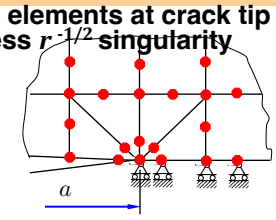
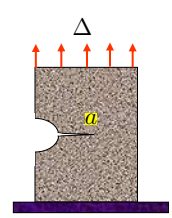
Use of FEA to compute J-K_I

- The finite element method is ideally suited to compute the energy release rate, J, and then K_I from the $J - K_I$ relationship
- Illustrate using displacement control loading of the FE model
- Real crack length is a, located at edge of some shape of hole
- Mode I conditions, model only upper ½ of specimen
- Use collapsed, 8-node 2-D elements at crack tip to best represent the strain-stress $r^{-1/2}$ singularity

$$\pi = U + \cancel{\Omega}^{\,0} = U = \frac{1}{2}\{u_s\}^T [K_s] \{u_s\}$$



¼ point singularity elements. See Anderson Appendix 12



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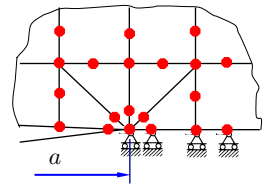
Use of FEA to compute J-K_I

$$\pi = U + \Omega = U = \frac{1}{2}\{u_s\}^T [K_s] \{u_s\}$$

- Run 2 analyses with slightly longer and shorter crack lengths: just perturb the horizontal position of the 4 singularity elements
- Compute the strain energy, U, in each case. Most FEA codes print U if asked.



$$a_1 = a - \Delta a \; ; \quad a_2 = a + \Delta a$$
$$\Delta a \approx 0.02a$$

$$J_{symm} = -\frac{\partial \pi}{\partial A_c} = -\frac{\partial U}{\partial A_c} = -\frac{1}{B}\frac{\partial U}{\partial a}$$

central difference estimate of derivative

$$J_{symm} = -\frac{1}{B}\left[\frac{U(a+\Delta a) - U(a-\Delta a)}{2\Delta a}\right]$$

$$J > 0 \quad \text{since } U_{a-\Delta a} > U_{a+\Delta a}$$

$$J = J_{total} = 2 \times J_{symm}$$

(the FE model has only ½ of the specimen!)

$$K_I = \sqrt{E' J}$$

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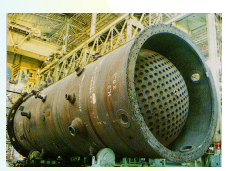
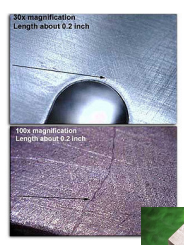
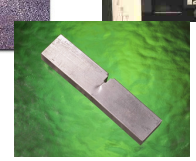
Elastic-plastic fracture mechanics

- Anderson
  - Chp. 3 (section 3.1)
  - Chp. 7 (section 7.5)
- Sanford
  - Chp. 11 (sections 11.1, 11.2)







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Elastic-plastic fracture mechanics

- Overview
- CTOD Methodology
- CTOD Estimation
- *J* Methodology
- *J* Integral
- HRR Solutions
- Features of HRR Solutions
- Asymptotic Dominance

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Overview (1)

- Late 1950s-mid 1960s, nearly all research focused on Linear Elastic Fracture Mechanics (LEFM)
- Structures of most interest at the time were made of high-strength materials: Ti, Al, high-strength steels all with $\sigma_{ys} > 100$ ksi
  - Rockets, ICBMs
  - B-1 bomber

ure toughness, plastic zones very
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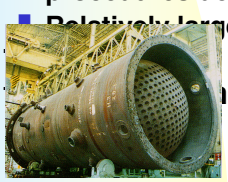
Overview (2)

- In mid-1960s additional focus on (often driven by regulators)
  - Commercial & military nuclear power production
  - Offshore platforms
  - Welded pipelines
  - Low strength, high toughness steels (A36, A572, A516, A508, A533B, …)
  - Large plastic zones at cleavage fracture, ductile instability
  - Invalidated analysis, experimental methods and assessment procedures developed for LEFM
  - Relatively large F S.
- "c od ous nd



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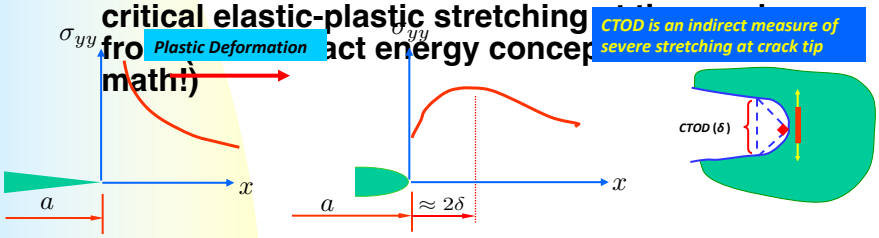
CTOD motivation

- Anderson Sections 3.1, 3.3, 7.5, 9.8
- Developed initially in the U.K. by Wells and colleagues at The Welding Institute (TWI, near Cambridge)
- Funding driven by safety of platforms in North Sea and associated pipelines
- Very physically appealing approach based on critical elastic-plastic stretching fro act energy concep math!)



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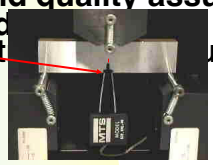
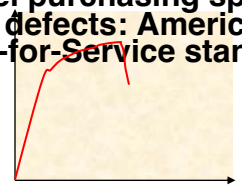
CTOD developments

- Somewhat less popular in the U.S. where the *J*-integral dominated research, thinking and funding
- UK researchers(and now Edison Welding Institute in U.S.) pushed development of extensive fracture toughness testing and defect assessment procedures
- Petro-chemical industry worldwide uses CTOD for weld quality assurance, steel purchasing specs and [illegible] of existing defects: American Pet[illegible]ute, Fitness-for-Service standards

Crack Mouth Opening Displacement



CTOD most commonly measured with SE(B) specimens



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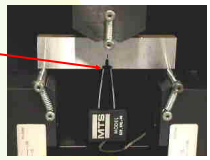
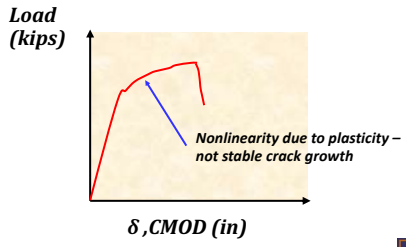
CTOD methodology

- CTOD intended as a *single value* of fracture toughness at instability
- Not meant to describe *R*–curve behavior
- For fracture after max load in a test, CTOD becomes quite ambiguous
- CTOA (crack tip opening angle) concept is an extension of CTOD for *R*–curve behavior (new ASTM E2472-06 standard for CTOA – much R&D activity)
- CTOD can be measured during a test by infiltration techniques, optically with high-resolution cameras, inferred from measured crack mouth opening displacement
- Plastic component of CTOD can be measured post-test directly from broken ends of specimen

Crack Mouth Opening Displacement



CTOD most commonly measured with SE(B) specimens



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CTOD methodology

- CTOD relies on its connection with the *J*-integral for a theoretical foundation (*J*-integral discussed soon)
- Major practical issue is *transferability* -- is the critical CTOD value measured using a deep notch SE(B) specimen in the lab the same value in a large, tension loaded structure?
- CTOD testing and assessment procedures attempt to "sidestep" this issue:
  - Specimen testing must also use the "application" thickness
  - Deep notch CTOD values are understood to be conservative
    alues pr itional (bu
    onservat ment proc

$$\delta_{crit}^{SE(B)} \stackrel{?}{=} \delta_{crit}^{Pipe}$$

Critical CTOD from deep-notch SE(B) test generally is smaller than critical CTOD measured in tension loaded components

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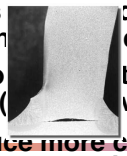
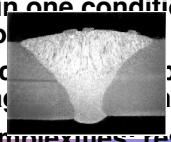
Use of CTOD methodology

- To use a CTOD methodology we need:
  - An un-ambiguous definition of CTOD both experimentally and numerically (CTOD changes with distance behind the crack front)
  - A computational procedure to estimate CTOD in a plastically deforming cracked body (at each location along crack front)
  - A experimental procedure to measure (or infer) values of CTOD at fracture instability that is robust, reliable and repeatable (can be standardized)
  - An understanding of potential effects of specimen size, loading mode (tension *vs.* bending), temperature, loading rate, prior ductile tearing, etc. on the critical value
  - Engineering (transferability) models for these effects to allow critical values d in one condition to be used ("adjusted") in co
  - Procedures to toc bility of measured CTOD values (va steels)
  - Welds introduce more complexities: residual stresses, inhomogeneous material flow properties (how to compute





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Use of CTOD methodology

- **Testing standards**
  - ASTM E1290-08 Standard Test Method for Crack-Tip Opening Displacement (CTOD) Fracture Toughness Measurement (also an ISO standard)
  - Must test at "application thickness"
  - Plate only – no welds: appendix for welds under development
  - Single point toughness values (no *R*-curve)
  - Deep notch SE(B)s required, shallow notch specimens being added to quantify constraint effects
  - No guidance on transferability, stochastic
  - Prescribes how to run the test, not how to [illegible]asured CTOD values at fracture
- **Defect assessment procedures**
  - Failure assessment diagrams (FADs)
  - UK-European: PD6493, R-6
  - US: API-579 (American Petroleum Institute)
  - See Anderson, Sections 9.3, 9.4, 9.8, 9.9

ASTM INTERNATIONAL Standards Worldwide

www.astm.org

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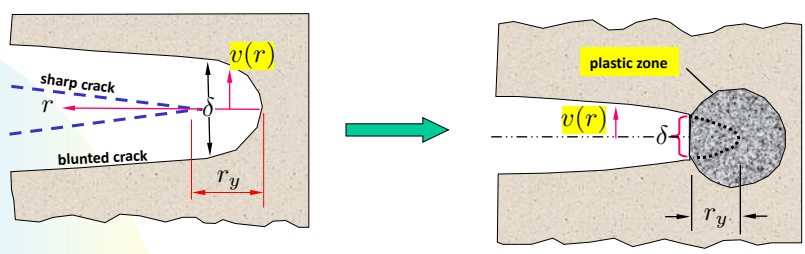
CTOD estimation



$$v(r) = \frac{4}{\pi} K_I \sqrt{\frac{r}{2\pi}}$$

Plane Stress

$$\delta = 2 \times v(r = r_y) = \frac{8}{E} K_I \sqrt{\frac{\frac{1}{2\pi}\left(\frac{K_I}{\sigma_0}\right)^2}{2\pi}}$$

$$\therefore \delta = \frac{4}{\pi} \frac{K_I^2}{\sigma_0 E} \Rightarrow$$

CTOD increases as K^2
(this is a nonlinear effect!)

The actual coefficient (not 4/π) is found from nonlinear FE analyses. Varies with material strain hardening and 3-D effects.

Suggestion: re-derive for plane-strain conditions, verify physical units are correct.

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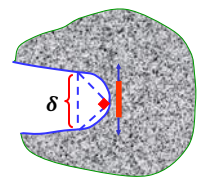
CTOD estimation (2)

- Compute J including plasticity (see later notes)
- Compute K_J from J (a K value computed from J)
- Finally get CTOD from K_J

$$J = \frac{K_J^2}{E'} \qquad K_J = \sqrt{E'J}$$

$$E' = E \ (plane\ stress)$$
$$E' = \frac{E}{1-\nu^2} \ (plane\ strain)$$



$$\therefore \delta = \frac{4}{\pi}\frac{K_I^2}{\sigma_0 E} \longrightarrow \therefore \delta = \frac{4}{\pi}\frac{J}{\sigma_0}$$

Detailed 2-D and 3-D finite element analyses show that a more accurate form is:

$$\therefore \delta = \frac{J}{m\sigma_{flow}}$$

m = 1.5 for plane strain and 3-D

$$\sigma_{flow} = \frac{1}{2}(\sigma_{ys} + \sigma_u)$$

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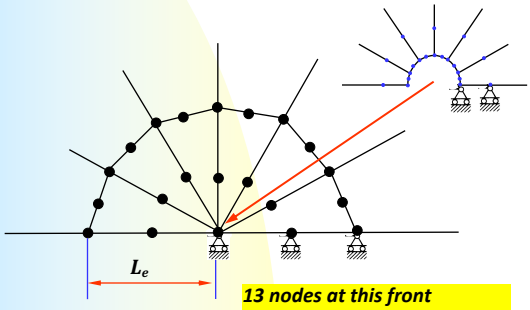
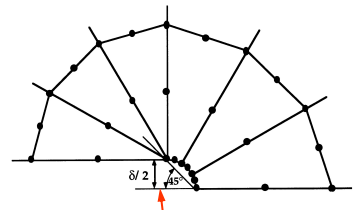
CTOD estimation (3)

- Use nonlinear finite element analysis to compute CTOD as function on applied loading
- Thermal loading, residual stresses, etc. introduce *NO* complications
- Just extract CTOD from displacements of nodes behind crack tip at each applied load level





More elements than this needed to resolve θ variation of fields

13 nodes at this front location- collapse to a point

Extract CTOD at each front location using 45° intercept method

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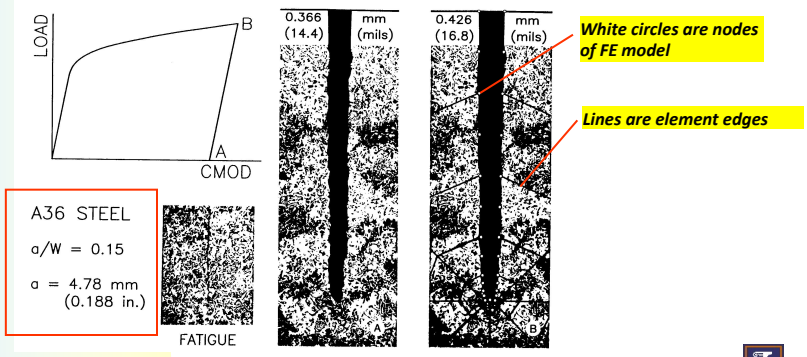
CTOD: example FEA model

- Example from work of Sorem, Rolfe, Dodds (*Int. J. Fracture*, V47, 1991)
- Test & analyses of A36 steel, deep notch SE(B) specimen
  - Load, then unload specimen as shown
  - Cool to liquid N2 and fracture
  - Remaining deformation is *plastic*
  - Superpose deformed finite element mesh (-elastic) on deformed specimen



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CTOD: Example data

- Example from work of Sorem, Rolfe, Dodds (*Int. J. Fracture*, V47, 1991)
- Illustrates major effect of ***constraint loss*** on measured fracture toughness (cleavage)
- Research still on-going to construct ***transferability*** models

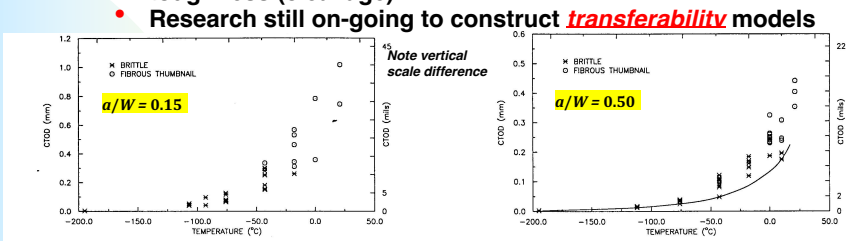
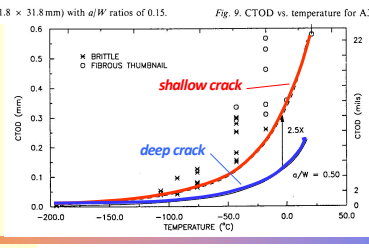
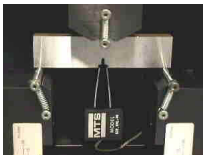


Fig. 8. CTOD vs. temperature for A36 steel specimens (31.8 × 31.8 mm) with a/W ratios of 0.15.

Fig. 9. CTOD vs. temperature for A36 steel specimens (31.8 × 31.8 mm) with a/W ratios of 0.50.

A36 Structural Steel





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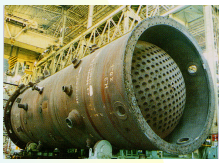
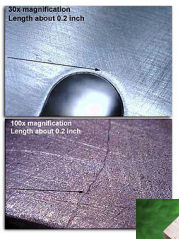
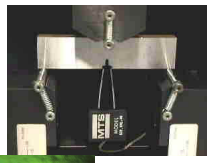
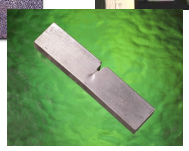
J integral for elastic-plastic fracture

- Anderson
  - Chp. 3 (3.2-3.5, App. A3.2-A3.4)
- Sanford
  - Chp. 11









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Elastic-plastic fracture mechanics

- What is *J* ?
  - Energy release rate concept extended to nonlinear material behavior in contained yielding
- *J* Methodology
- *J* Integral
- HRR Solutions
- Features of HRR Solutions
- Asymptotic Dominance

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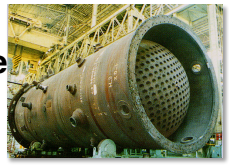
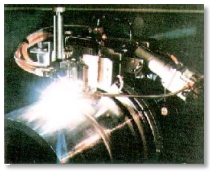
J methodology (1)





- Development started in early 1970s
- R & D driven and funded by safety issue of nuclear power generation
- High toughness, low strength materials
  - K_{ic} >> 100 at operating temperatures
  - Yield stresses 50-80 ksi
- Large plastic zones at fracture invalidate LEFM assumptions
  - Cleavage in the DBT under large-scale plasticity
  - Initiation of stable ductile tearing
  - Extensive stable tearing terminated by tearing instability
- *J* based fracture mechanics has the same goals as CTOD approach – perhaps more focus on ductile tearing

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J methodology (2)

- More abstract than CTOD (we cannot "see" a *J*), requires advanced math skills to understand, and nonlinear solid mechanics for analysis
- *J* approach used extensively in advanced R & D efforts – it has more "headroom" than CTOD given the fundamental mechanics basis
- Major practical issue is *transferability* -- is the critical *J* value or J-Δa curve measured using a deep-notch SE(B) specimen in the lab the same toughness in a large, tension loaded structure?
- Residual stresses, inhomogeneous materials (welds) are more difficult to address with *J* approach
- *J* approach mathematically (rigorously) simplifies to K_I "correlative" fracture mechanics for SSY and LEFM – this is a major appeal of *J*

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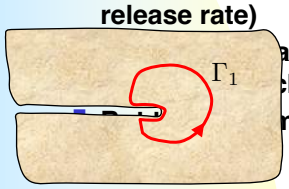
J & HRR Solutions

- Three key developments occurred over a short time period (1966- 1970)
- James Rice (then at Brown U.) explored a particular path independent (conservation) integral for 2-D sharp crack tip configurations
  - The scalar value of the integral has units of $F\text{-}L/L^2$ (energy release rate)



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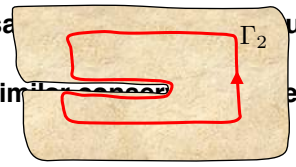

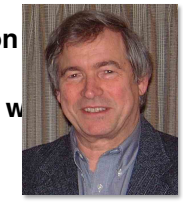

James (Jim) Rice

$$J_1 = \oint_{\Gamma_1} (\ldots)\, \mathrm{d}\Gamma_1 \qquad J_2 = \oint_{\Gamma_2} (\ldots)\, \mathrm{d}\Gamma_2$$

$J_1 = J_2 \rightarrow$ path independence

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J & HRR Solutions

- Rice and his student (Rosengren) at Brown and John Hutchinson at Harvard both solved the asymptotic, Mode I crack-tip fields for a nonlinear material, analogous to the Williams solution for linear elasticity (the now-termed HRR fields)
  - The leading stress-strain term of the field is singular
  - For a linear-elastic material the solution is identical to the Williams solution
  - The "amplitude" of the strain-stress-displacement field is again undetermined by the asymptotic solution (just like K_I)
- Rice then showed that the scalar value, J, is the amplitude of the asymptotic field analogous to K_I in the Williams solution
- He did this by evaluating the J-integral over a circular contour enclosing the crack tip with strains-stresses-displacements given by the HRR asymptotic field
- This remains today a remarkable breakthrough !

John Hutchinson

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Conservation integrals (1)

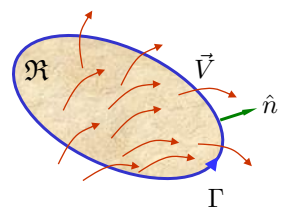
Consider a simple 2-D region $\Re$ through through which an incompressible fluid flows in steady-state.

The velocity vector of the fluid at each point (*x,y*) is:

$$\vec{V}(x,y) = V_x(x,y)\,\hat{i} + V_y(x,y)\,\hat{j}$$

Let ρ denote the mass density of the fluid. Then the net ***outflow*** of fluid from the region is given by

$$I_\Gamma = \rho \oint_\Gamma \vec{V}(x,y)\cdot\hat{n}\,ds = \text{net outflow}$$



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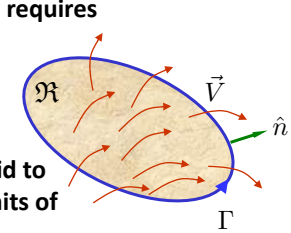
Conservation integrals (2)

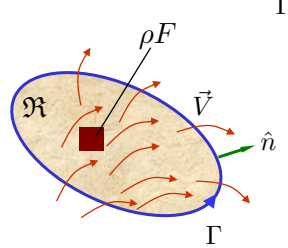
If there are no sources or sinks of fluid in the region $\Re$. Then conservation of mass for incompressible, steady flow requires that

$$I_\Gamma = \rho \oint_\Gamma \vec{V}(x,y)\cdot\hat{n}\,ds = 0$$



Suppose now there exists a point source that adds fluid to the region $\Re$. The ***strength*** of the source is ρF (has units of mass / time). Then

$$I_\Gamma = \rho \oint_\Gamma \vec{V}(x,y)\cdot\hat{n}\,ds = \rho F$$



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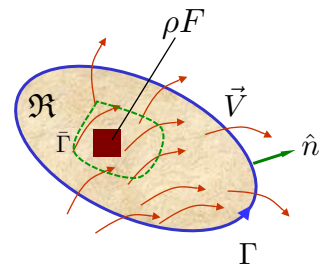
Conservation integrals (3)

Now consider some other closed contour in $\Re$ that encloses the point source. Then mass conservation also requires that

$$I_{\bar{\Gamma}} = \rho \oint_{\bar{\Gamma}} \vec{V}(x,y) \cdot \hat{n}\, ds = \rho F$$

Then,

$$\rho \oint_{\Gamma} \vec{V}(x,y) \cdot \hat{n}\, ds = \rho \oint_{\bar{\Gamma}} \vec{V}(x,y) \cdot \hat{n}\, ds$$



It is clear that all such contour integrals "compute" the same *strength* of the point source in the fluid

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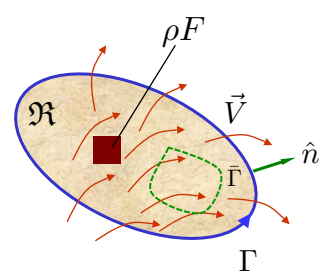
Conservation integrals (4)

If we evaluate the integral over a contour within $\Re$ that does not contain the point source, we see that

$$I_{\bar{\Gamma}} = \rho \oint_{\bar{\Gamma}} \vec{V}(x,y) \cdot \hat{n}\, ds = 0$$



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Extension to cracks (1)

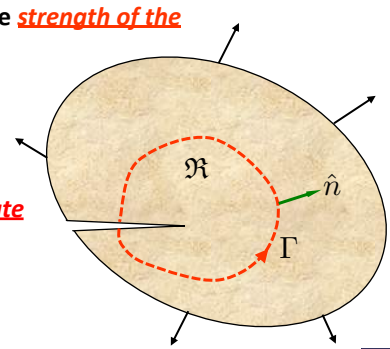
Suppose we have an elastic body in equilibrium that contains a sharp crack as shown. The crack tip represents a *point singularity* with an effect on the mechanical fields that radiates outward from the tip location and diminishes with distance.

We readily imagine that a integral defined over a contour enclosing the crack tip could compute the *strength of the singularity*.

$$I_\Gamma = \oint_\Gamma \vec{F}\left(T_x, T_y, W, \ldots\right) \cdot \hat{n}\, ds$$

A plausible measure for *the strength of the singularity is the energy release rate*

$$I_\Gamma = J = -\frac{\partial \pi}{\partial A_c}$$



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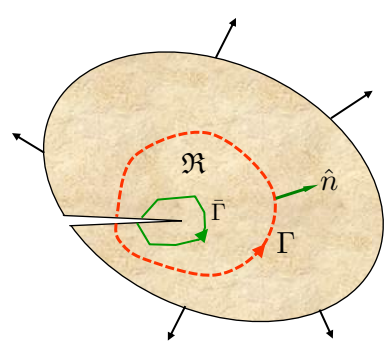
Extension to cracks (2)

Based on the analogy with incompressible fluid flow, we anticipate that *J* computed over a different contour will have the same value under suitable restrictions (*e.g.*, no body forces applied inside the contour, no crack face loading, etc.)

$$J_\Gamma = -\frac{\partial \pi}{\partial A_c} = \oint_\Gamma \vec{F}\left(T_x, T_y, W, \ldots\right) \cdot \hat{n}\, ds$$

$$J_{\bar{\Gamma}} = -\frac{\partial \pi}{\partial A_c} = \oint_{\bar{\Gamma}} \vec{F}\left(T_x, T_y, W, \ldots\right) \cdot \hat{n}\, ds$$

$$J_{\bar{\Gamma}} = J_\Gamma = -\frac{\partial \pi}{\partial A_c}$$



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Extension to cracks (3)

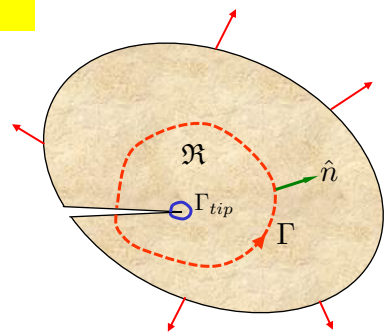
If we shrink the contour to the immediate crack tip region, we can define a "crack tip" value for *J*

$$J_{tip} = -\frac{\partial \pi}{\partial A_c} = \oint_{\Gamma_{tip}} \vec{F}\left(T_x, T_y, W, \ldots\right) \cdot \hat{n}\, ds$$

In the limit of shrinking contour sizes, the issues of body forces, crack face loading, etc. disappears.

Under these restrictions (no body forces inertial forces, isothermal, no crack-face tractions), we have

$$J_{tip} = J_{far} = -\frac{\partial \pi}{\partial A_c}$$



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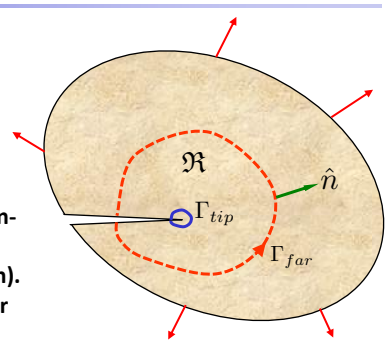
Extension to cracks (4)

$$J_{tip} = J_{far} = -\frac{\partial \pi}{\partial A_c}$$



Why of major importance?

Value of J_{tip} is clearly determined by stress-strain-displacement fields very near the crack tip (the crack-tip loading that causes material separation). Hard to compute accurate fields very near tip for nonlinear conditions!

J_{far} is determined by stress-strain-displacement values remote from the crack tip -- they reflect loading, geometry, boundary conditions. Should be able to compute these values accurately !!!

J relates quantitatively near-tip stress-strain-displacement fields to remote geometry, loading, boundary conditions

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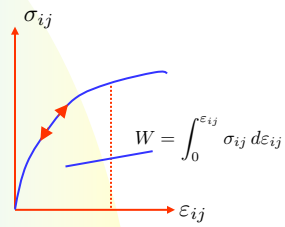
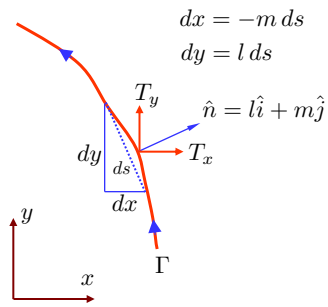
J-integral (1)

$$J = \int_\Gamma \left(W\,dy - T_i \frac{\partial u_i}{\partial x} ds \right) = \int_\Gamma \left(W\,dy - T_x \frac{\partial u}{\partial x} ds - T_y \frac{\partial v}{\partial x} ds \right)$$



Nonlinear elastic material



Rice, J.R., "A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks," *Journal of Applied Mechanics*, Vol. 35, pp. 379-386, 1968.

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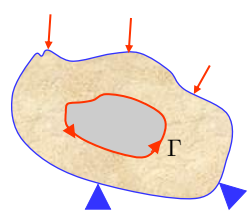
J-integral (2)

$$J = \int_\Gamma \left(W\,dy - T_i \frac{\partial u_i}{\partial x} ds \right) = \int_\Gamma \left(W\,dy - T_x \frac{\partial u}{\partial x} ds - T_y \frac{\partial v}{\partial x} ds \right)$$



***J* = 0 when Γ *contains no cracks or singularities* !**

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J-integral (3)

$$J = \int_{\Gamma} \left(W dy - T_i \frac{\partial u_i}{\partial x} ds \right) = \int_{\Gamma} \left(W dy - T_x \frac{\partial u}{\partial x} ds - T_y \frac{\partial v}{\partial x} ds \right)$$

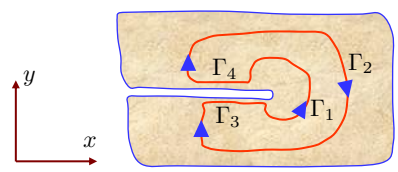
$$J = \int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4} \left(W dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

$J = J_1 + J_2 + J_3 + J_4$

$J_3 = J_4 = 0 \;\; because \;\; dy = 0, \; T_x = T_y = 0$

$J_1 = -J_2$

Reverse direction of contour 2 to match 1, then $J_1 = J_2$



Crack faces assumed traction free (no internal pressure)

J_1 turns out to provide the positive value so we adopt it

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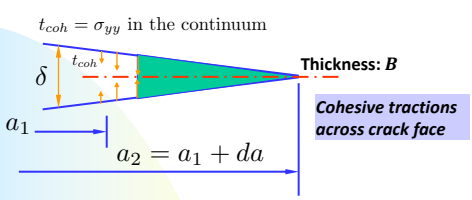
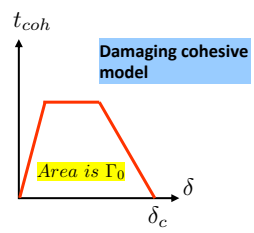
J-integral = energy release rate?





From previous notes

$$U_{sp} = U_{coh} = \int_0^{\delta_c} \underbrace{[t(\delta)\; Bda]}_{\text{traction x area = force}} d\delta = Bda \int_0^{\delta_c} t(\delta)\; d\delta$$

$$J = \frac{U_{coh}}{Bda} = \int_0^{\delta_c} t(\delta)\, d\delta = \Gamma$$

Energy release per new crack face area

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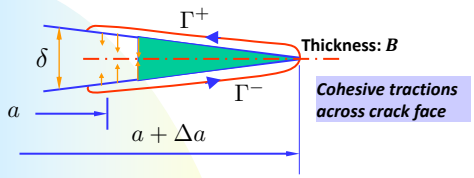
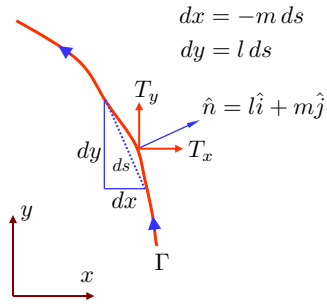
J = energy release rate (2)





$$J = \int_\Gamma \left(W\,dy - T_x \frac{\partial u}{\partial x} ds - T_y \frac{\partial v}{\partial x} ds \right) = - \int_{\Gamma^- + \Gamma^+} T_y \frac{\partial v}{\partial x}\, ds$$

$dy = 0$ and $T_x = 0$
on Γ^- and on Γ^+

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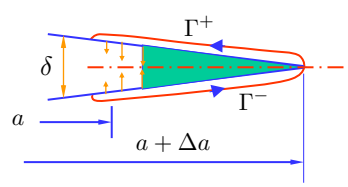
J = energy release rate (3)

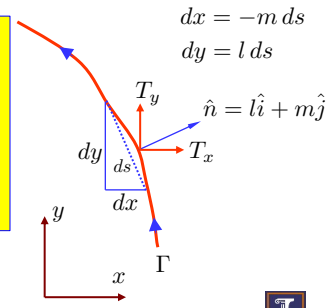
$$J = - \int_{\Gamma^- = 0}^{\Gamma^- = \Delta a} T_y \left[\frac{\partial v^-}{\partial x} \right] ds \; - \int_{\Gamma^+ = 0}^{\Gamma^+ = \Delta a} T_y \left[\frac{\partial v^+}{\partial x} \right] ds$$



The only non-zero stress acting on material points along Γ is $\sigma_{yy} = t(\delta)$ and it is positive (tension) algebraically

on Γ^- :
$\hat{n} = l\hat{i} + m\hat{j} = (0)\,\hat{i} + (-1)\,\hat{j}$
$T_x = \sigma_{xx} l + \tau_{xy} m = 0$
$T_y = \tau_{xy} l + \sigma_{yy} m = t\,(-1) = -t$
$dx = -m\,ds \quad \Rightarrow ds = dx$
$for \quad \Gamma^- = 0 \Rightarrow x = a$
$for \quad \Gamma^- = \Delta a \Rightarrow x = a + \Delta a$

on Γ^+ :
$\hat{n} = l\hat{i} + m\hat{j} = (0)\,\hat{i} + (+1)\,\hat{j}$
$T_x = \sigma_{xx} l + \tau_{xy} m = 0$
$T_y = \tau_{xy} l + \sigma_{yy} m = t\,(+1) = t$
$dx = -m\,ds \quad \Rightarrow ds = -dx$
$for \quad \Gamma^+ = 0 \Rightarrow x = a + \Delta a$
$for \quad \Gamma^+ = \Delta a \Rightarrow x = a$



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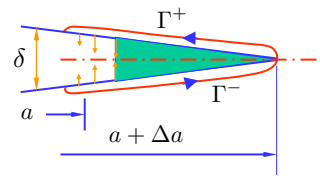
J = energy release rate (4)

$$J = -\int_a^{a+\Delta a} -t(\delta)\left[\frac{\partial v^-}{\partial x}\right] dx \; - \int_{a+\Delta a}^{a} t(\delta)\left[\frac{\partial v^+}{\partial x}\right](-dx)$$



$$J = -\int_a^{a+\Delta a} -t(\delta)\left[\frac{\partial v^-}{\partial x}\right] dx \; - \int_a^{a+\Delta a} t(\delta)\left[\frac{\partial v^+}{\partial x}\right] dx$$

$$J = -\int_a^{a+\Delta a} t(\delta)\left[\frac{\partial v^+}{\partial x} - \frac{\partial v^-}{\partial x}\right] dx \quad \Leftarrow \quad \delta = v^+(x) - v^-(x)$$

$$J = -\int_a^{a+\Delta a} t(\delta)\left[\frac{\partial \delta}{\partial x}\right] dx$$

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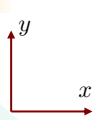
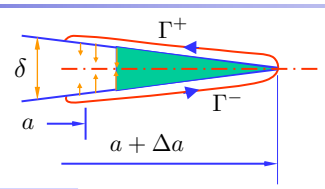
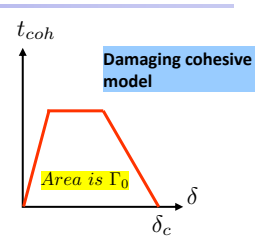
J = energy release rate (5)







$$J = -\int_a^{a+\Delta a} t(\delta)\left[\frac{\partial \delta}{\partial x}\right] dx$$

When

$\Delta a \to 0$

$\delta(a) = \delta_c$ and $\delta(a + \Delta a) = 0$ *(not yet opening)*

$$\left[\frac{\partial \delta}{\partial x}\right] dx = d\delta$$

$$\Rightarrow \quad J = -\int_{\delta_c}^{0} t(\delta)\, d\delta = \int_0^{\delta_c} t(\delta)\, d\delta$$

Same result as before for energy release rate !

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J-integral limitations

- Note that J is a 2-D concept (the integration is around a *planar contour* !)
- There are fairly strict requirements to insure that J remains independent of the integration contour Γ
  - The material must be *elastic* (linear or nonlinear)
  - The material properties (E, ν) can vary in Y but not X direction
  - The contour contains only 1 crack tip
  - No body forces inside the contour (including inertia loading)
  - No thermal or other initial strains or stresses inside the contour
  - No loading on crack faces that lie inside the contour
- Additional terms can be added to the definition of J to "remove" contributions from these other effects that destroy the path independence
- When this is done, the connection of J to the energy release rate is maintained but connection to the HRR field is not guaranteed

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