Commitment Equilibria in Bertrand Games: Commitment v. Flexibility

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Abstract

Recently, some papers have examined commitment equilibria, which are subgame perfect equilibria when agents can limit the sets of actions they can play later in the game. In this paper, we show the sufficient condition to support Pareto efficient outcome by commitment equilibria in Bertrand price competition. After that, we also consider the trade-off between welfare improvement by commitment and flexibility, by adding stochastic nature into the game. The first best outcome is always attainable regardless of probabilistic or belief structure of the game, when the states are mutually different enough from each other.

1 Introduction

It is well known that in many economic situations, we can obtain more preferable result to Nash Equilibrium by limiting our action space, in other words, making a commitment not to take best response. By including this commitment procedure into the whole game, we can analyze more general situation.

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Lazarev (2012) considered the game consists of two stages, which are the commitment stage and the action stage. First, at the commitment stage, players can simultaneously and independently choose an action set, which is all the available actions at the action stage. Observing the result of the commitment stage, that is, the player’s own action set and the opponent’s action set, agents play the newly created simultaneous game at the action stage.

Naturally, the equilibrium concept for this game is subgame perfect equilibrium, since the entire game can be regarded as a sequential game. As we explain the detail in later section, this equilibrium is called commitment equilibrium. However, some conditions are needed to assure the existence of a commitment equilibrium, because possibly there may not exist any Nash equilibrium at the action stage, if some strange action sets are chosen at the commitment stage.

There are several assumptions to avoid this problem. Bade et al. (2009) assumed that players can only commit to a compact convex action set. In this case, the existence of a commitment equilibrium is proved. Lazarev (2012) chose another way. In this paper, he considered the game with players who have supermodular utility functions. In this case, by using the theorem in Topkis (1979), the existence of Nash Equilibrium at the action stage is assured. Since our main interest is price competition, we follow Lazarev’s specification.

2 Non-Stochastic Commitment Equilibria

2.1 Model

First, we introduce several definitions we use in this paper. Most notations for non-stochastic commitment equilibrium is borrowed from Lazarev (2012).

The original game $G$ Agents play a one-shot simultaneous game $G = (\mathcal{I}, \mathcal{A}, \pi)$ where $\mathcal{I} = \{1, 2, ..., n\}$ is a set of players, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times ... \times \mathcal{A}_n$ is a collection of players’ action spaces and $\pi = \pi_1 \times \pi_2 \times ... \times \pi_n$ is a collection of payoff functions ($\pi_i : \mathcal{A} \to \mathbb{R}$ for each $i \in \mathcal{I}$). We denote $a^{NE} \in \mathcal{A}$ as the Nash equilibrium (NE) outcome if there exists a pure-strategy Nash equilibrium.
The commitment game $C(G)$ For any simultaneous game $G$, we can construct following two-stage commitment game $C(G)$ generated by $G$.

Commitment Stage Each agent $i \in \mathcal{I}$ simultaneously chooses a non-empty compact subset $A_i \subset \mathcal{A}_i$.

Action Stage After observing $A = A_1 \times \ldots \times A_n$, the agents play $G_C = (\mathcal{I}, A, \pi)$.

Generally, $C(G)$ is just a deterministic sequential game, so our solution concept is the subgame perfect equilibrium. Following Lazarev’s notation, we call a subgame perfect equilibrium of $C(G)$ as a commitment equilibrium.

Definition 1 A commitment equilibrium is a profile of strategies $(A^*, \sigma^*)$, where $A^*$ is an action set profile and $\sigma^*_i : A \rightarrow a_i$ is a strategy at the action stage, which satisfies

(i). for all $i$ and any compact subset of $\mathcal{A}_i$ and action stage strategy $\sigma_i$,

$$\pi_i(\sigma^*(A^*)) \geq \pi_i(\sigma^*(A_i, A_{-i}^*))$$

(ii). for all $A$, $\sigma^*(A)$ is a Nash equilibrium of $G_C = (\mathcal{I}, A, \pi)$.

2.2 Bertrand Duopoly with Differentiated Products; Non-stochastic

Although we can define the commitment equilibrium for wider class of games, from now on, we restrict our attention on two-player symmetric price competition, i.e., $\mathcal{A}_1 = \mathcal{A}_2$ and $\pi_1 = \pi_2$. We also assume that $\mathcal{A}_i = [\underline{p}, \bar{p}] \subset \mathbb{R}$, and $\bar{p}$ is sufficiently large in the sense that $\bar{p} > p^M$ where $p^M$ represents monopoly price, that is, Pareto efficient price. Moreover, we assume that $\pi_i(p)$ is continuous, supermodular in $p$ and strictly increasing in $p_{-i}$ for each $i$. Next example shows many Bertrand price competition settings satisfy these assumptions.

Example 1 Consider a differentiated Bertrand duopoly model with linear demand functions. Let $\mathcal{I} = \{1, 2\}$, $\mathcal{A}_i = [\underline{p}, \bar{p}]$, $\underline{p} > 0$ and $\pi_i(p_i, p_j) = p_i(1 - p_i + \alpha p_j)$ for each $i$, where $\alpha \in (0, 1)$.
Since $\pi_i$ is $C^2$, we can calculate
\[
\frac{\partial \pi_i(p_i, p_j)}{\partial p_j} = \alpha p_i \geq \alpha > 0
\]
\[
\frac{\partial^2 \pi_i(p_i, p_j)}{\partial p_i \partial p_j} = \alpha > 0
\]
and the first equation implies $\pi_i$ is increasing in $p_j$ and $\pi_i$ has increasing
difference in $(p_i, p_j)$, thus $\pi_i$ is supermodular in $(p_i, p_j)$.

In this case, in general, the monopoly price $p^M$ is bigger than the Nash
equilibrium price $p^{NE}$. Next lemma shows this.

**Lemma 1** If the game is symmetric, $\pi_i$ is supermodular in $(p_i, p_j)$ and strictly
increasing in $p_j$, then there exists symmetric Pareto efficient prices $(p^M, p^M)$
and $p^M \geq p^{NE}$.

**Proof.**
First, we show that there exists Pareto efficient symmetric prices, because
by supermodularity and symmetry, we have
\[
\pi_1(p_1, p_2) + \pi_2(p_2, p_1) \leq \pi_1(p_1, p_1) + \pi_2(p_2, p_2) \leq 2 \max_{i=1,2} \{\pi_i(p_i, p_i)\}.
\]

Then, $p^M$ is defined as
\[
p^M = \arg \max_p \pi_i(p, p) + \pi_i(p, p) = \arg \max_p 2\pi_i(p, p) = \arg \max_p \pi_i(p, p).
\]

Thus, monopoly price is the price which maximizes $\pi_i(p, p)$.

Suppose that $p^{NE} > p^M$. Then, since $\pi_i$ is increasing in the opponents’
prices, $\pi_i(p^M, p^M) < \pi_i(p^M, p^{NE})$ holds. Moreover, by definition of Nash
equilibrium, $p^{NE}$ is a best response for the opponents’ $p^{NE}$. Therefore,
$\pi_i(p^M, p^{NE}) \leq \pi_i(p^{NE}, p^{NE})$. Thus, we have $\pi_i(p^M, p^M) < \pi_i(p^{NE}, p^{NE})$ but
this contradicts Pareto optimality of $p^M$. ■

1 There exist multiple Nash equilibria, or $\arg \max_p \pi_i(p, p)$ is not a singleton, then pick
and fix any Nash price and monopoly price.
We are interested in the case $p^M \neq p^{NE}$. In this case, if agents play the original game $G$ naively, agents cannot enjoy the Pareto efficient result because the Nash price is smaller than the monopoly price.

However, if agents can make commitment before taking actions, in other words, if agents play $C(G)$, $p^M$ can be supported by some commitment equilibria. Lazarev (2012) introduced the idea of the temptation and the punishment.

In the original game, $(p^M, p^M)$ cannot become Nash equilibrium because of the existence of prices $p^T$ s.t. $\pi_i(p^T, p^M) > \pi_i(p^M, p^M)$ (since $(p^M, p^M)$ is Pareto efficient, this $i$'s deviation hurts the other player). Conversely, if players exclude these temptations from the action set at the commitment stage, players can achieve the monopoly price.

We need one more trick to support the monopoly price, since players are also able to deviate in the commitment stage. To prevent the deviation at the commitment stage, the equilibrium action set must include punishment price $p^P$, which satisfies $\pi_i(p^P, p^M) = \pi_i(p^M, p^M)$. If $\pi_i(p^P, p^M) > \pi_i(p^M, p^M)$, this player has incentive to deviate to $p^P$ at the action stage, so $p^M$'s are not supported by any commitment equilibria. In this case, $p^P$ itself become a temptation, so this is not an appropriate punishment. If $\pi_i(p^P, p^M) < \pi_i(p^M, p^M)$, even if player $i$ deviates at the commitment stage and adds $p^M - \epsilon$ to $i$’s action set, taking $p^M$ gives strictly greater payoff than $p^P$ to $j$.

Therefore, there exists exploiting at the commitment stage and the other players cannot credibly punish the deviator, $p^M$ cannot be supported by any commitment equilibria.

We found all the commitment equilibria which support monopoly price, by next theorem.

**Theorem 1** Consider a symmetric two-player non-stochastic commitment game. Suppose that $A_i = [p, \bar{p}] \subset \mathbb{R}$ where $\bar{p} > p^M$, $\pi_i$ is supermodular in $p$ and increasing in $p_i$. Then for any commitment equilibria $(A^*, \sigma^*)$, $(p^M, p^M)$ is supported if and only if,

(i). $p^M \in A^*_i$ for all $i$.

(ii). $p^P \in A^*_i$ for all $i$, where $p^P \neq p^M$ is the punishment price which satisfies $\pi_i(p^P, p^M) = \pi_i(p^M, p^M)$.

(iii). $(p^P, p^M) \cap A^*_i = \phi$ for all $i$. 

(iv) \( \sigma(A^*) = (p^M, p^M) \).

(v) For all \( A \neq A^* \), \( \sigma^*(A) \) is an arbitrary Nash equilibrium of subgame \( G_C = (\{1, 2\}, A, \pi) \).

Before showing the proof, we explain the implication of this theorem. This theorem shows that, to achieve the monopoly price, the important part is to include the monopoly price \( p_M \), the appropriate punishment price \( p^P \) and exclude all the temptation \((p^P, p^M)\). Actually the other parts, \([p, p^P]\) and \((p^M, \bar{p})\) are irrelevant. If we are discussing non-stochastic case, it’s slight thing, however, this theorem has very important role when we are considering the stochastic case.

**Proof.**

**(Sufficiency)**

First, we see that \((p^M, p^M)\) is a Nash equilibrium of the subgame \((\{1, 2\}, A^*, \pi)\). In fact,

\[
\pi_i(p^L, p^M) \leq \pi_i(p^P, p^M) = \pi_i(p^M, p^M) \geq \pi_i(p^U, p^M)
\]

for all \( p^L \leq p^P \) and \( p^U \geq p^M \), since \( \pi_i(p, p_M) \) is increasing in \( p \) if \( p \leq BR(p^M) \) and decreasing in \( p \) if \( p \geq BR(p^M) \), and \( BR(p^M) \in (p^P, p^M) \). Therefore, \((p^M, p^M)\) is actually a Nash equilibrium.

Now we consider the incentive condition at the commitment stage. It’s enough to show that there exists no \( p \in [p, \bar{p}] \) which satisfies \( \pi_i(p, BR^*_j(p)) > \pi_i(p^M, p^M) \), where \( BR^*_j(p) \in \arg \max_{p' \in A_j^*} \pi_i(p', p) \).

If \( p < p^M \), player \( j \) takes \( p^L \in A_j^* \) which satisfies \( p^L \leq p^P \) since for any \( p^U \geq p^M \),

\[
0 \leq \pi_i(p^P, p^M) - \pi_i(p^U, p^M) \leq \pi_i(p^P, p) - \pi_i(p^U, p).
\]

First inequality holds because \( p^P \) is a best response to \( p^M \) and second inequality holds because of supermodularity.

Then, we have

\[
\pi_i(p, p^L) \leq \pi_i(p, p^{NE}) \leq \pi_i(p^{NE}, p^{NE}) \leq \pi_i(p^M, p^M),
\]

because \( p^L \) is increasing in the opponent’s action, \( p^{NE} \) is a best response to \( p^{NE} \) and \( p^M \in \arg \max_p \pi_i(p, p) \). Thus, this deviations are not profitable.
If $p > p^M$, player $j$ takes $p^U \in A^*_j$ which satisfies $p^U \geq p^M$ since for any $p^L \leq p^M$,

$$0 \leq \pi_i(p^M, p^M) - \pi_i(p^L, p^M) \leq \pi_i(p^M, p) - \pi_i(p^L, p).$$

The logic is the same as the case of $p < p^M$.

Suppose that there exists $p$ s.t. $\pi_i(p, p^U) > \pi_i(p^M, p^M)$ and $p^U$ is a best response to $p$ among $A^*$. Then,

$$\pi_j(p^U, p) \geq \pi_j(p^M, p) \geq \pi_j(p^M, p^M)$$

holds by optimality of $p^U$ and the increasing nature of $\pi_j$ in $p_i$. Now, we have $\pi_i(p, p^U) > \pi_i(p^M, p^M)$ and $\pi_j(p^U, p) \geq \pi_j(p^M, p^M)$, but this contradicts Pareto optimality of $(p^M, p^M)$.

Therefore, each player doesn’t have the incentive to deviate in the commitment stage. Thus, if the strategy profile satisfies conditions above, it’s a commitment equilibrium which supports $(p^M, p^M)$. ■

(Necessity)

We haven’t completed yet, but the idea is shown in the previous paragraph.

Therefore, we can always achieve the monopoly price by adding the commitment stage, if the payoff function $\pi_i$ is continuous, supermodular in $(p_i, p_j)$ and increasing in $p_j$. However, if the action set includes a temptation or does not include the punishment, then the monopoly price cannot be achieved.

3 Stochastic Commitment Equilibria

3.1 Stochastic Commitment Equilibria

From now on, we start considering the stochastic case. In the real world, demand function is usually affected by external factors, for example, booms, fashion, weather, e.t.c. If firms have an abundant action sets, they can respond to change of states flexibly, but as we saw, to achieve monopoly price, it’s necessary to waive all the temptations. Here exists trade-off between flexibility and commitment. If we want to analyze the benefit of commitment, we must consider this problem.
Definition 2 A stochastic commitment game is $\Gamma = (\mathcal{I}, (A_i, B_i)_{i \in \mathcal{I}}, \Theta, (\pi_i)_{i \in \mathcal{I}}, \mu)$, where:

- $\mathcal{I} = \{1, 2, ..., n\}$: the set of players.
- $A_i$: the set of potentially available actions of player $i$.
- $B_i \subset 2^{A_i}$: the set of all possible commitments for player $i$.
- $\Theta$: the set of states\(^2\). We denote a generic element of $\Theta$ by $\theta$.
- $\pi_i: A_1 \times ... \times A_n \times \Theta \rightarrow \mathbb{R}$: the payoff function of player $i$. For each $a \in A$, $\pi_i(a, \cdot)$ is measurable.
- $\mu$: the probability measure from which Nature draws the state.

The game proceeds as follows:

Stage 0: Nature draws the state $\theta \in \Theta$ according to $\mu$.

Stage 1: player $i$ chooses $A_i \in B_i$ without knowing the state chosen by Nature.

Stage 2: the players observe the realization of $\theta$, and play a normal form game

$$G^X(\theta) = (\mathcal{I}, (A_i)_{i \in \mathcal{I}}, (\pi_i^A(\cdot|\theta))_{i \in \mathcal{I}}),$$

where $\pi_i^A(\cdot|\theta): A \rightarrow \mathbb{R}$ is defined as $\pi_i^A(a|\theta) = \pi_i(a, \theta)$ for all $a \in A$. The payoff from the entire game is the payoff from the game played in stage 2.

The original game is just a normal form game since players can choose their actions after they observe the realized states. However, if they want to commit an action set, they should make decision before the realization of the state, since there is time lag between the commitment stage and the action stage.

Here we define the strategy and the equilibrium of the stochastic commitment game.

Definition 3 A strategy of a stochastic commitment game $\Gamma$ is a pair $(A_i, \sigma_i)$ where

(i). $A_i \in B_i$

(ii). $\sigma_i: B \times \Theta \rightarrow A_i$ with $\sigma_i(A, \theta) \in A_i$ for any $A \in B$ and $\sigma(A, \cdot)$ is measurable for each $A \in B$.

\(^2\)We implicitly assume that $\Theta$ has a $\sigma$-algebra $\Sigma$
Definition 4 A strategy profile \((A^*, \sigma^*)\) is a stochastic commitment equilibrium, or SCE of \(\Gamma\) if it is a perfect Bayesian equilibrium\(^3\) of \(\Gamma\). Namely, \((A^*, \sigma^*)\) is a SCE if the following (i) and (ii) hold.

(i). For all \(i \in I\) and for all \(A_i \in B_i\),
\[
E_\theta[\pi_i(\sigma^*(A^*, \theta), \theta)] \geq E_\theta[\pi_i(\sigma^*(A_i, A^*_{-i}, \theta), \theta)].
\]

(ii). For all \(\theta \in \Theta\) and for all \(A \in B\), \(\sigma^*(A, \theta) \in NE(G_A(\theta))\).

Definition 5 Take any \\(((a^*(\theta))_{\theta \in \Theta}, A^*)\) satisfying \(a^*(\theta) \in A^*\) for any \(\theta \in \Theta\) and \(A^* \in B\). Then, \\(((x^*(\theta))_{\theta \in \Theta}, A^*)\) can be supported by a SCE if there exists a SCE \((A^*, \sigma^*)\) such that
\[
\sigma^*(A^*, \theta) = a^*(\theta) \\forall \theta \in \Theta
\]

These definitions are natural expansion of the non-stochastic case. Next theorem shows that the necessary and sufficient condition for supportability of a tuple of action profiles \\(((a^*(\theta))_{\theta \in \Theta}, A^*)\).

Theorem 2 \\(((a^*(\theta))_{\theta \in \Theta}, A^*)\) can be supported by a stochastic commitment equilibrium if the following two conditions hold:

(i). For all \(i\),
\[
\max_{A_i \in B_i} E_\theta[\min_{a(\theta) \in NE(G^{A_i}(\theta))} \pi_i(a(\theta)|\theta)] \leq E_\theta[\pi_i(a^*(\theta)|\theta)].
\]

(ii). For all \(\theta \in \Theta\), \(a^*(\theta) \in NE(G^{A^*}(\theta))\).

Moreover, if the left hand side of (i) is well-defined, the converse holds.

Proof.
Suppose there exists an action profile \(\sigma^* = (\sigma^*_i)_{i \in I}\) such that \((A^*, \sigma^*)\) is a SCE and \(\sigma^*(X^*, \theta) = a^*(\theta)\) for all \(\theta \in \Theta\). Then, obviously, (ii) holds. Then, by definition of the SCE, we have
\[
\max_{A_i \in B_i} E_\theta[\pi_i(a(\theta)|\theta)] \leq E_\theta[\pi_i(a^*(\theta)|\theta)].
\]

\(^3\)We omit the belief system \(\mu\) since it is obvious
Since $\sigma^*(A_i, A_{-i}^*, \theta) \in \text{NE}(G^{X_i \times X_{-i}^*}(\theta))$ for all $\theta \in \Theta$ and $A_i \in B_i$, (i) holds.

Let us now show the opposite and assume that (i) and (ii) hold. Define an action profile $\sigma^* = (\sigma^*_i)_{i \in I}$ so that $\sigma^*(A, \theta) \in \text{NE}(G^A(\theta))$ for all $A \in B$ and $\theta \in \Theta$, $\sigma^*(A^*, \theta) = a^*(\theta)$ for all $\theta \in \Theta$ and

$$\sigma^*(A_i, A_{-i}^*, \theta) \in \arg \min_{x(\theta)} x(\theta) \in \text{NE}(G^{A_i \times A_{-i}^*}(\theta)) \pi_i(a(\theta) | \theta)$$

for all $i \in I$, $A_i \in B_i$ and $\theta \in \Theta$. Then $(A^*, \sigma^*)$ is obviously a SCE that supports $\{(a^*(\theta))_{\theta \in \Theta}, A^*\}$. ■

In general, if we want to check some action profiles’ supportability in a stochastic commitment game, we have to take expectations to check the incentive of deviation at the commitment stage. However, if we are very fortunate, we can easily show the supportability by considering ex post equilibria.

**Corollary 1** $\{(a^*(\theta))_{\theta \in \Theta}, A^*\}$ can be supported by a SCE if, for each $\theta \in \Theta$, $\{a^*(\theta), A^*\}$ can be supported by a (non-stochastic) commitment equilibrium.

**Proof.**
Suppose, for each $\theta \in \Theta$, $\{a^*(\theta), A^*\}$ can be supported by a commitment equilibrium. Then it follows that

$$\max_{A_i \in B_i} \min a(\theta) \in \text{NE}(G^{A_i \times A_{-i}^*}(\theta)) \pi_i(a(\theta) | \theta) \leq \pi(a^*(\theta) | \theta)$$

from Theorem 2. Then taking expectations with respect to $\theta$ yields

$$E_\theta[\max_{A_i \in B_i} \min a(\theta) \in \text{NE}(G^{A_i \times A_{-i}^*}(\theta)) \pi_i(a(\theta) | \theta)] \leq E_\theta[\pi(a^*(\theta) | \theta)]$$

$$\Leftrightarrow \max_{A_i \in B_i} E_\theta[\min a(\theta) \in \text{NE}(G^{A_i \times A_{-i}^*}(\theta)) \pi_i(a(\theta) | \theta)] \leq E_\theta[\pi(a^*(\theta) | \theta)]$$

Therefore, the result follows from Theorem 2. ■

### 3.2 Bertrand Duopoly with Differentiated Products; Stochastic

Now, we are going to expand the Bertrand duopoly model we discussed in the previous section. Suppose that $I = \{1, 2\}$, $A_1 = A_2$, $\pi_1 = \pi_2$, where $\pi$ is continuous, supermodular in $(p_1, p_2)$, increasing in $p_j$, and $\Theta = (\theta_1, \theta_2, ..., \theta_S)$. 10
Note that this game is ex post symmetric, since after the realization of any states, the payoff functions are the same. Let $A_i = [\bar{p}, \bar{p}]$ like before. In this stochastic situation, we have different monopoly prices for different states. We denote $p^M_s$ as

$$p^M_s = \arg \max_p \pi_i(p, p, \theta).$$

We can justify the Pareto efficiency of $p^M_s$ by the same logic as non-stochastic case. Without loss of generality, we assume $p^M_1 \leq p^M_2 \leq \ldots \leq p^M_S$. $\bar{p}$ is sufficiently large in the sense that $\bar{p} > p^M_S$.

In this case, if the states are different enough, some commitment equilibria support $(p^M_s, p^M_s)$ for each $\theta_s$ regardless of the probabilistic or belief structure of the states.

**Theorem 3** Let $p^P_s \neq p^M_s$ is the punishment price in $\theta_s$ s.t. $\pi_i(p^P_s, p^M_s, \theta_s) = \pi_i(p^M_s, p^M_s, \theta_s)$. Suppose that $p^P_1 \leq p^M_1 \leq p^P_2 \leq p^M_2 \leq \ldots \leq p^P_S \leq p^M_S$. Then, $\forall \mu$, there exists a commitment equilibrium which supports the Pareto efficient outcome $(p^M_s, p^M_s)$ for all $\theta_s$.

**Proof.**

Let $A^*_1 = A^*_2 = \{p^P_1, p^M_1, p^P_2, p^M_2, \ldots, p^P_S, p^M_S\}$. Then, by Theorem 1, $(p^M_s, p^M_s)$ can be supported as a non-stochastic commitment equilibrium for each $\theta_s$. By Corollary 1, we can support $(p^M_s, p^M_s)$ for all $\theta_s$ as a stochastic commitment equilibrium.

### 4 Conclusion

Although we obtained some interesting results, still little is known in the field of stochastic commitment games. Especially, when we consider discrete, but very close states or continuous states, the problem become much more complicated because we have no hope to support monopoly price by some ex post equilibria.

This is just a draft and we are working on analyzing more general case. Please wait for our successive results.

### References

