Eigenvalue estimates and localization of the first Dirichlet eigenfunction

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First Dirichlet eigenfunction

Let $u$ be the first Dirichlet eigenfunction, with eigenvalue $\lambda$:

$$
\begin{cases}
(\Delta + \lambda)u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Throughout, $\Omega \subset \mathbb{R}^n$ will be a convex domain of inner radius 1.

$u = 0$ on $\partial \Omega$, and we normalize $u > 0$ inside $\Omega$.

Superlevel sets: $\Omega_c := \{x \in \Omega : u(x) > c\}$.

**Question**

How small a subset of the domain $\Omega$ can the eigenfunction localize to?

**Aim:** Study localization via the quantity

$$
\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}}.
$$
An estimate of Chiti

Theorem (Chiti ’82)

There exists a constant $c_n^*$ (independent of $\Omega$) such that

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n^*.$$  

Equality holds precisely if $\Omega$ is the ball of radius 1.

As the diameter of $\Omega$ increases, is the eigenfunction forced to spread out along the diameter?

Does there exist $\alpha > 0$ (independent of the domain $\Omega$) such that

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n \text{diam}(\Omega)^{\alpha}.$$  

Conjecture (van den Berg ’00)

There exists a constant $c_n$ (independent of the domain $\Omega$) such that

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n \text{diam}(\Omega)^{1/6}.$$
The two dimensional case

Two explicit examples:

1) Rectangles, $\Omega = [0, N] \times [0, 1]$: Solve via separation of variables, and

$$u(x, y) = \sin \left( \frac{\pi x}{N} \right) \sin(\pi y).$$

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$$ is comparable to $N^{1/2} = \text{diam}(\Omega)^{1/2}$.

2) Circular sectors, $\Omega = \{(r, \theta) : 0 \leq r \leq \alpha_1(N), 0 \leq \theta \leq \frac{1}{N}\}$: Solve via separation of variables, and

$$u(x, y) = J_N(r) \sin \left( \pi N \theta \right).$$

$J_N$ is the $N$th Bessel function, $\alpha_1(N) \sim N$ is its first zero.

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$$ is comparable to $N^{1/6} = \text{diam}(\Omega)^{1/6}$.

Georgiev-Mukherjee-Steinerberger ’18 showed that the sector is the most localized case and proved the conjecture in two dimensions, using work of Grieser and Jerison, ’95, ’96, ’98.
There exists $c_n > 0$ (independent of $\Omega$) such that 

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n \text{diam}(\Omega)^{1/6}.$$ 

The $\frac{1}{6}$ power cannot be improved in any dimension (attained by the cone).

When $\Omega$ extends in more than one direction a stronger version of the inequality holds:

Let $K$ be a John ellipsoid for $\Omega$, with axes lengths

$$N_1 \geq N_2 \geq \cdots \geq N_n \sim 1.$$ 

Then,

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n \prod_{j=1}^{n-1} N_j^{1/6}.$$
Localization of the eigenfunction

The crucial estimate in the proof:

**Proposition**

There exists a constant $C_n$ (independent of $\Omega$) such that

$$
\int_{\Omega} |\partial_{x_1} u|^2 \leq C_n \text{diam}(\Omega)^{-2/3} \int_{\Omega} u^2.
$$

The theorem follows from this proposition combined crucially with the convexity of the superlevel sets (Brascamp and Lieb ’76).

Idea of the proof of the proposition:

1) Reduce to eigenvalue bounds by writing $x = (x_1, x')$ and combining:

$$
\int_{\Omega} |\partial_{x_1} u|^2 + \int_{\Omega} |\nabla_{x'} u|^2 = \lambda \int_{\Omega} u^2, \quad \int_{\Omega} |\nabla_{x'} u|^2 \geq \mu^* \int_{\Omega} u^2.
$$

Here $\mu^*$ is the minimal first eigenvalue of a $(n - 1)$-dimensional cross-section of $\Omega$, denoted by $\Omega^*$. 
**Proposition**

There exists a constant $C_n$ (independent of $\Omega$) such that

$$\int_{\Omega} |\partial_{x_1} u|^2 \leq C_n \text{diam}(\Omega)^{-2/3} \int_{\Omega} u^2.$$ 

1) Combining implies $\int_{\Omega} |\partial_{x_1} u|^2 \leq (\lambda - \mu^*) \int_{\Omega} u^2$, and it remains to estimate $\lambda - \mu^*$ from above.

2) Do this by showing $\mu^* \leq \lambda \leq \mu^* + C_n \text{diam}(\Omega)^{-2/3}$ using the variational formulation of the first eigenvalue.

Let $\psi^*(x')$ be the first Dirichlet eigenfunction of $\Omega^*$, eigenvalue $\mu^*$. Use the test function

$$w(x_1, x') = \chi(x_1) \psi(x'N_1/(N_1 - x_1)),$$

with $\chi(x_1)$ supported in an interval of length $N_1^{1/3} \sim \text{diam}(\Omega)^{1/3}$ around $\Omega^*$. 
Theorem (B. ‘19)

There exists $c_n > 0$ (independent of $\Omega$) such that

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \geq c_n \text{diam}(\Omega)^{1/6}.$$ 

There is also the trivial upper bound

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}} \leq \text{vol}(\Omega)^{1/2}.$$

Question

Can we use the geometry of the domain $\Omega$ to determine comparable upper and lower bounds on

$$\frac{\|u\|_{L^2(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$$

In 2 dimensions, Jerison (’95), introduced a length scale $L = L(\Omega)$ to give a positive answer to this question.

Open in higher dimensions.
Thank you for your attention!