Lecture 22: Abstractions
Counterexample-guided abstraction refinement

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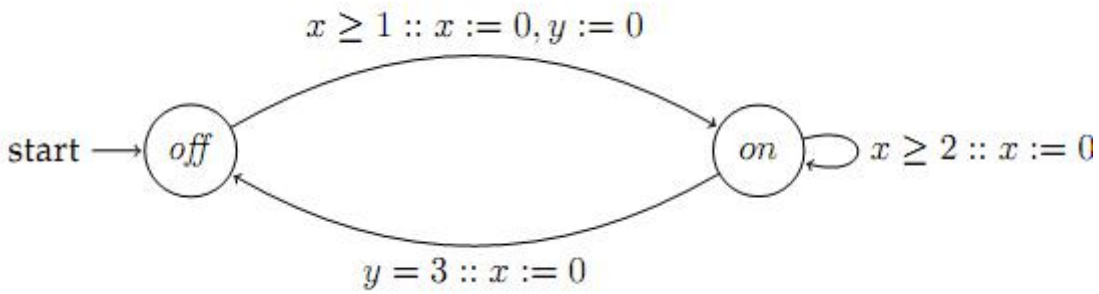
Slides adapted from Prof. Sayan Mitra's slides in Fall 2021

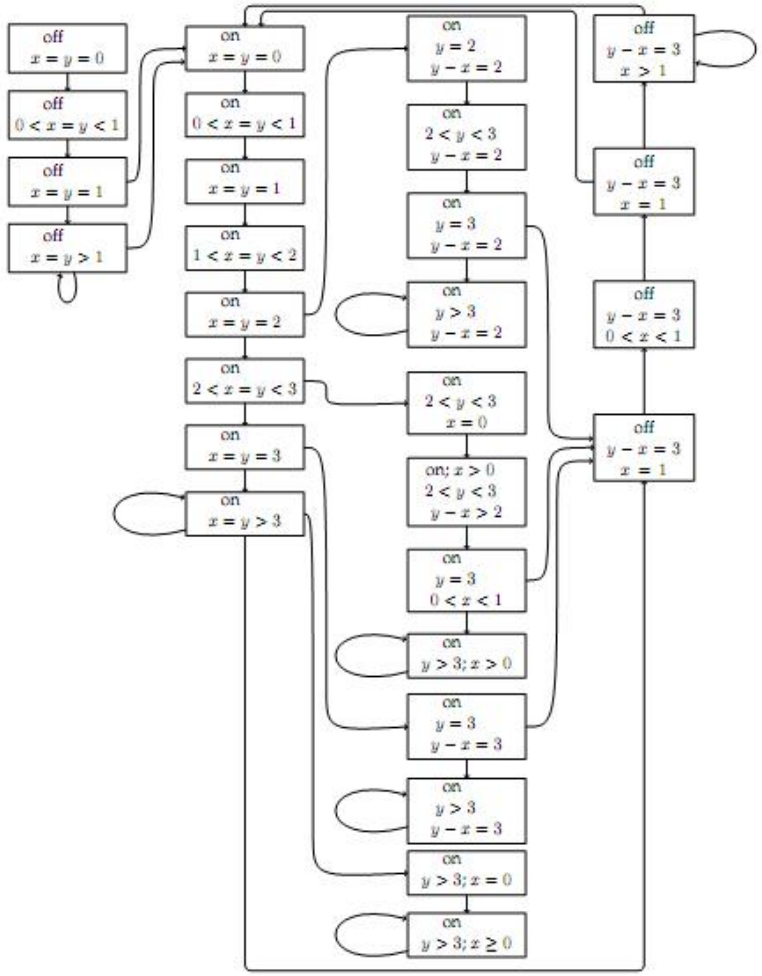
Be prepared for your final project presentation!

- **HW 3 due 4/22**
- Final project presentations: Week of **4/29** and **5/1**
- Same presentation schedule as the mid-term presentation:
  - People who presented mid-term on Tuesday will present on Thursday
  - People who presented mid-term on Thursday will present on Tuesday
- Provide feedback for other students' projects (5% of final grades); feedback form will be available on Canvas

Review: reachability of Integral Time Automaton



Integral Time Automaton



Region Automaton

Review: Special Classes of Hybrid Automata

- Finite Automata
- Integral Timed Automata
- Rational time automata
- Multirate automata
- Initialized Rectangular HA
  - continuous variables re-initialized on each transition (no memory)
- Rectangular HA
  - continuous variables evolve in rectangular regions
- Linear HA
  - dynamics, invariants, and transitions are defined using linear constraints
- Nonlinear HA

For control state reachability poroblem, which ones are decidable?

Practical reachability: propagate **set** of states

Algorithm: BasicReach
Input: $\mathbf{A} = \langle V, \Theta, A, \mathbf{D}, \mathbf{T} \rangle$, $d > 0$
  $Rt, Reach: val(V)$
  $Rt := \Theta$;
  $Reach := \emptyset$;
  While $(Rt \not\subseteq \text{Reach})$
    $Reach := Reach \cup Rt$;
    $Rt := Rt \cup Post_{\mathbf{D}}(Rt)$;
    $Rt := Post_{\mathbf{T}(d)}(Rt)$;
Output: $Reach$

Algorithm: Post_{D}
\\ **computes post of all transitions**
Input: $A, \mathbf{D}, S_{in}$
  $S_{out} = \emptyset$
  For each $a \in A$
    For each $\langle g_1, g_2 \rangle \in S_{in}$
      If $[\![g_1, g_2]\!] \cap [\![g_{ga1}, g_{ga2}]\!] \neq \emptyset$
        $S_{out} := S_{out} \cup \langle g_{ra1}, g_{gra2} \rangle$
Output: S_{out}

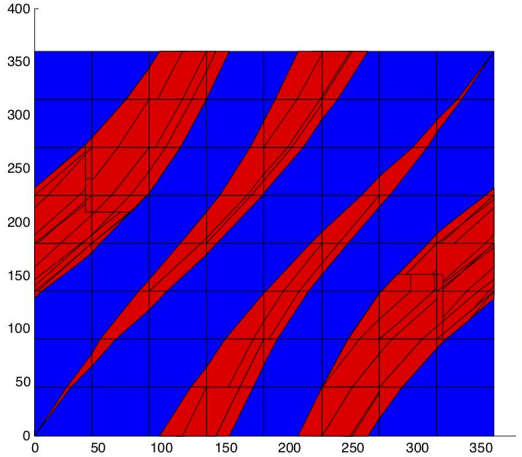
Algorithm: $\text{Post}_{\mathbf{T}(d)}$
\\ **computes post of all trajectories**
Input: $A, \mathbf{T}, S_{in}, d$
$S_{out} = \emptyset$
  For each $\ell \in L$
    For each $\langle g_1, g_2 \rangle \in S_{in}$
      $P := \cup_{t \leq d} [\![g_1, g_2]\!] \oplus [\![t g_{\ell 1}, t g_{\ell 2}]\!]$
      $S_{out} := S_{out} \cup Approx(P)$
Output: S_{out}

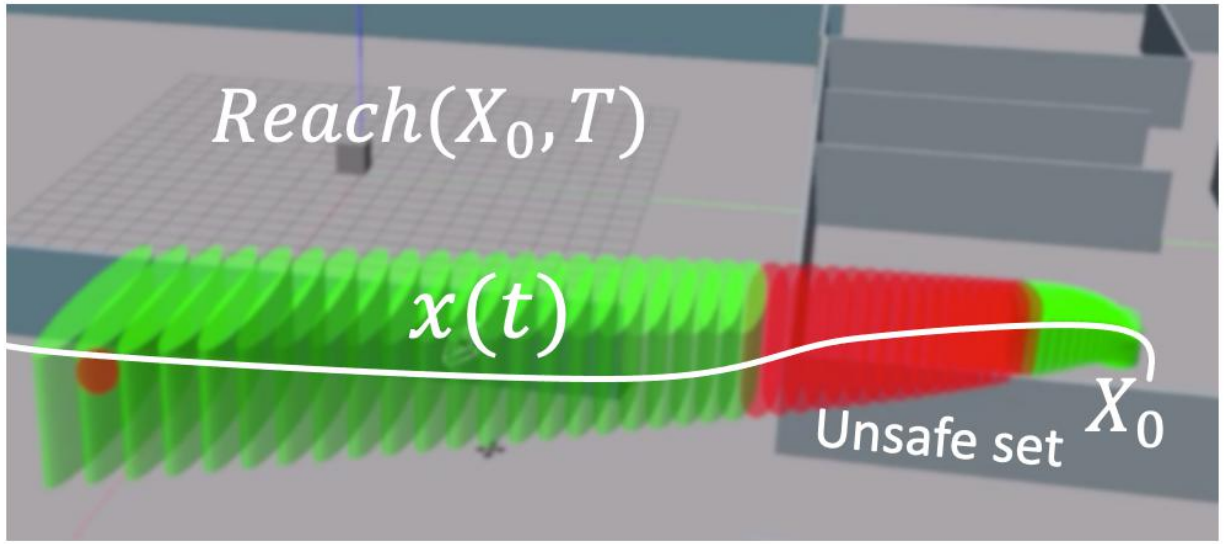
Tools:
SpaceEX
CORA
C2E2
Flow*
DryVR

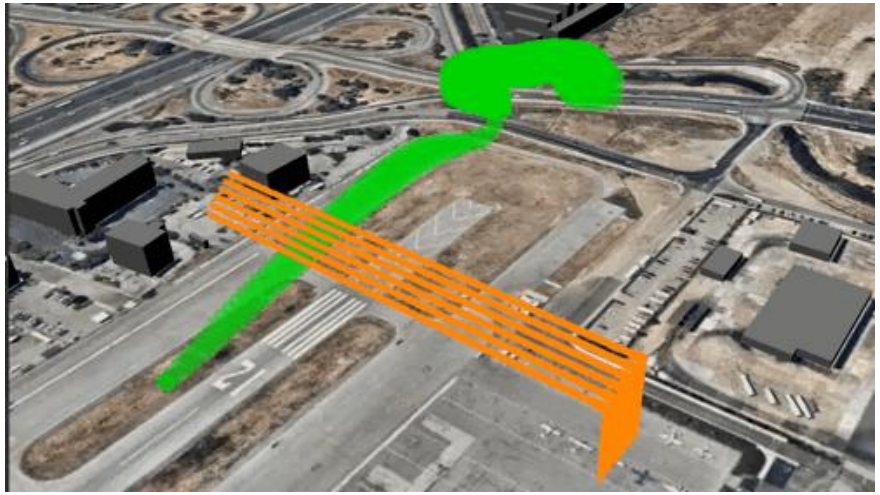
Data structures critical for reachability

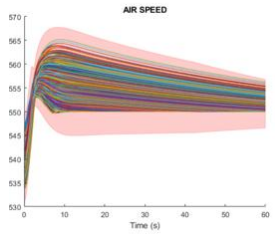
- Hyperrectangles
  - $[[g_1; g_2]] = \{x \in R^n \mid \|x - g_1\|_\infty \leq \|g_2 - g_1\|_\infty\} = \Pi_i [g_{1i}, g_{2i}]$
- Polyhedra
- Zonotopes [Girard 2005]
- Ellipsoids [Kurzhanskiy 2001]
- Support functions [Guernic et al. 2009]
- Generalized star set [Duggirala and Viswanathan 2018]

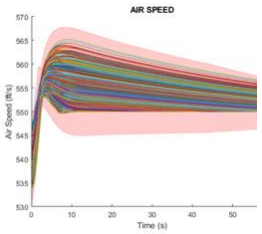
Reachability in practice



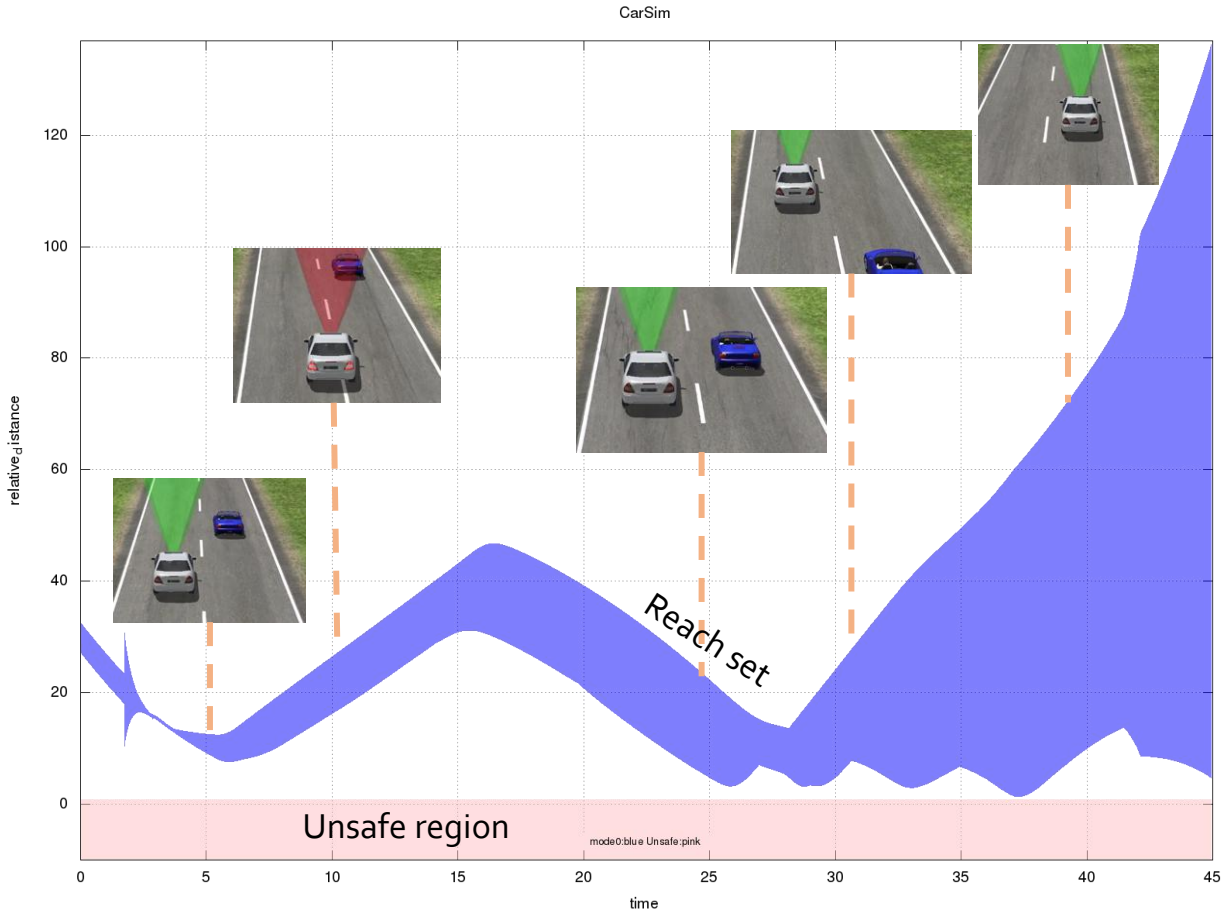


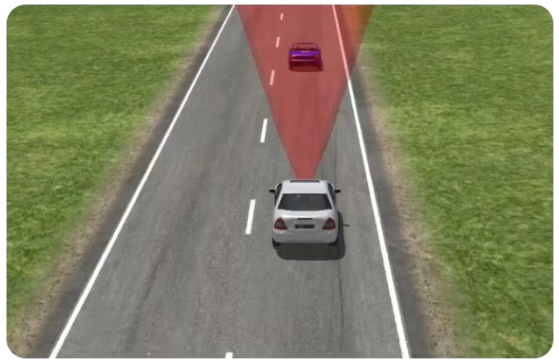






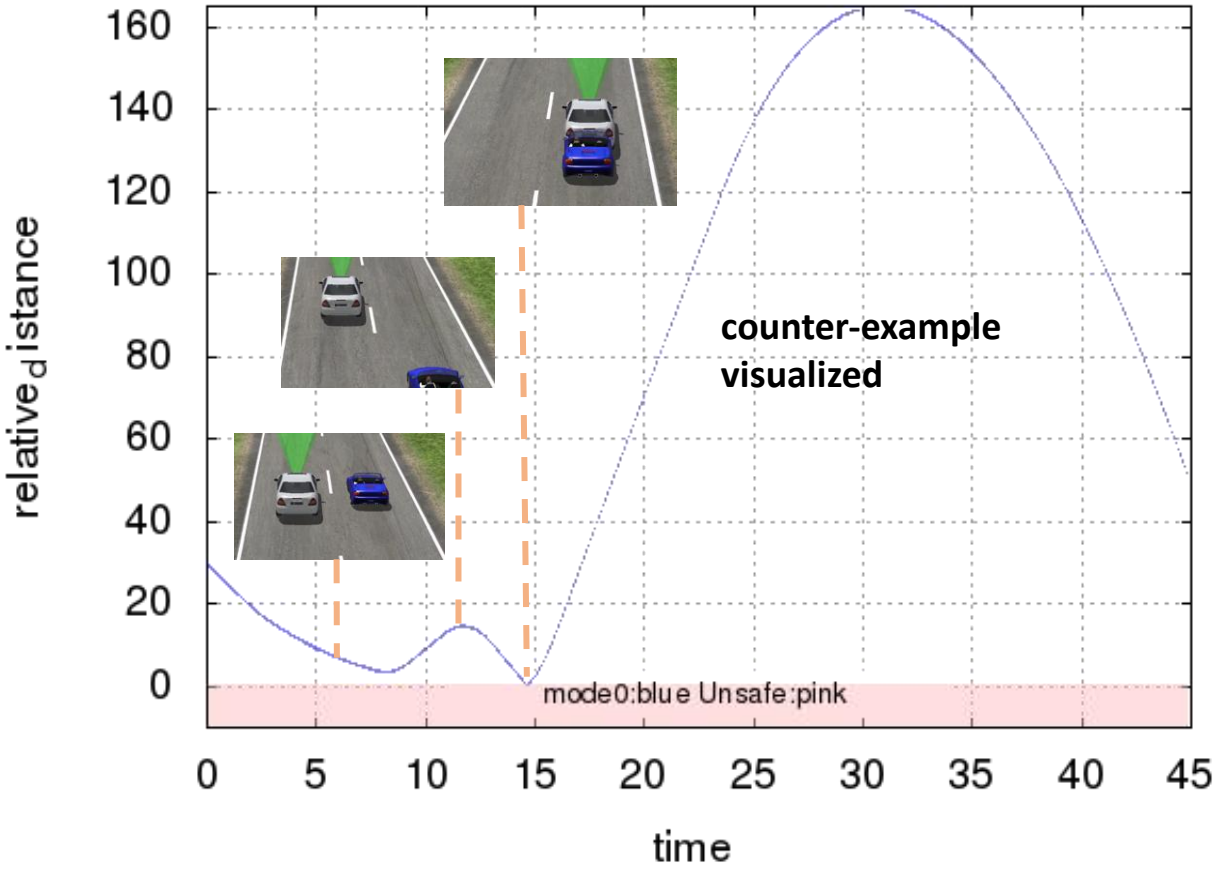
C2E2 generated safety certificate for a given user model

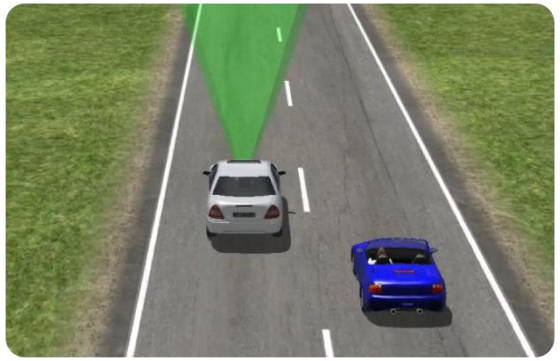




Verify no collision with **uncertainties**: speeds in [70, 85] mph and acceleration range of NPC

For a different user model C2E2 finds a corner case





Verify no collision with **uncertainties** like speeds in [70, 85] mph and **bigger** acceleration range of NPC

Abstractions and Simulations

Consider models that have the same external interface (input/output variables and actions)

We would like to *approximate* one (hybrid) automaton H_1 with another one H_2

- We can **over-approximate** the reachable states of H_1 with those of H_2
- This would ensure that invariants of H_2 *carry over* to H_1

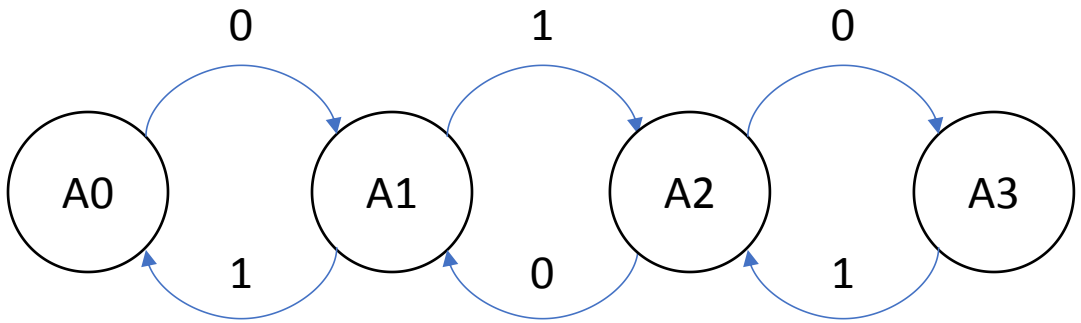
We would like to go beyond invariants, and want to have more general requirements (e.g., CTL) carry over

H_2 should be ***simpler*** (smaller description, fewer states, transitions, linear dynamics, etc.) and preserve **some** properties of H_1 (and not others)

Verifying some requirements of H_2 can then carry over requirements to H_1

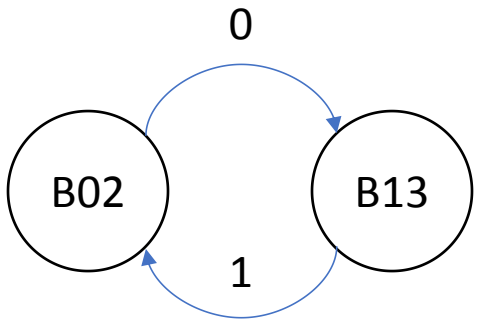
Finite state examples

A



$Traces_A = (01)^*$

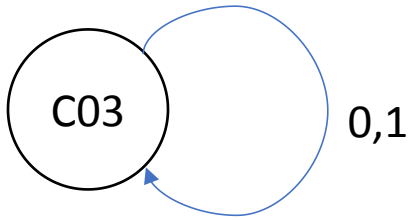
B



$Traces_B = (01)^*$

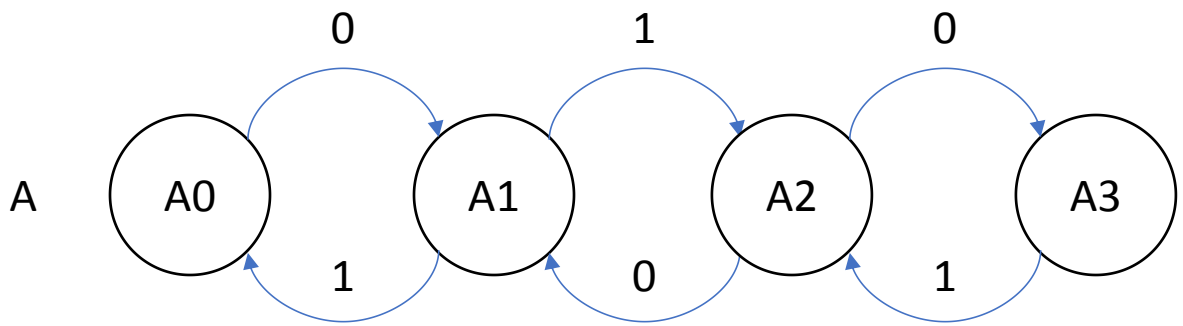
Trace := sequence of actions for some execution
Traces := set of all trace

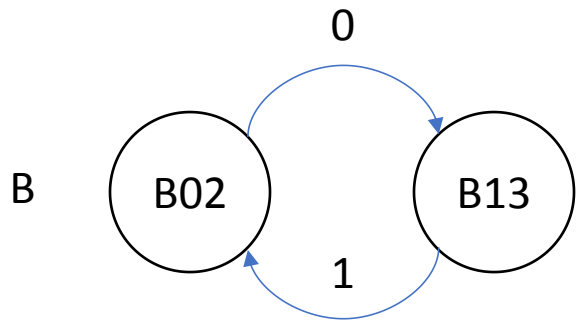
C

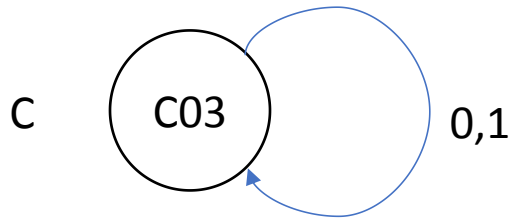


$Traces_C = \{0,1\}^*$

Finite state examples







B **simulates** A and vice versa.
A and B are **bisimilar**.

C simulates both A and B.
C is an **abstraction** of both A and B.
A **implements** C.
B implements C.

Definitions

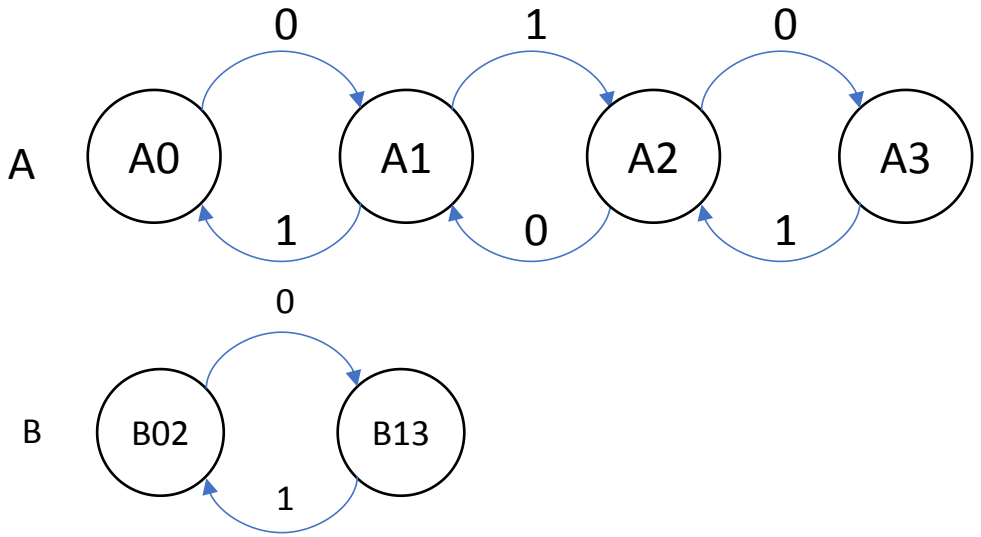
Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be **comparable** (identical I/O varaibles and actions) HA. If R_1 is a forward simulation from $\mathcal{A}$ to $\mathcal{B}$ and R_2 is a forward simulation from $\mathcal{B}$ to $\mathcal{C}$, then $R_1 \circ R_2$ is a forward simulation from $\mathcal{A}$ to $\mathcal{C}$

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are comparable and $Traces_{\mathcal{A}_1} \subseteq Traces_{\mathcal{A}_2}$, we say $\mathcal{A}_1$ implements $\mathcal{A}_2$, and $\mathcal{A}_2$ is an abstraction of $\mathcal{A}_1$

The **implementation relation** is a preorder of the set of all (comparable) hybrid automata

(A preorder is a reflexive and transitive relation)

How to prove B simulates A?



Show there exists a **simulation relation** from states of A to states of B.
Say, $R = ((A0, B02), (A2, B02), (A1, B13), (A3, B13))$

Show that for every transition $Ai \rightarrow_A Ai'$ and $(Ai, Bj) \in R$ there exists Bj' such that

1. $Bj \rightarrow_B Bj'$
2. $(Ai', Bj') \in R$ (also written as $Ai'\ R\ Bj'$)
3. $Trace(Bj \rightarrow_B Bj') = Trace(Ai \rightarrow_A Ai')$

Forward simulation relation

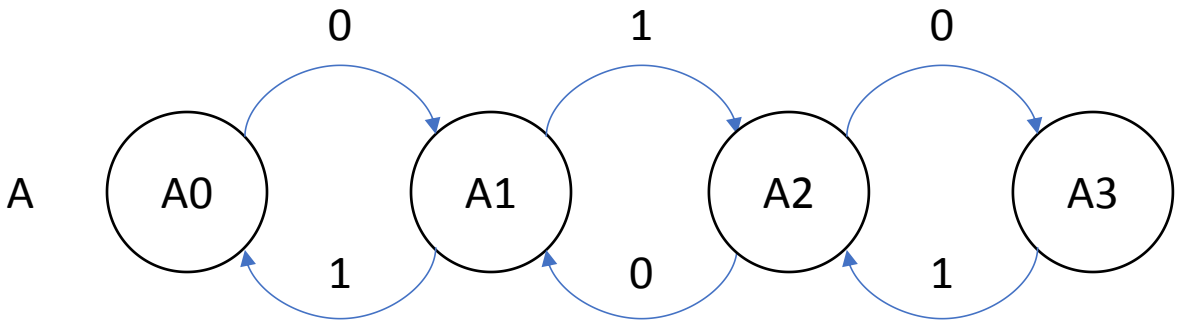
Consider a pair of automata $\mathcal{A}_1 = \langle Q_1, \Theta_1, A_1, D_1 \rangle$ and $\mathcal{A}_2 = \langle Q_2, \Theta_2, A_2, D_2 \rangle$.

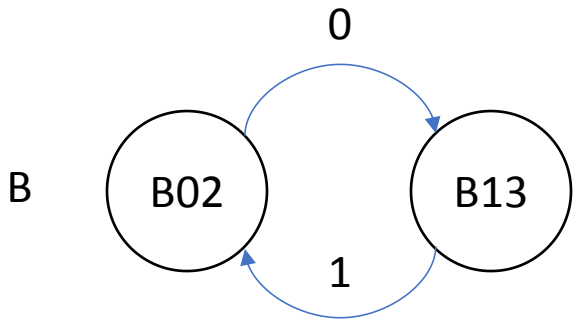
Definition. A **relation** $R \subseteq Q_1 \times Q_2$ is a forward simulation relation from $\mathcal{A}_1$ to $\mathcal{A}_2$ if

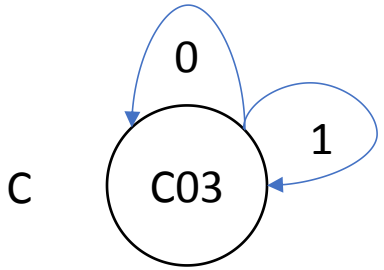
1. For every $q_1 \in \Theta_1$ there exists a $q_2 \in \Theta_2$ such that $q_1 R q_2$
2. For every transition $q_1 \rightarrow_{a_1} q_1'$ and $q_1 R q_2$ there exists q_2', a_2 such that
   - $q_2 \rightarrow_{a_2} q_2'$
   - $q_1' \; R \; q_2'$
   - $Trace(q_1, a_1, q_1') = Trace(q_2, a_2, q_2')$

Theorem. If there exists a forward simulation from $\mathcal{A}_1$ to $\mathcal{A}_2$ then $Traces_{A1} \subseteq Traces_{A2}$.

Finite state examples







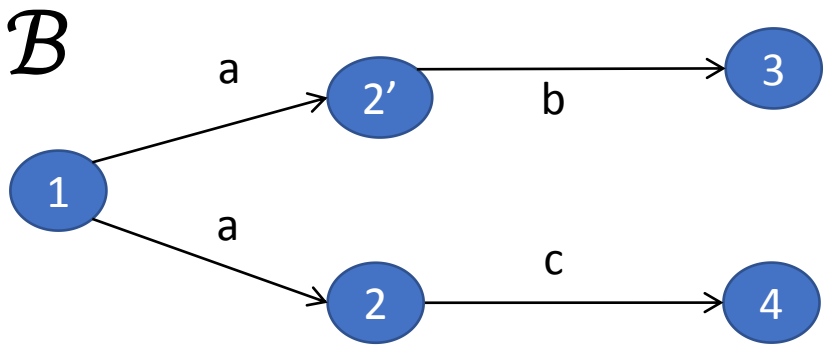
Check that A also simulates B and that C simulates both A and B.

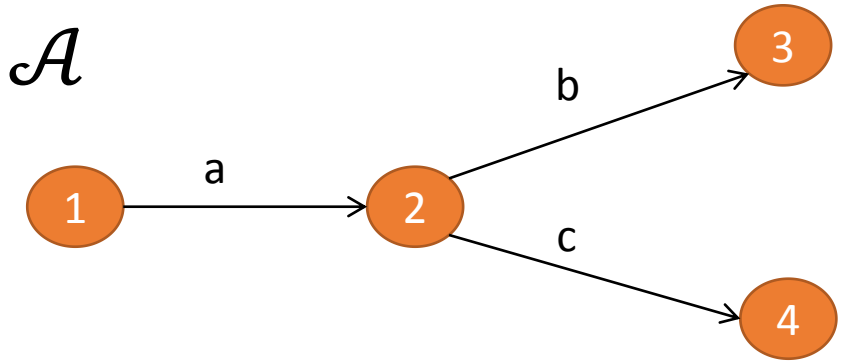
Therefore, $\text{Traces}_A = \text{Traces}_B \subseteq Traces_C$

Does A simulate C?

A Simulation Example

- Is there a forward simulation from $\mathcal{A}$ to $\mathcal{B}$?

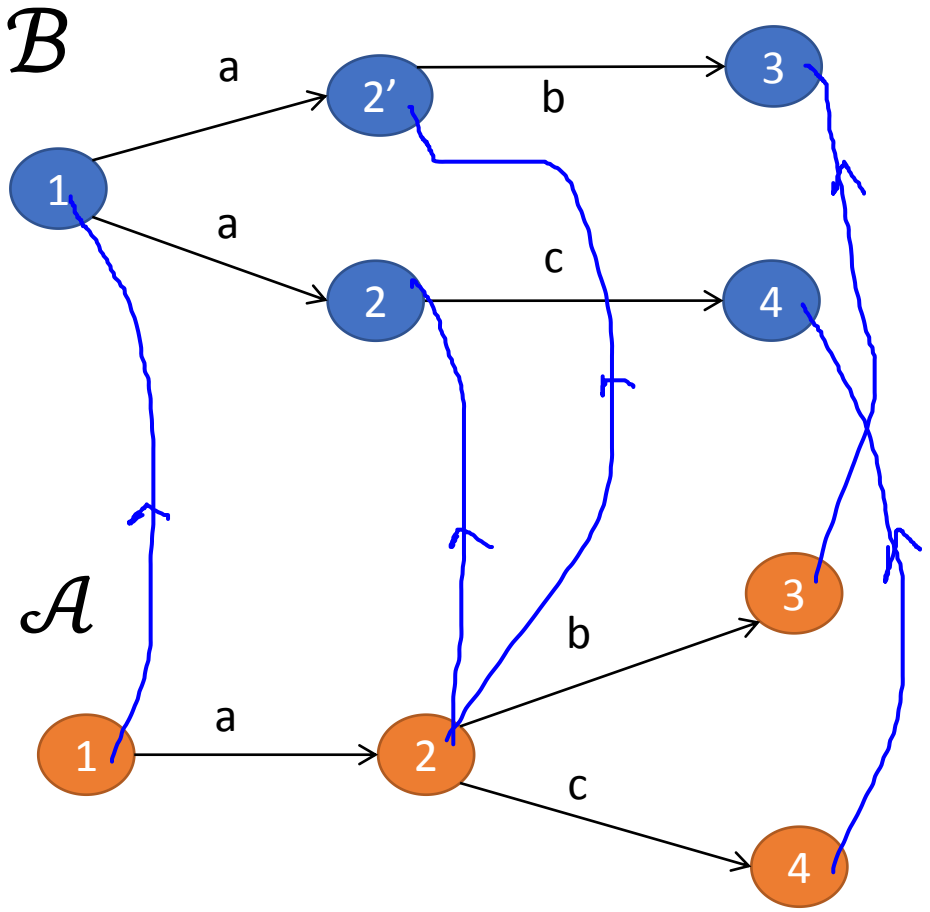




A Simulation Example

- Is there a forward simulation from $\mathcal{A}$ to $\mathcal{B}$?
- Consider the forward simulation relation

For **every** transition $q_1 \rightarrow_{a_1} q_1'$ and $\boldsymbol{q_1 R q_2}$ there exists q_2', a_2 such that $q_2 \rightarrow_{a_2} q_2'$, $q_1'\ R\ q_2'$, $Trace(q_1, a_1, q_1') = Trace(q_2, a_2, q_2')$



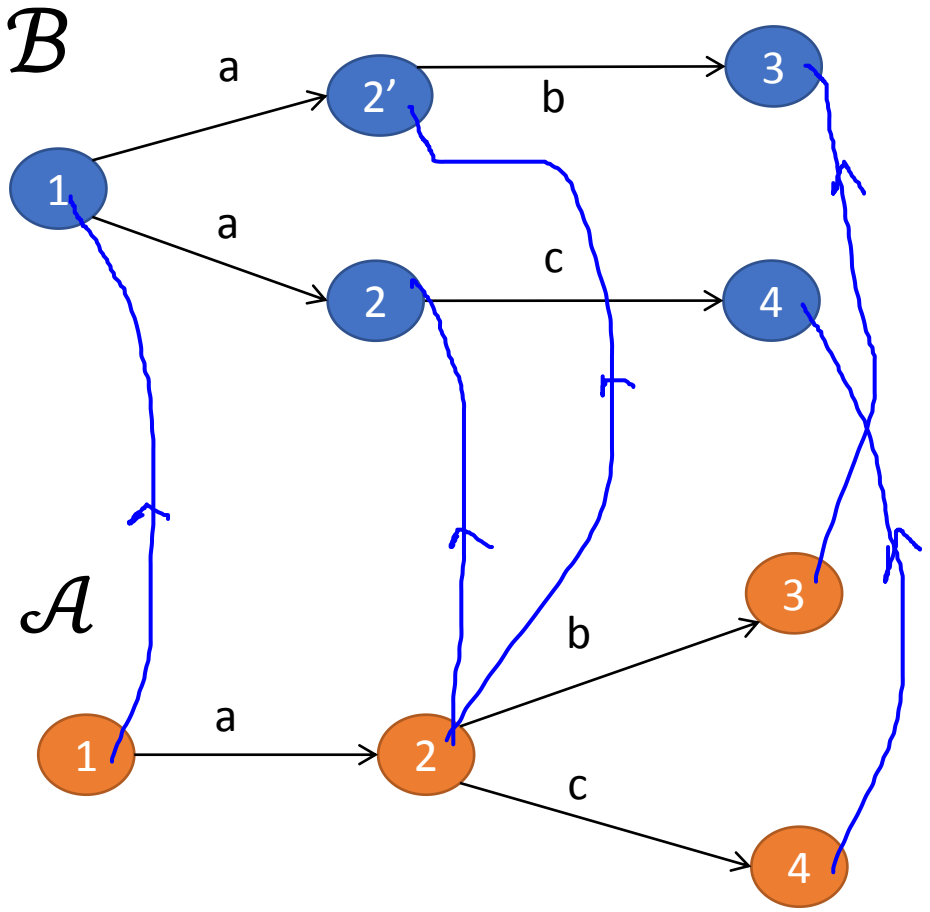
A Simulation Example

- Is there a forward simulation from $\mathcal{A}$ to $\mathcal{B}$?

- Consider the forward simulation relation

  $\mathcal{A}$: $2 \rightarrow_c 4$ cannot be simulated by $\mathcal{B}$. Since (2,2') is related, and from 2' we cannot go to 4.

  Intuitively, starting from 1 in $\mathcal{B}$, we may go to 2', then it cannot go to 4, although $\mathcal{A}$ can



Simulations for hybrid systems

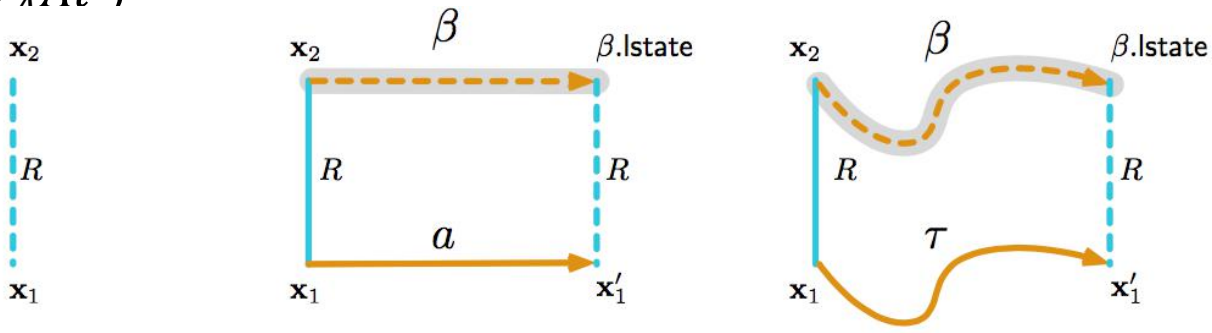
Forward simulation relation from $\mathcal{A}_1$ **to** $\mathcal{A}_2$ is a **relation** R $\subseteq val(X_1) \times val(X_2)$ such that

1. For every $\mathbf{x_1} \in \Theta_1$ there exists $\mathbf{x_2} \in \Theta_2$ such that $\mathbf{x_1}$ R $\mathbf{x_2}$
2. For every $\mathbf{x_1} \rightarrow_{a_1} \mathbf{x_1'} \in \mathcal{D}$ and $\mathbf{x_2}$ such that $\mathbf{x_1}$ R $\mathbf{x_2}$, there exists $\mathbf{x_2'}$ such that
   - $\mathbf{x_2} \rightarrow_{a_1} \mathbf{x_2'}$ and
   - $\mathbf{x_1'}$ R $\mathbf{x_2'}$
3. For every $\boldsymbol{\tau_1} \in \mathcal{T}_1$ and $\mathbf{x_2}$ such that $\tau_1.fstate$ R $\mathbf{x_2}$, there exists $\tau_2 \in \mathcal{T}_2$ that
   - $\mathbf{x_2} = \tau_2.fstate$ and
   - $\mathbf{x_1'}$ R $\tau_2.lstate$
   - $\tau_2.\mathrm{dom} = \tau_1.dom$

Theorem. If there exists a forward simulation relation from hybrid automaton $\mathcal{A}_1$ to $\mathcal{A}_2$ then for every execution of $\mathcal{A}_1$ there exists a corresponding execution of $\mathcal{A}_2$.

Simulation relations for hybrid automata

- Recall condition 3 in definition of simulation relation: $Trace(Bj \rightarrow_B Bj') = Trace(Ai \rightarrow_A Ai')$



- Hybrid automata have transitions and trajectories
- Different types of simulation depending on different notions for "Trace"
  - Match for all variable values, action names, and time duration of trajectories (abstraction)
  - Match variables but not time (time abstract simulation)
  - Match a subset (external) of variables and actions (trace inclusion)
  - Match single action/trajectory of A with a sequence of actions and trajectories of B

Timer simulates Ball (w.r.t. timing of bounce actions)

Automaton Ball(c,v_0,g)
  variables:
    x: Reals := 0
    v: Reals := v_0
  actions: bounce
    transitions:
      bounce
        pre *x = 0 /\ v < 0*
        eff v := -cv
    trajectories:
        evolve d(x) = v; d(v) = -g
        invariant $x \geq 0$

Automaton Timer(c, v_0, g)
  variables: analog
  timer: Reals := $2v_0/g$,
  n:Naturals=0;
  actions: bounce
    transitions:
      bounce
        pre *timer = 0*
        eff n:=n+1; timer := $\frac{2v_0}{gc^n}$
    trajectories:
        evolve d(timer) = -1
        invariant timer ≥ 0

Some nice properties of Forward Simulation

Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be **comparable** (identical I/O varaibles and actions) HA.

If R_1 is a forward simulation from $\mathcal{A}$ to $\mathcal{B}$ and R_2 is a forward simulation from $\mathcal{B}$ to $\mathcal{C}$, then $R_1 \circ R_2$ is a forward simulation from $\mathcal{A}$ to $\mathcal{C}$

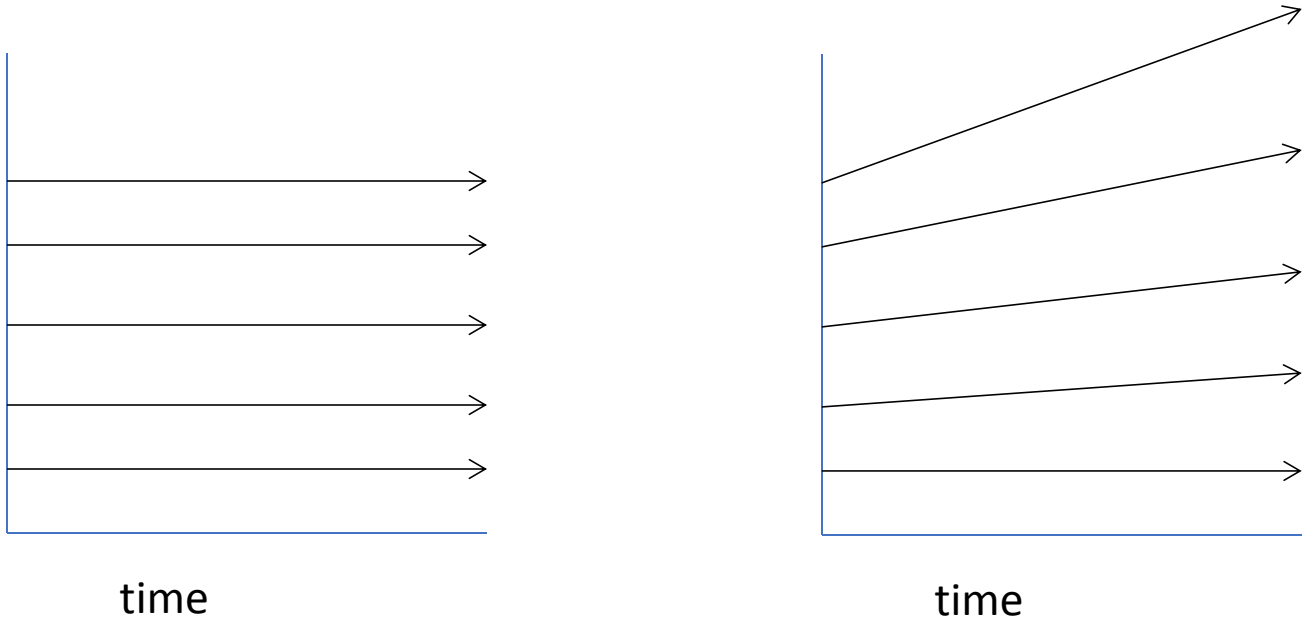
If R is a forward simulation from $\mathcal{A}$ to $\mathcal{B}$ and R^{-1} is a forward simulation from $\mathcal{B}$ to $\mathcal{A}$ then R is called a **bisimulation** and $\mathcal{B}$ are $\mathcal{A}$ **bisimilar**

Bisimilarity is an **equivalence relation**

(reflexive, transitive, and symmetric)

Remark on Simulations and Stability

Stability not preserved by ordinary simulations and bisimulations [Prabhakar, et. al 15]



Stability Preserving Simulations and Bisimulations for Hybrid Systems, Prabhakar, Dullerud, Viswanathan IEEE Trans. Automatic Control 2015

Backward Simulations

Backward simulation relation from $\mathcal{A}_1$ **to** $\mathcal{A}_2$ is a relation R $\subseteq$ $Q_1 \times Q_2$ such that

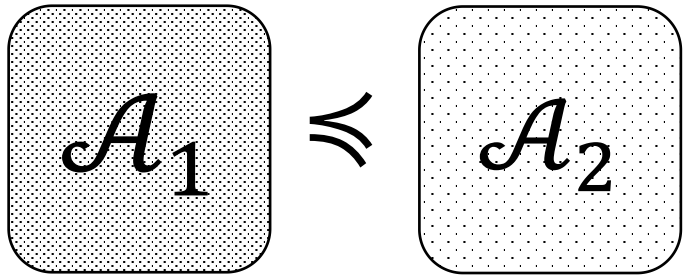
1. If $\mathbf{x_1} \in \Theta_1$ and $\mathbf{x_1}$ R $\mathbf{x_2}$ then $\mathbf{x_2} \in \Theta_2$ such that
2. If $\mathbf{x'_1}$ R $\mathbf{x'_2}$ and $x_1 \text{—}\mathbf{a}\rightarrow x_1'$ then there exists an execution fragment $\boldsymbol{\beta}$
   - $\mathbf{x_2} \text{–}\boldsymbol{\beta}\rightarrow \mathbf{x_2'}$ and
   - $\mathbf{x_1}$ R $\mathbf{x_2}$
   - **Trace(**$\boldsymbol{\beta}$**)** = **a**
3. For every $\boldsymbol{\tau} \in \mathcal{T}$ and $\mathbf{x_2} \in Q_2$ such that $\mathbf{x_1'}$ R $\mathbf{x_2'}$, there exists $\mathbf{x_2}$ such that
   - $\mathbf{x_2} \text{–}\boldsymbol{\beta}\rightarrow \mathbf{x_2'}$ and
   - $\mathbf{x_1}$ R $\mathbf{x_2}$
   - **Trace(**$\boldsymbol{\beta}$**)** = $\boldsymbol{\tau}$

Theorem. If there exists a backward simulation relation from $\mathcal{A}_1$ to $\mathcal{A}_2$ then $\text{ClosedTraces}_1 \subseteq \text{ClosedTraces}_2$

"Closed" means: Finite execution with final trajectory with closed domain $\tau_0\ a_1\ \tau_1 a_2 \tau_2\ \ldots \tau_k$ and $\tau_k.dom = [0, T]$

Abstraction recap

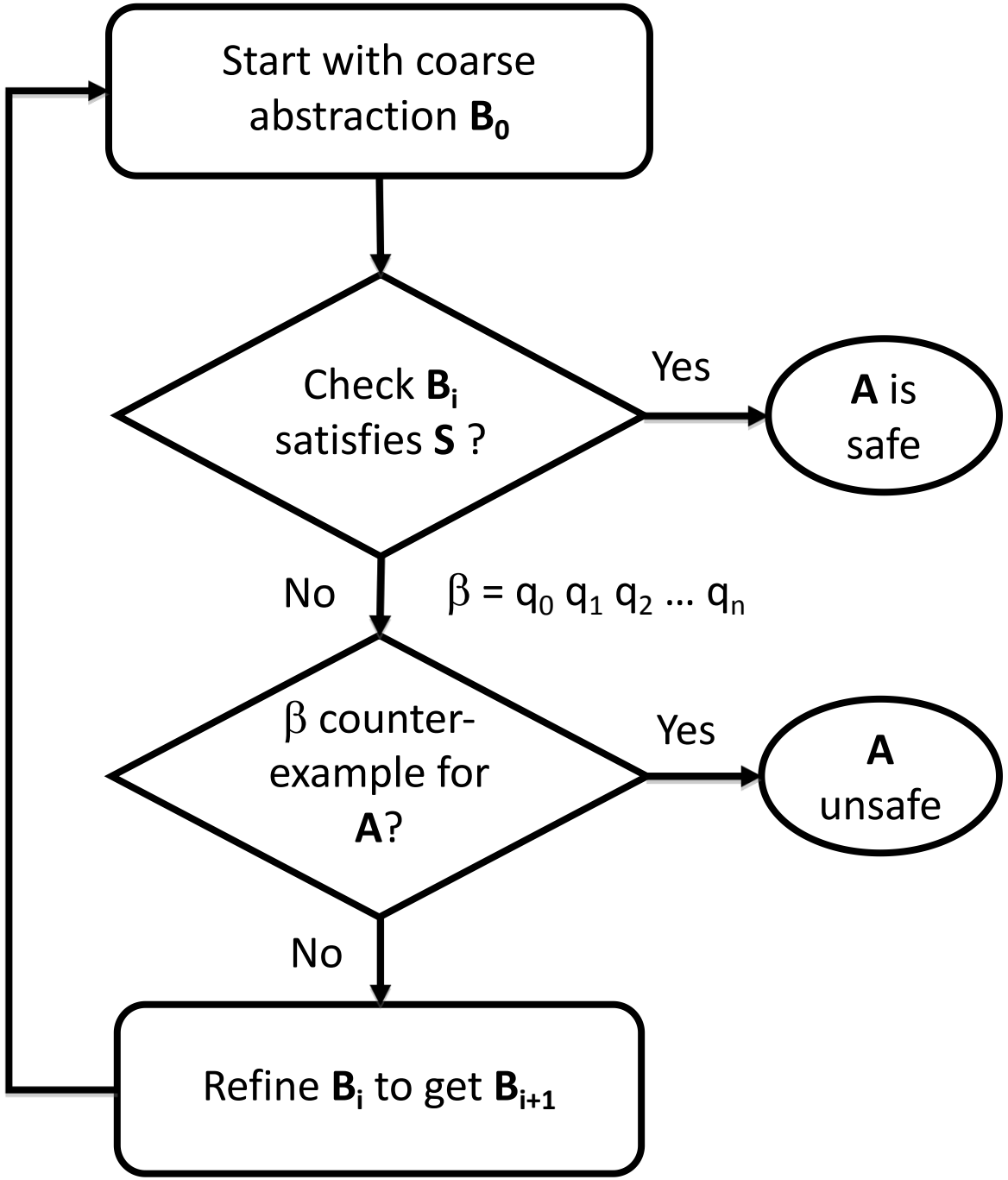
- Defined what it means for $\mathcal{A}_2$ to be abstraction of$\mathcal{A}_1$
- $Traces_{\mathcal{A}_1} \subseteq Traces_{\mathcal{A}_2}$
- $\mathcal{A}_1 \preccurlyeq_T \mathcal{A}_2$
- If $\mathcal{A}_1 \preccurlyeq_T \mathcal{A}_2$ and $\mathcal{A}_2 \preccurlyeq_T \mathcal{A}_1$ then $\mathcal{A}_1 \preccurlyeq_T \mathcal{A}_3$
- Transitive, $\preccurlyeq_T$ defines a preordering on compatible automata
- We saw methods for proving $\mathcal{A}_1 \preccurlyeq_T \mathcal{A}_2$
  - *Forward simulation and backward simulation*
- $\preccurlyeq_T$ defines a preorder



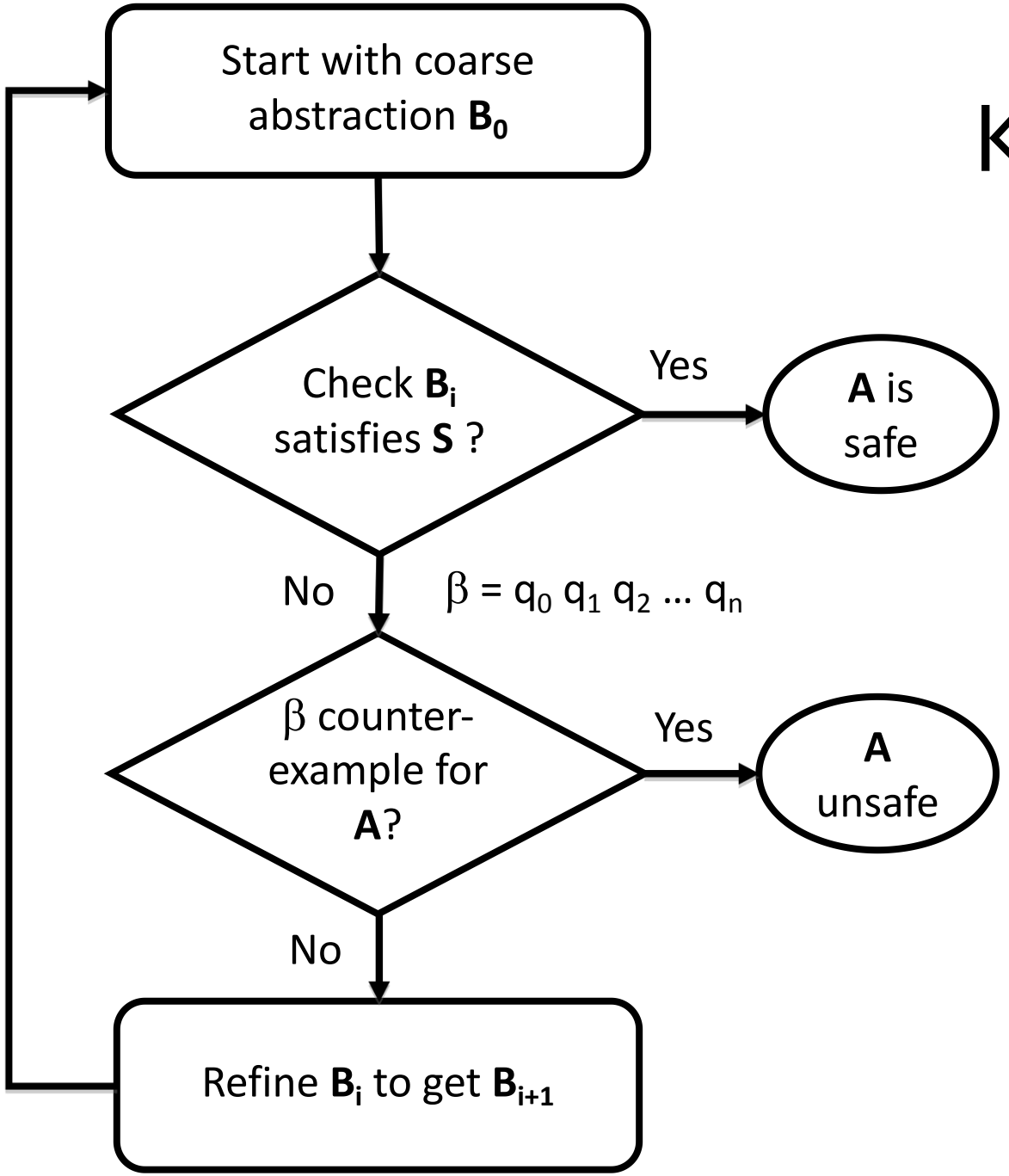
Counter-example guided abstraction-refinement

Counterexample guided abstraction refinement (CEGAR)

- A general algorithmic framework for automatically constructing and verifying property-specific abstractions [Clarke:2000]
- CEGAR has been applied to discrete automata, software, and hybrid systems [Holzman 00,Ball 01, Alur 2006,Clarke 2003, Fehnker2005, Prabhakar 15, Roohi 17]
- We will discuss the basic idea of the CEGAR and the key design choices, and their implications.

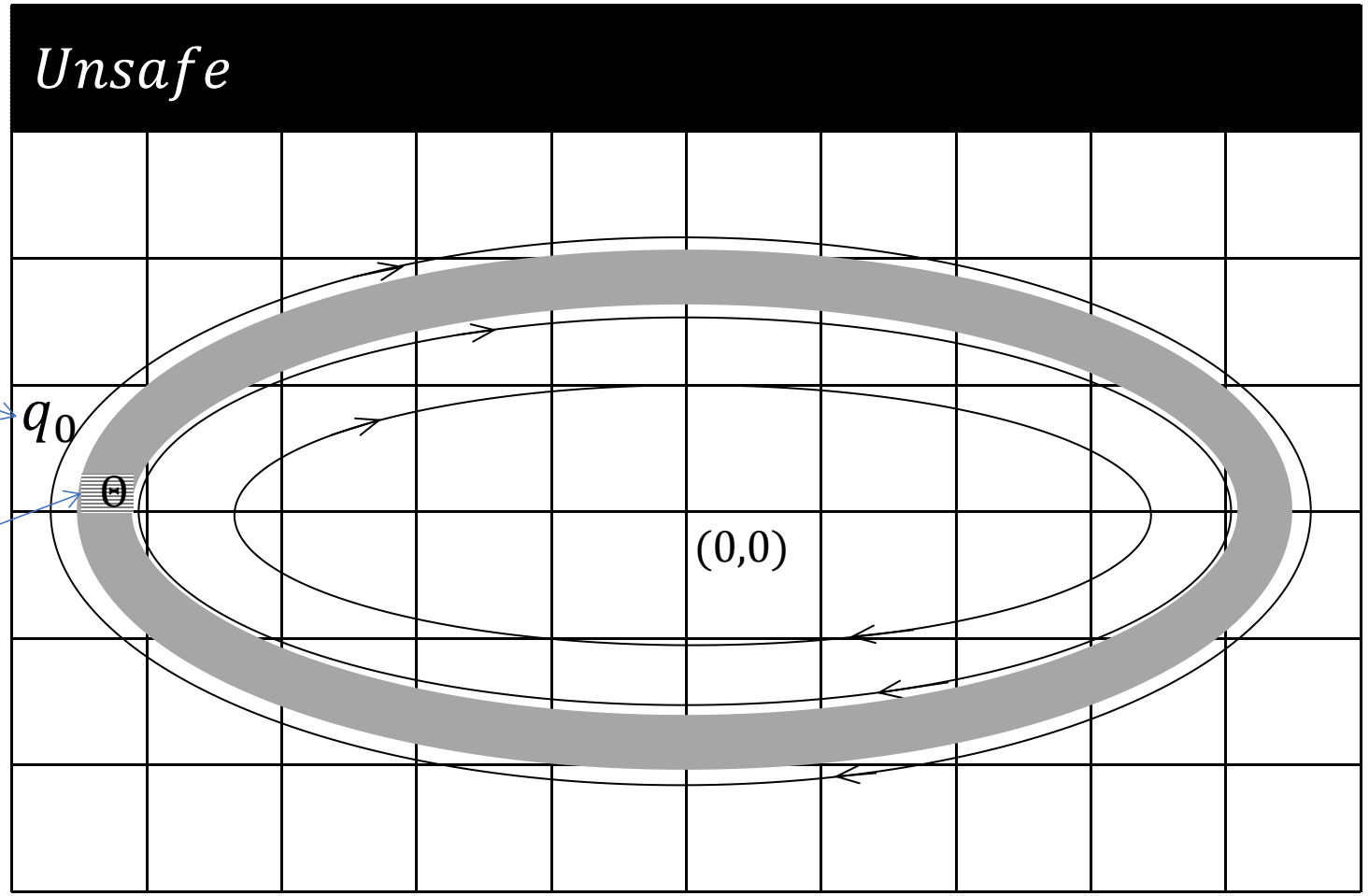


Idea of CEGAR



Key design choices

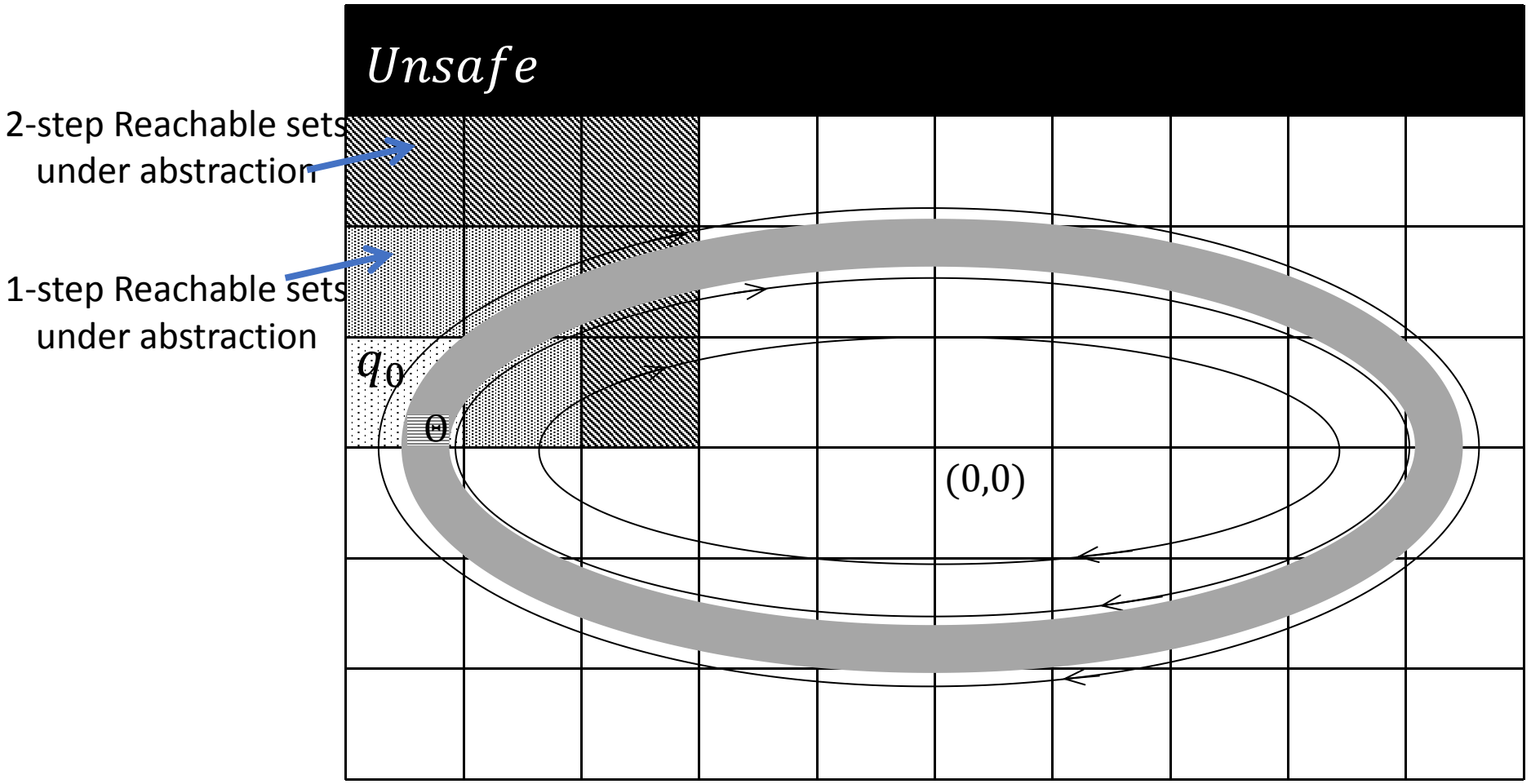
- Space of the abstract automata (finite, timed, linear)
- Model checker for abstract automaton (decidable?)
- Counter-example validation procedure
- Refinement strategy



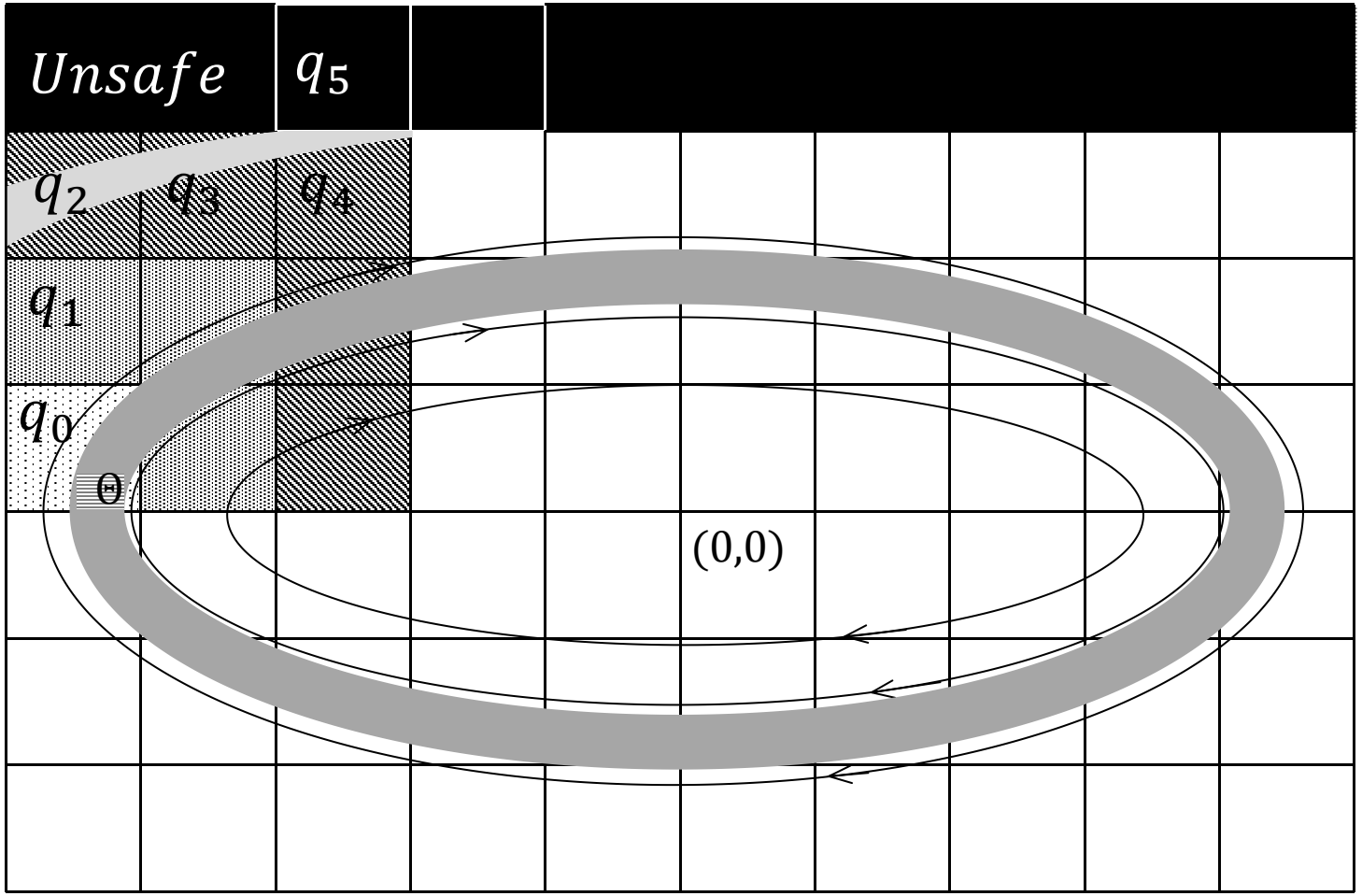
Example: dynamical systems with elliptical orbits

Abstraction: maps a box in state space to a discrete state q_i

Verification goal: will we reach unsafe regions on the top?


Unsafe
2-step Reachable sets under abstraction
1-step Reachable sets under abstraction
q_0
0
(0,0)

Counterexample with abstraction: q0 q1 q2 q3 q4 q5 (unsafe)



Is it a real counterexample?
Check using backward-reachability

$$S_5 = R^{-1}(q_5)$$
$$S_4 = Pre_A(S_5) \cap R^{-1}(q_4) \neq \emptyset$$
$$S_3 = Pre_A(S_4) \cap R^{-1}(q_3) \neq \emptyset$$
$$S_2 = Pre_A(S_3) \cap R^{-1}(q_2) \neq \emptyset$$
$$S_1 = Pre_A(S_2) \cap R^{-1}(q_1) = \emptyset$$

Impossible from q2 to q1!

$R^{-1}(q_i)$ is the box region in **original** dynamical system state space, corresponding to the discrete abstraction state q_i