# OPEN PROBLEMS FROM THE MINICONFERENCE ON SHARP EIGENVALUE ESTIMATES FOR PARTIAL DIFFERENTIAL OPERATORS (APRIL 2020)

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. In the Neumann and Steklov cases, assume the boundary is Lipschitz.

$\lambda_j = j^{\text{th}}$ Dirichlet eigenvalue, $j \geq 1$
$\mu_j = j^{\text{th}}$ Neumann eigenvalue, $j \geq 0$, $\mu_0 = 0$
$\sigma_j = j^{\text{th}}$ Steklov eigenvalue, $j \geq 0$, $\sigma_0 = 0$

Write $P_k \subset \mathbb{R}^2$ for an open $k$-gon, and $P_k^* \subset \mathbb{R}^2$ for an open regular $k$-gon.

## 1. Dirichlet Laplacian

**Open Problem 1** (proposed by Richard Laugesen)**.** Is $\lambda_{n+1}(\Omega) \geq \lambda_{n+1}(\Omega^*)$? Here $\Omega^*$ is a ball having the same volume as $\Omega$. In 2-dimensions this says $\lambda_3(\Omega) \geq \lambda_3(\Omega^*)$. See the list of open problem by Ashbaugh [5], and the work of Antunes and Freitas [3] and Antunes and Oudet [7, Chapter 11] for numerical optimization results on low Dirichlet eigenvalues. Nilima Nigam also did numerics on this problem (unpublished).

**Open Problem 2** (Asymptotic isoperimetric inequality, proposed by Richard Laugesen)**.** Let $\Omega_k$ be the minimizing domain for $\lambda_k$, among domains of a given volume. Does $\Omega_k$ converge to a ball as $k \to \infty$? The same question applies for the maximizing domain for $\mu_k$. The problem is due to Antunes and Freitas [4].

**Open Problem 3** (polygonal Faber–Krahn, proposed by Michael Loss)**.** For $k \geq 3$, is $\lambda_1(P_k)A(P_k) \geq \lambda_1(P_k^*)A(P_k^*)$? Pólya and Szegő proved this inequality for triangles and quadrilaterals (the cases $k = 3$ and $k = 4$).

A weak version of the question is: for $k \geq 5$, does an $m = m(k) \geq k$ exist such that $\lambda_1(P_k)A(P_k) \geq \lambda_1(P_m^*)A(P_m^*)$?

**Open Problem 4** (polygonal Ashbaugh–Benguria–PPW)**.** For $k \geq 3$, is $\lambda_2/\lambda_1(P_k) \leq \lambda_2/\lambda_1(P_k^*)$? Siudeja [10] has proved this inequality for acute triangles, but it seems to remain open for obtuse triangles. The general case was raised by Laugesen and Siudeja [7, Conjecture 6.31].

## 2. Neumann Laplacian

**Open Problem 5** (polygonal Szegő–Weinberger, proposed by Mark Ashbaugh)**.** For $k \geq 3$, is $\mu_1(P_k)A(P_k) \leq \mu_1(P_k^*)A(P_k^*)$? Laugesen and Siudeja [9] proved the triangular case ($k = 3$).

**Open Problem 6** (Obstacle on triangle, proposed by Michael Loss)**.** Consider the operator $-\Delta + V(x - a)$ on an equilateral triangle with Neumann boundary conditions. Assume that the potential is radial and supported in a small ball at the origin that fits inside the triangle. The ground state energy $E(a)$ will depend on the position $a$ of the potential. It can be shown that this function has its minimum when the potential touches the boundary, but one does not know where. (This works for any convex domain and $V$ does not have to be radial.) The problem is to show that $E(a)$ is smallest when the potential is pushed into a corner.

## 3. Robin Laplacian

**Open Problem 7** (Spectral gap and ratio, proposed by Richard Laugesen)**.** Conjectures for a fixed domain with varying Robin parameter:

- Monotonicity of the spectral gap with respect to the Robin parameter?
- Monotonicity of the spectral ratio with respect to Robin parameter?
- Concavity of the second eigenvalue with respect to Robin parameter, for convex domains?

Conjectures for a fixed, positive Robin parameter and varying domains:

- Does the segment minimize the spectral gap among convex domains, under diameter normalization? (Gap conjecture)
- Does the ball maximize the ratio of the first two Robin eigenvalues (PPW), among convex domains of given volume?
- Polygonal versions of these problems?

These conjectures are described by Laugesen [8].

## 4. Steklov Problem

**Open Problem 8** (polygonal Weinstock-type conjecture, proposed by Nilima Nigam)**.** Is $\sigma_1 L$ maximal for the regular $k$-gon, among all $k$-gons? Here $L$ is the perimeter (boundary length) of the domain. Work of Dominguez, Nigam, and Shahriari [6] provides numerical evidence for the conjecture. Even the triangular case ($k = 3$) appears to be open.

**Open Problem 9** (Exponential decay of Steklov eigenfunctions, proposed by David Sher and Jean Lagacé)**.** Suppose $\Omega \subset \mathbb{R}^2$ has smooth boundary. For each compact set $K \subset \Omega$, does the $j^{\text{th}}$ Steklov eigenfunction ($L^2$-normalized on $\partial\Omega$) decay exponentially on $K$ as $j \to \infty$? Perhaps one needs $\partial\Omega$ to be analytic?

**Open Problem 10** (proposed by Jean Lagacé)**.** Let $M$ be a surface, or domain in $\mathbb{R}^2$. Does a point $x \in M$ exist such that all eigenfunctions $u$ associated to the first eigenvalue $\sigma_1$ (with $u$ being normalized in $L^2$ on the boundary) satisfy $|u(x)| < 2^{-1/2}$, or some other constant? In dimension $d$, the same question applies with $d^{-1/2}$. Note: if the first eigenvalue is simple (multiplicity 1), then the problem is trivial because one could take the point in the nodal set, where $u(x) = 0$.

## 5. Dirac Operator

**Open Problem 11** (Faber–Krahn for Dirac, proposed by Rafael Benguria)**.** Does a Faber–Krahn inequality hold for the free (zero potential) Dirac operator with infinite mass boundary condition? See [2] for background and recent results. See [1] for the statement of this specific problem formulated in more detail.

## 6. Concluding remarks

For further lists of open problems, see Ashbaugh's survey paper [5], the book edited by Henrot [7], and the open problems from the American Institute of Mathematics workshop "Shape optimization with surface interactions" [1].

## References


[1] American Institute of Mathematics. Workshop on Shape Optimization with Surface Interactions. 2019. `https://aimath.org/pastworkshops/shapesurface.html`.

[2] P. R. S. Antunes, R. D. Benguria, V. Lotoreichik, and T. Ourmières-Bonafos. A variational formulation for Dirac operators in bounded domains. Applications to spectral geometric inequalities. *ArXiv:2003.04061.*

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[5] M. S. Ashbaugh. Open problems on eigenvalues of the Laplacian. In *Analytic and Geometric Inequalities and Applications*, volume 478 of *Math. Appl.*, pages 13–28. Kluwer Acad. Publ., Dordrecht, 1999. (Please note that inequalities (27)-(30) of this paper were withdrawn as conjectures before the paper was published, as there are counterexamples when $n = 2$ and presumably also for all higher dimensions $n$).

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