Eigenvalue estimates and localization of the first Dirichlet eigenfunction

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First Dirichlet eigenfunction

u first Dirichlet eigenfunction, with eigenvalue λ:

$$\begin{cases} (\Delta + \lambda)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Throughout, $\Omega \subset \mathbb{R}^n$ will be a convex domain of inner radius 1.

$u = 0$ on $\partial\Omega$, and we normalize $u > 0$ inside Ω.

Superlevel sets: $\Omega_c := \{x \in \Omega : u(x) > c\}$.

Question

How small a subset of the domain Ω can the eigenfunction localize to?

Aim: Study localization via the quantity $\dfrac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$.

An estimate of Chiti

Theorem (Chiti '82)

There exists a constant c_n^* (independent of Ω) such that $\dfrac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n^*$.

Equality holds precisely if Ω is the ball of radius 1.

As the diameter of Ω increases, is the eigenfunction forced to *spread out* along the diameter?
Does there exist $\alpha > 0$ (independent of the domain Ω) such that

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n \mathrm{diam}(\Omega)^\alpha.$$

Conjecture (van den Berg '00)

There exists a constant c_n (independent of the domain Ω) such that

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n diam(\Omega)^{1/6}.$$

The two dimensional case

Two explicit examples:

1) Rectangles, $\Omega = [0, N] \times [0, 1]$: Solve via separation of variables, and

$$u(x, y) = \sin\left(\frac{\pi x}{N}\right) \sin(\pi y).$$

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \text{ is comparable to } N^{1/2} = \mathrm{diam}(\Omega)^{1/2}.$$

2) Circular sectors, $\Omega = \{(r, \theta) : 0 \leq r \leq \alpha_1(N), 0 \leq \theta \leq \frac{1}{N}\}$: Solve via separation of variables, and

$$u(x, y) = J_N(r) \sin(\pi N \theta).$$

J_N is the Nth Bessel function, $\alpha_1(N) \sim N$ is its first zero.

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \text{ is comparable to } N^{1/6} = \mathrm{diam}(\Omega)^{1/6}.$$

Georgiev-Mukherjee-Steinerberger '18 showed that the sector is the most localized case and proved the conjecture in two dimensions, using work of Grieser and Jerison, '95, '96, '98.

Localization of the eigenfunction

Theorem (B. '19)

There exists $c_n > 0$ (independent of Ω) such that $\dfrac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n diam(\Omega)^{1/6}$.

The $\frac{1}{6}$ power cannot be improved in any dimension (attained by the cone).

When Ω extends in more than one direction a stronger version of the inequality holds:

Let K be a John ellipsoid for Ω, with axes lengths

$$N_1 \geq N_2 \geq \cdots \geq N_n \sim 1.$$

Then,

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n \prod_{j=1}^{n-1} N_j^{1/6}.$$

Localization of the eigenfunction

The crucial estimate in the proof:

Proposition

There exists a constant C_n (independent of Ω) such that

$$\int_\Omega |\partial_{x_1} u|^2 \leq C_n diam(\Omega)^{-2/3} \int_\Omega u^2.$$

The theorem follows from this proposition combined crucially with the convexity of the superlevel sets (Brascamp and Lieb '76).

Idea of the proof of the proposition:

1) Reduce to eigenvalue bounds by writing $x = (x_1, x')$ and combining:

$$\int_\Omega |\partial_{x_1} u|^2 + \int_\Omega |\nabla_{x'} u|^2 = \lambda \int_\Omega u^2, \qquad \int_\Omega |\nabla_{x'} u|^2 \geq \mu^* \int_\Omega u^2.$$

Here μ^* is the minimal first eigenvalue of a $(n-1)$-dimensional cross-section of Ω, denoted by Ω^*.

Localization of the eigenfunction

Proposition

There exists a constant C_n (independent of Ω) such that

$$\int_\Omega |\partial_{x_1} u|^2 \leq C_n diam(\Omega)^{-2/3} \int_\Omega u^2.$$

1) Combining implies $\int_\Omega |\partial_{x_1} u|^2 \leq (\lambda - \mu^*) \int_\Omega u^2$, and it remains to estimate $\lambda - \mu^*$ from above.

2) Do this by showing $\mu^* \leq \lambda \leq \mu^* + C_n \mathrm{diam}(\Omega)^{-2/3}$ using the variational formulation of the first eigenvalue.

   Let $\psi^*(x')$ be the first Dirichlet eigenfunction of Ω^*, eigenvalue μ^*. Use the test function

$$w(x_1, x') = \chi(x_1)\psi(x' N_1/(N_1 - x_1)),$$

   with $\chi(x_1)$ supported in an interval of length $N_1^{1/3} \sim \mathrm{diam}(\Omega)^{1/3}$ around Ω^*.

Future Directions

Theorem (B. '19)

There exists $c_n > 0$ (independent of Ω) such that $\dfrac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n \mathrm{diam}(\Omega)^{1/6}$.

There is also the trivial upper bound $\dfrac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \leq \mathrm{vol}(\Omega)^{1/2}$.

Question

Can we use the geometry of the domain Ω to determine comparable upper and lower bounds on $\dfrac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$*?*

In 2 dimensions, Jerison ('95), introduced a length scale $L = L(\Omega)$ to give a positive answer to this question.

Open in higher dimensions.

Thank you for your attention!