Lecture 21: Timed automata and its reachability (cont.)

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Slides adapted from Prof. Sayan Mitra's slides in Fall 2021

Be prepared for your final project presentation!

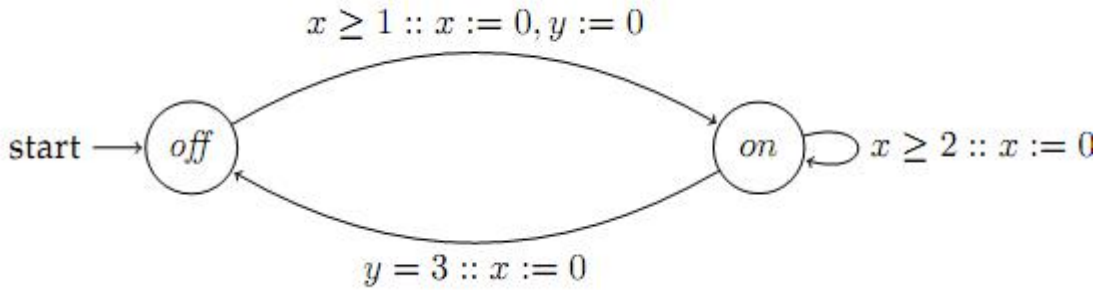
- Final project presentations: Week of **4/29** and **5/1**
- Same presentation schedule as the mid-term presentation:
  - People who presented mid-term on Tuesday will present on Thursday
  - People who presented mid-term on Thursday will present on Tuesday
- No regular class on 5/6 (will become an office hour for final report, or a guest lecture)

Review: Integral Timed Automata (ITA)

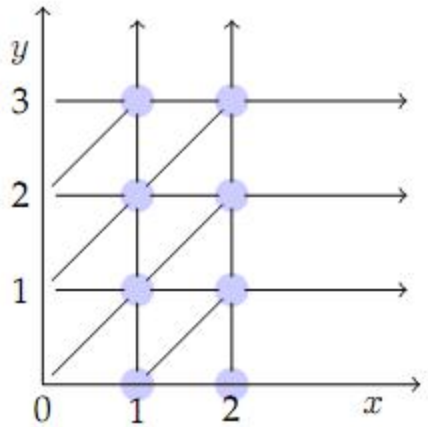
- **Definition.** A **integral timed automaton** $\mathcal{A} = \langle V, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$ where
  - V = X ∪ $\{l\}$, where X is a set of n clocks and l is a discrete state variable of finite type L. The **stata space is** val(V) × L
  - A is a finite set
  - $\mathcal{D}$ is a set of transitions such that
    - The preconditions are described by clock constraings $\Phi(X)$
    - $\langle x, l \rangle_a \rightarrow \langle x', l' \rangle$ implies either $x' = x$ or $x' = 0$ (time is reset to 0, or no change)
  - $\mathcal{T}$ set of clock trajectories for the clock variables in X

Review: Control-state Reachability of ITA: construct a bisimilar FA

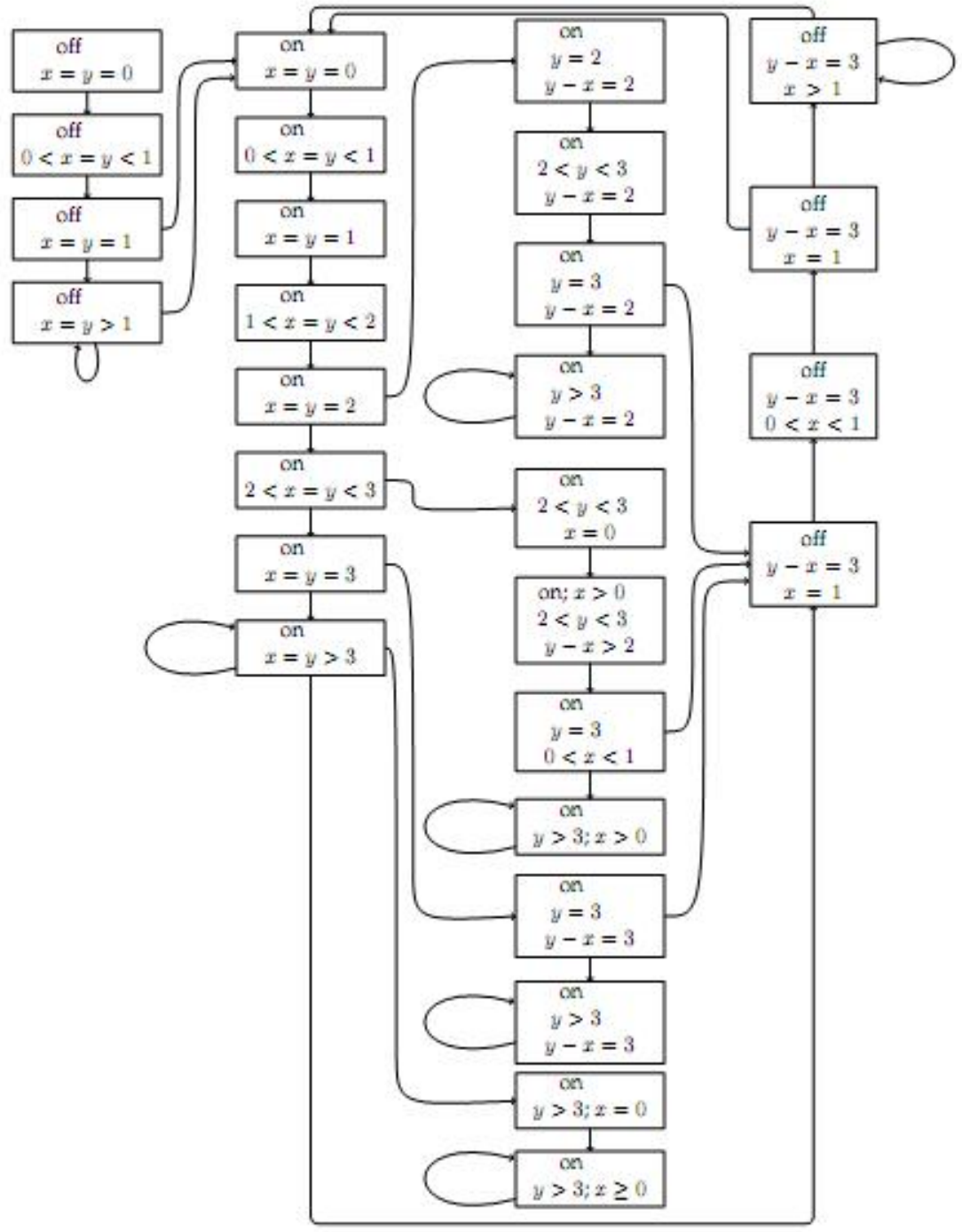
ITA



Clock Regions



Corresponding FA



$$|X|!\, 2^{|X|} \prod_{z \in X} (2c_{\mathcal{A}z} + 2)$$

Drastically increasing with the number of clocks

Special Classes of Hybrid Automata

- Finite Automata
- Integral Timed Automata ←
- Rational time automata
- Multirate automata
- Rectangular Initialized HA
  - continuous variables re-initialized on each transition (no memory)
- Rectangular HA
  - continuous variables evolve in rectangular regions
- Linear HA
  - dynamics, invariants, and transitions are defined using linear constraints
- Nonlinear HA

Step 1. Rational Timed Automata

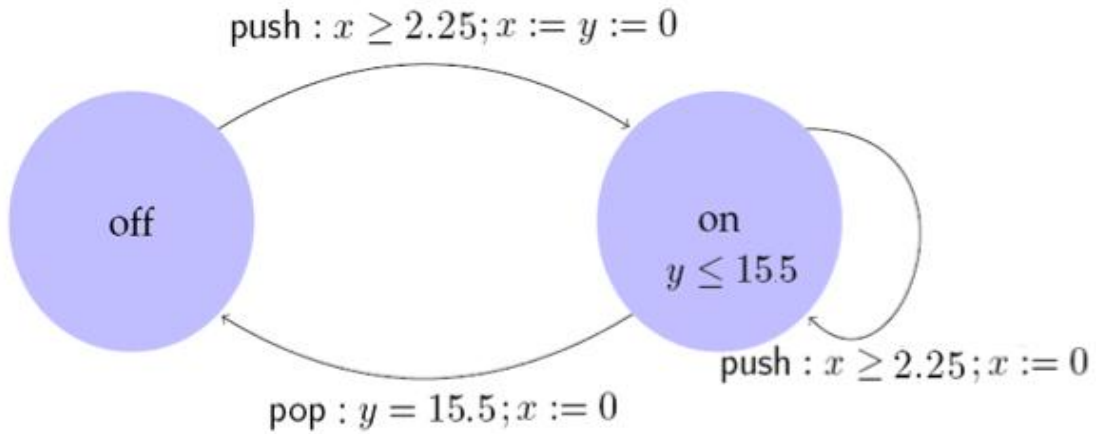
Definition. A ***rational timed automaton*** is a HA $\mathcal{A} = \langle V, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$ where

- V = X ∪ $\{loc\}$, where X is a set of n clocks and l is a discrete state variable of finite type L
- A is a finite set
- $\mathcal{D}$ is a set of transitions such that
  - The guards are described by **rational** clock constraings $\Phi(X)$
  - $\langle x, l \rangle \rightarrow_a \langle x', l' \rangle$ implies either $x' = x$ or $x = 0$
- $\mathcal{T}$ set of clock trajectories for the clock variables in X

Example: **Rational** Light switch

Switch can be turned on whenever at least 2.25 time units have elapsed since the last turn off or on. Switch can be turn off 15.5 time units after the last on.

```
automaton Switch
 internal push; pop
  variables
    internal x, y:Real := 0, loc:{on,off} := off
  transitions
    push
      pre x >=2.25
      eff if loc = on then y := 0 fi; x := 0;
       loc := off
    pop
      pre y = 15.5 ∧ loc = off
      eff x := 0; loc := off
  trajectories
    invariant loc = on ∨ loc = off
    stop when y = 15.5
    evolve d(x) = 1; d(y) = 1
```



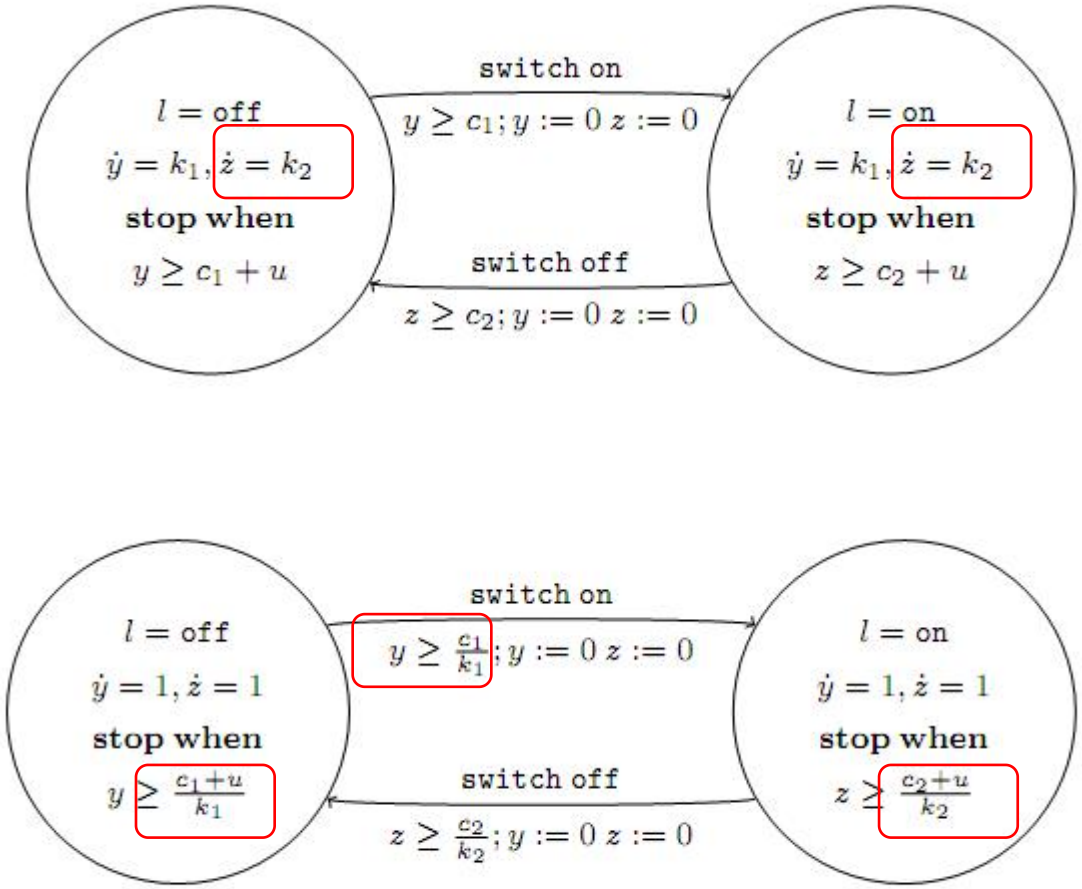
Step 2. Multi-Rate Automaton

- **Definition.** A **multirate automaton** is $\mathcal{A} = \langle V,\ Q,\ \Theta,\ A,\ \mathcal{D},\ \mathcal{T} \rangle$ where
  - V = X ∪ $\{loc\}$, where X is a set of n **continuous variables** and loc is a discrete state variable of finite type L
  - A is a finite set of actions
  - $\mathcal{D}$ is a set of transitions such that
    - The guards are described by **rational** clock constraings $\Phi(X)$
    - $\langle x, l \rangle \rightarrow_a \langle x', l' \rangle$ implies either $x' = c \ or \ x' = x$
  - $\mathcal{T}$ set of trajectories such that

    for each variable $x \in X \ \exists k \ such \ that \ \tau \in \mathcal{T},\ t \in \tau.dom$

    $$\tau(t).x = \tau(0).x + k\ t$$

Example: Multi-rate to rational TA



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Step 3. Rectangular HA

Definition. A **rectangular hybrid automaton (RHA)** is a HA $\mathcal{A} = \langle V, A, \mathcal{T}, \mathcal{D} \rangle$ where

- V = X ∪ $\{loc\}$, where X is a set of n **continuous variables** and loc is a discrete state variable of finite type L
- A is a finite set
- $\mathcal{T} = \cup_\ell \mathcal{T}_\ell$ set of trajectories for X
  - For each $\tau \in \mathcal{T}_\ell$, $x \in X$ either (i) $d(x) = k_\ell$ or (ii) $d(x) \in [k_{\ell 1}, k_{\ell 2}]$
  - Equivalently, (i) $\tau(t)\lceil x = \tau(0)\lceil x + k_\ell t$
    (ii) $\tau(0)\lceil x + k_{\ell 1} t \leq \tau(t)\lceil x \leq \tau(0)\lceil x + k_{\ell 2} t$

d(x) can vary with time, as long as it is bounded

- $\mathcal{D}$ is a set of transitions such that
  - Guards are described by **rational** clock constraings
  - $\langle x, l \rangle \rightarrow_a \langle x', l' \rangle$ implies $x' = x$ *or* $x' \in [c_1, c_2]$

bounded in a rectangle

CSR Decidable for RHA?

- Given an RHA, check if a particular location is reachable from the initial states?
- Is this problem decidable? No
  - **[Henz95]** Thomas Henzinger, Peter Kopke, Anuj Puri, and Pravin Varaiya. What's Decidable About Hybrid Automata?. Journal of Computer and System Sciences, pages 373–382. ACM Press, 1995.
  - CSR for RHA reduction to Halting problem for 2 counter machines
  - Halting problem for 2CM known to be undecidable
  - Reduction in **next lecture**

Step 4. Initialized Rectangular HA

Definition. ***An initialized rectangular hybrid automaton*** **(IRHA)** is a RHA $\mathcal{A}$ where

- V = X ∪ $\{loc\}$, where X is a set of n continuous variables and $\{loc\}$ is a discrete state variable of finite type Ł
- A is a finite set
- $\mathcal{T} = \cup_\ell \mathcal{T}_\ell$ set of trajectories for X
  - For each $\tau \in \mathcal{T}_\ell$, $x \in X$ either (i) $d(x) = k_\ell$ or (ii) $d(x) \in [k_{\ell 1}, k_{\ell 2}]$
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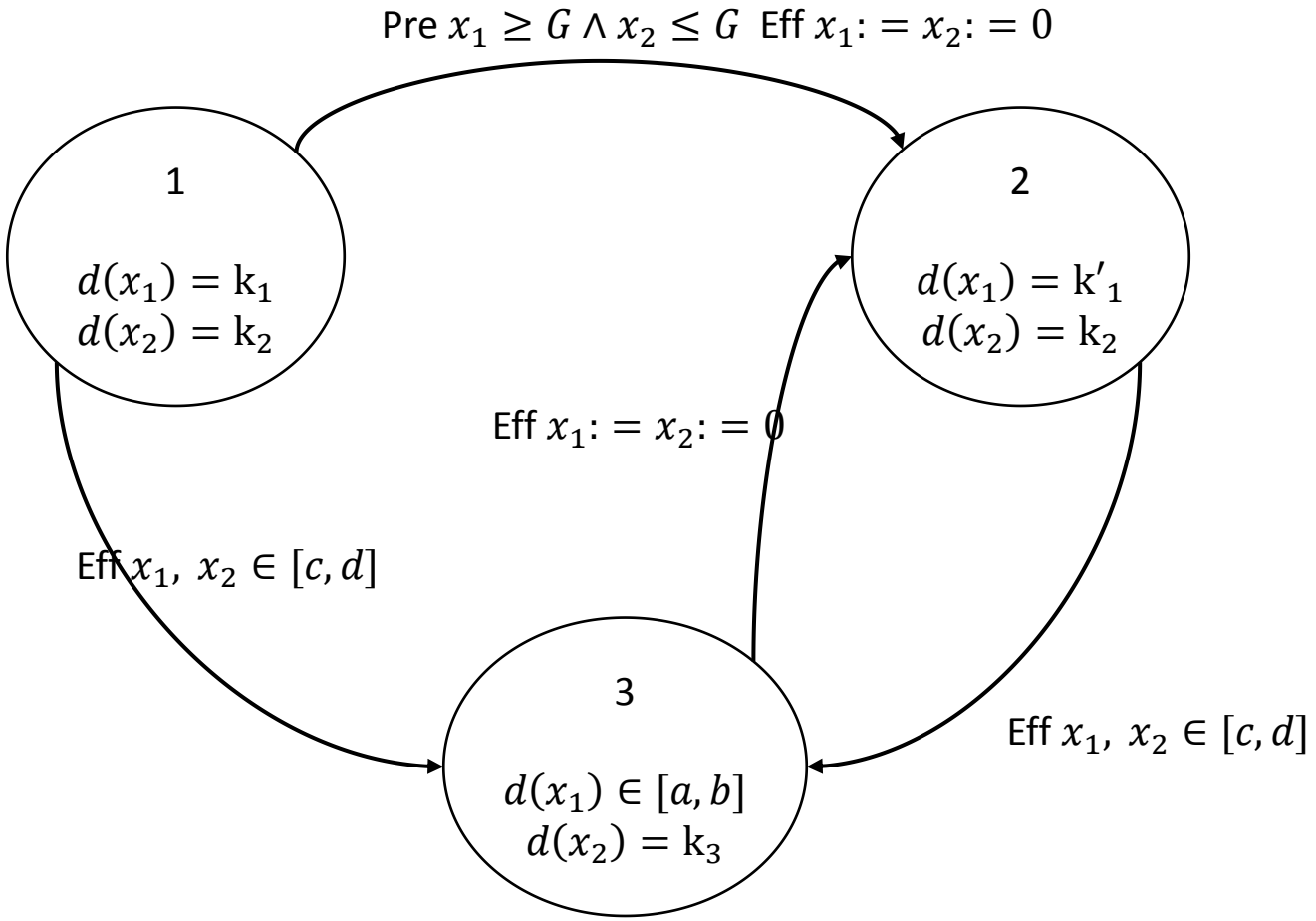
- $\mathcal{D}$ is a set of transitions such that
  - Guards are described by **rational** clock constraings
  - $\langle x, l\rangle \rightarrow_a \langle x', l'\rangle$ implies if dynamics d(x) changes from ℓ to another $\ell' \neq \ell$ then $x' \in [c_1, c_2]$ (**initialized, rather than keeping the old value**), otherwise $x' = x$ (keep the old value)

Example: Initialized Rectangular HA



Both $x_1,\ x_2$ have to be reset on transaction to a different mode

CSR Decidable for IRHA?

- Given an IRHA, check if a particular location is reachable from the initial states
- Is this problem decidable? Yes
- Key idea:
  - Construct a 2n-dimensional **initialized** multi-rate automaton that is bisimilar to the given IRHA
  - Construct a ITA that is bisimilar to the Singular TA
  - More details later

Rectangular HA: undecidable

Definition. A **rectangular hybrid automaton (RHA)** is a HA $\mathcal{A} = \langle V, A, \mathcal{T}, \mathcal{D} \rangle$ where

- V = X ∪ $\{loc\}$, where X is a set of n **continuous variables** and loc is a discrete state variable of finite type L
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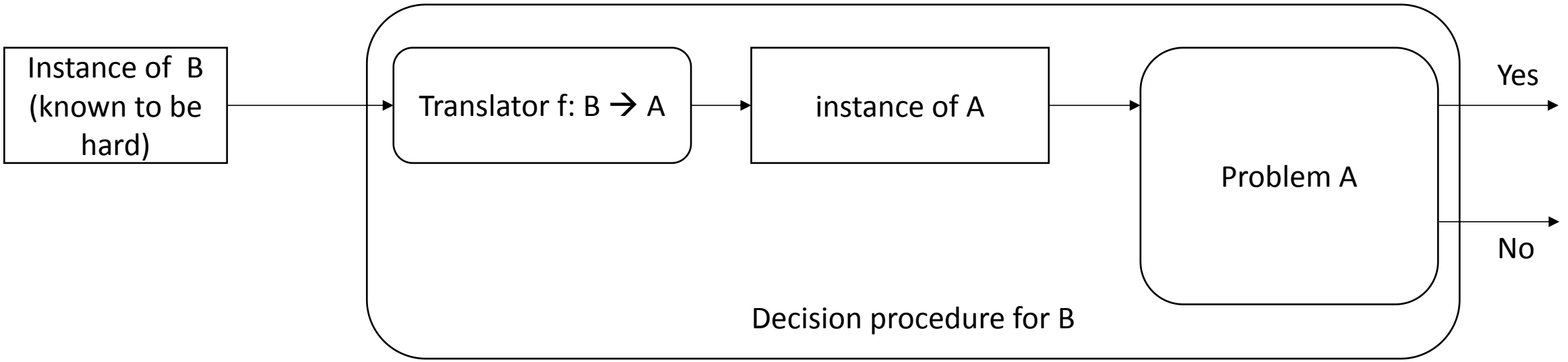
  - $\langle x, l \rangle \rightarrow_a \langle x', l' \rangle$ implies $x' = x$ *or* $x' \in [c_1, c_2]$

keeping the old value

Reachability of Rectangular HA

- Is this problem decidable? No
  - **[Henz95]** Thomas Henzinger, Peter Kopke, Anuj Puri, and Pravin Varaiya. What's Decidable About Hybrid Automata?. Journal of Computer and System Sciences, pages 373–382. ACM Press, 1995.
- We will see that the control state reachability (CSR) problem for rectangular hybrid automata (RHA) is undecidable
- This implies that automatic verification of invariants and safety properties is also impossible for this class of models
- The result was shown by Henzinger et al. [1995] through a *reduction from* the Halting problem of two counter machines

General reductions: Using known hard problem B to show hardness of A

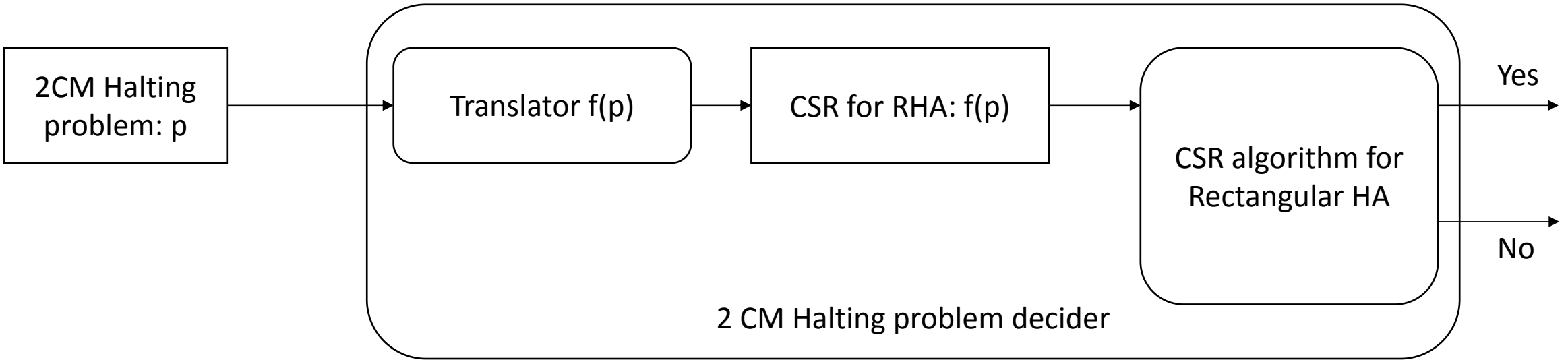


Given B is known to be hard

Suppose (for the sake of contradiction) A is solvable

If we can construct a reduction f: B→A (from B to A) then B becomes easy, which is a contradiction

Reduction from Halting Problem for 2CM



Suppose CSR for RHA is decidable

If we can construct a reduction from 2CM Halting Problem to CSR for RHA then 2CM Halting problem is also decidable

Counter Machines

An n-counter machine is an elementary computer with n-unbounded counters and a finite program written in a minimalistic assembly language.

More precisely: A 2-counter machine (2CM) is a discrete transition system with the following components:

- Two **nonnegative** integer counters C and D. Both are **initialized to 0**.
- A finite program with one of these instructions at each location (or line):
  - INCC, INCD: increments counter C (or D)
  - DECC, DECD: decrements counter C (or D), provided it is not 0,
  - JNZC, JNZD [label]: moves the program control to line *label* provided that counter C (or D) is not zero.

Example 2CM for multiplication

A 2-counter machine for multiplying 2x3 is shown below.

```
% C = D = 0 initially
INCC;
INCC;          % C = 2
INCD;          % LOOP
INCD;
INCD;
DECC;
JNZC 3;        % Jump to LOOP
               % HALT
```

Exercise: Show that any k-counter machine can be simulated by a 2CM.

Halting problem for 2CM

- A **configuration** of a 2CM is a triple (pc, C, D)
  - pc is the program counter that stores the next line to be executed
  - C, D are values of the counter
- A sequence of configurations (pc0, D0, C0), (pc1, D1, C1), … is an **execution** if the ith configuration goes to the (i+1)st configuration in the sequence executing the instruction in line pci
- Given a 2CM **M** and a special halting location (pc_halt), the Halting problem requires us to decide whether all executions of **M** reach the halting location
- Theorem [Minsky 67]. The Halting problem for 2CMs is undecidable.

Reduction from 2CM to CSR-RHA

We have to construct a function (reduction) that maps instances of 2CM-Halt to instances of CSR-RHA

Reduction from 2CM to CSR-RHA

- Program counter pc
- Counters C, D
- Instructions (program)
- Halting location

- Locations, sequence of locations
- Clocks c, d that can go at some constant rates k_1, k_2
- Transitions: *widgets* to simulation these instructions
- Particular location / control state (to which we will check CSR)

Idea of reduction (an RHA compiler)

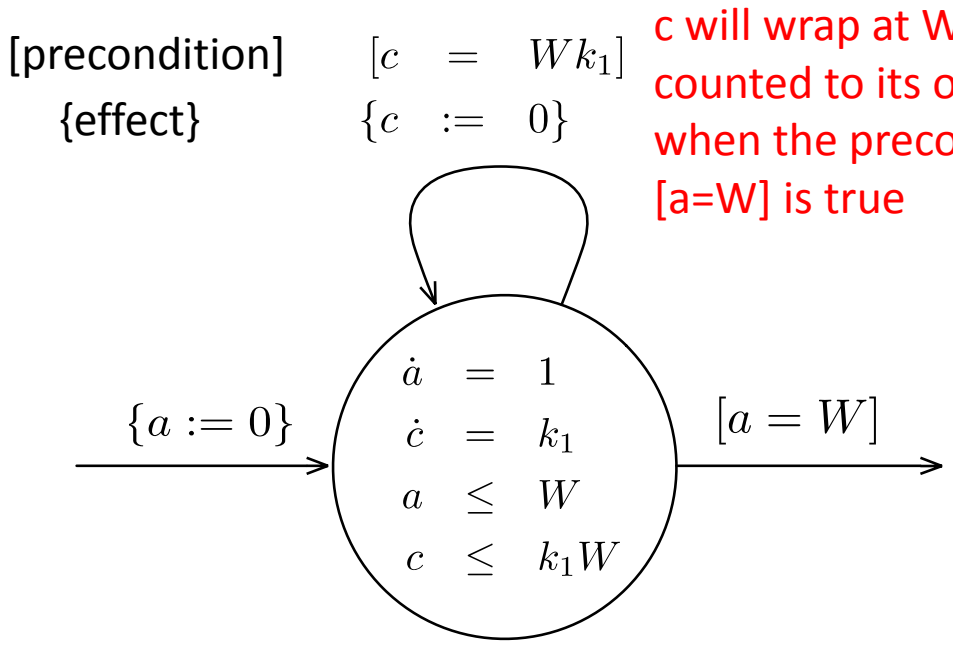
- Two clocks (k_2>k_1)
  - $c = k_1\left(\frac{k_2}{k_1}\right)^C$
  - d $= k_1\left(\frac{k_2}{k_1}\right)^D$

  Register values in C, D represented by the value of clock variables

- INCC: increase c by $\left(\frac{k_2}{k_1}\right)$
  - $k_1\left(\frac{k_2}{k_1}\right)^{C+1} = c\left(\frac{k_2}{k_1}\right)$
- DECC: decrease c by $\left(\frac{k_2}{k_1}\right)$
  - $k_1\left(\frac{k_2}{k_1}\right)^{C-1} = c\left(\frac{k_1}{k_2}\right)$
- checking nonzero:
  - check $c > k_1$ (JNZC)

  since $k_1\left(\frac{k_2}{k_1}\right)^0 = k_1$

A widget that preserves the value of clock c

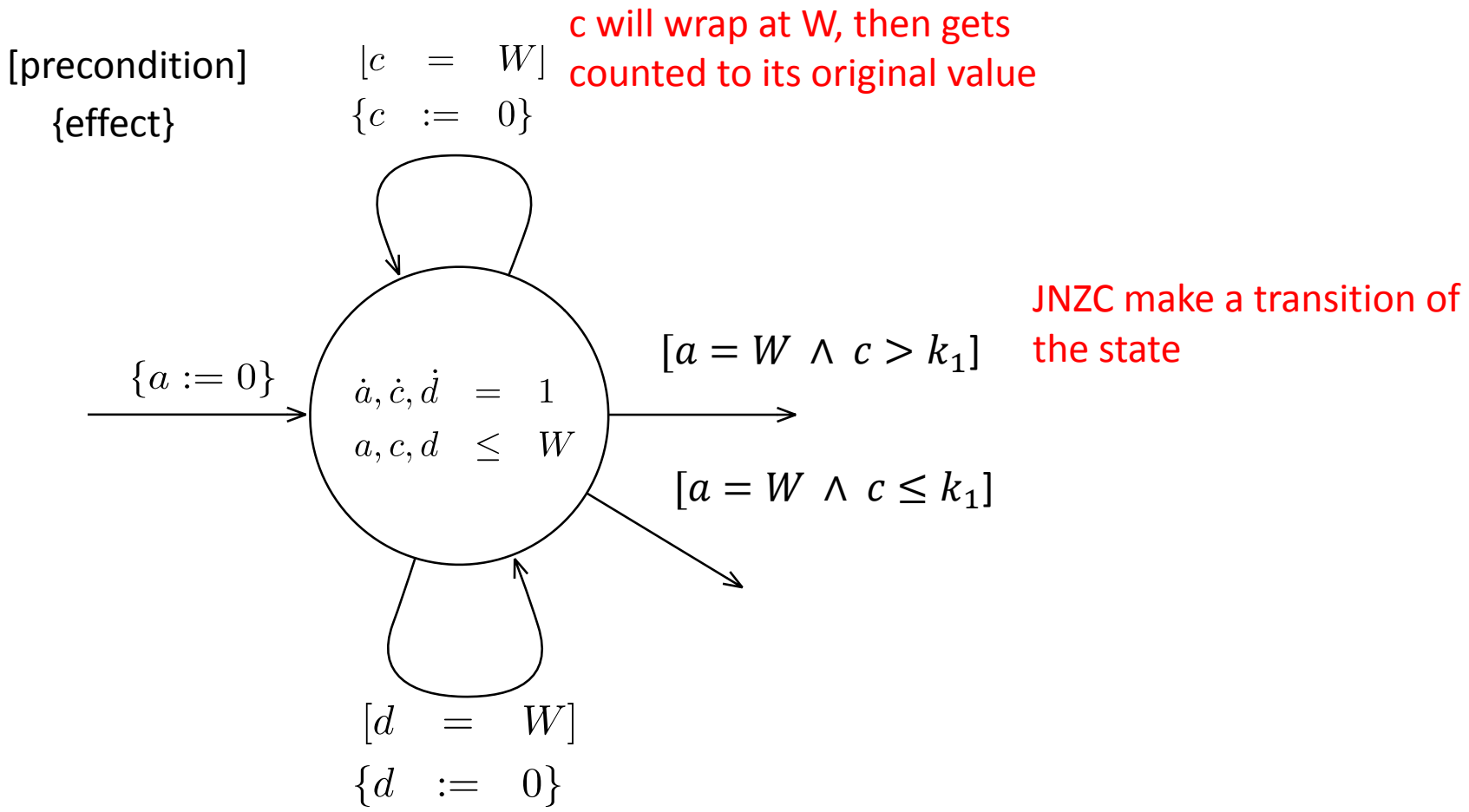


c will wrap at Wk, then gets counted to its original value when the precondition [a=W] is true

Transitions and clock guards for this control state

- We use clock variable c to represent register value C
- We need to ensure c does not change when leaving a state

A widget for checking JNZC ($c < k_1$)



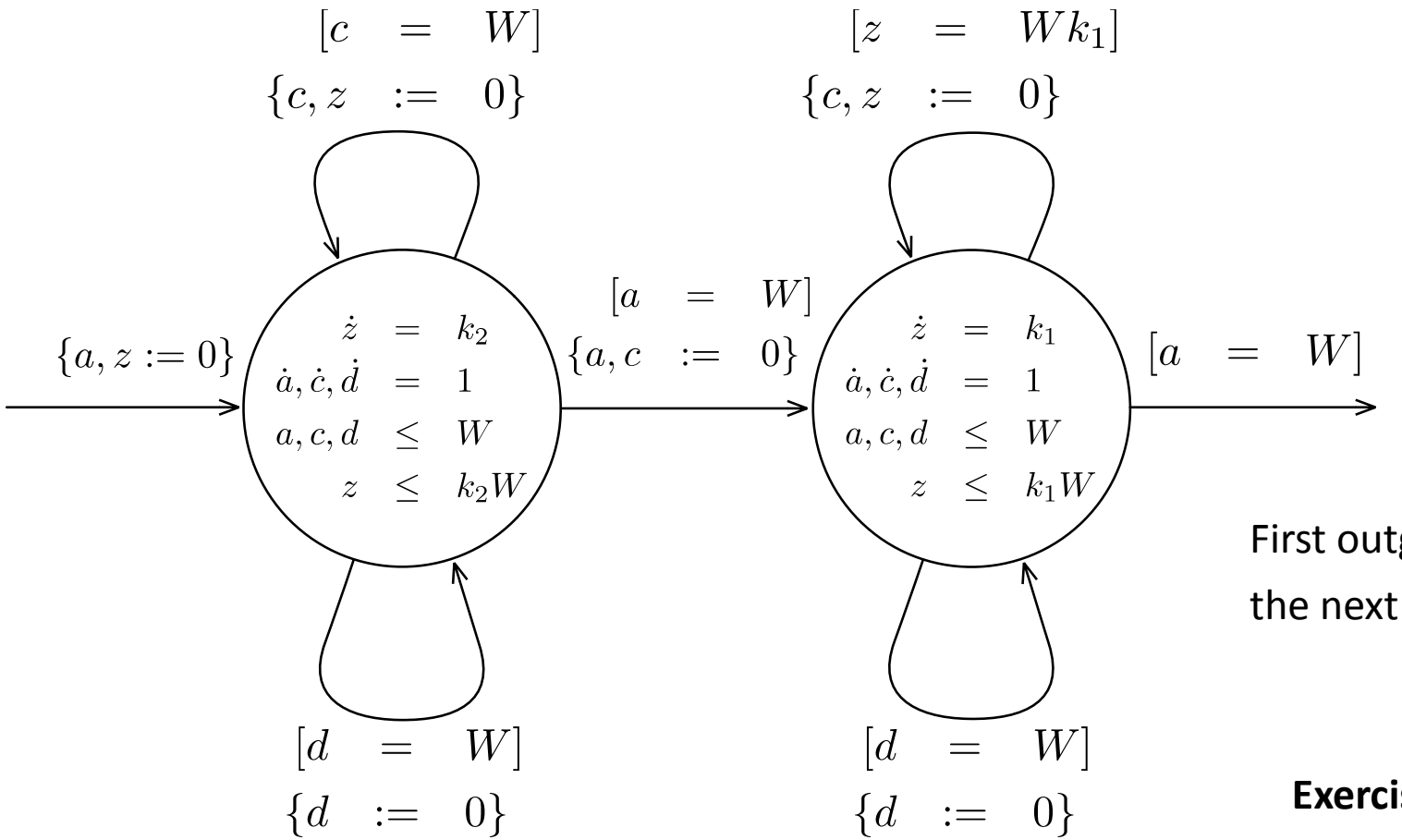
c will wrap at W, then gets counted to its original value

JNZC make a transition of the state

A widget implementing INCC

c will wrap at W, then gets counted to its original value; in the same time, z counts to k times of the vale of c

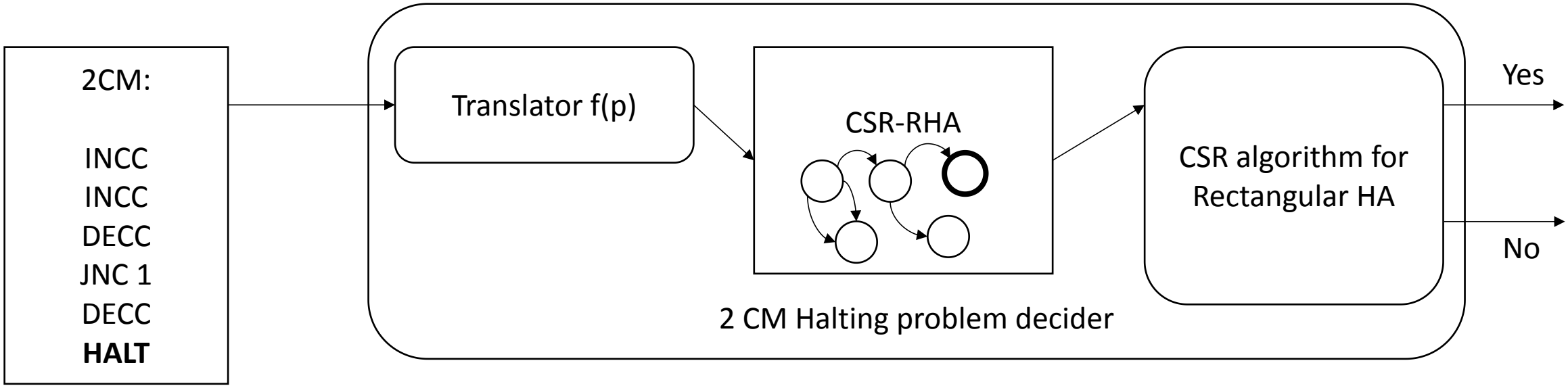
z will wrap at Wk1, then gets to 0 and then its original value when leaving this control state. In the same time, c counts k1 times slower.



First outgoing transition sets $z = k_2 c$ and the next outgoing transition sets c $= z * \left(\frac{1}{k_1}\right)$

Exercise: Show the widget for DECC

Putting it all together



Suppose CSR for RHA is decidable

If we can construct a reduction from 2CM Halting Problem to CSR for RHA then 2CM Halting problem is also decidable

Theorem: CSR for RHA is undecidable

Initialized Rectangular HA

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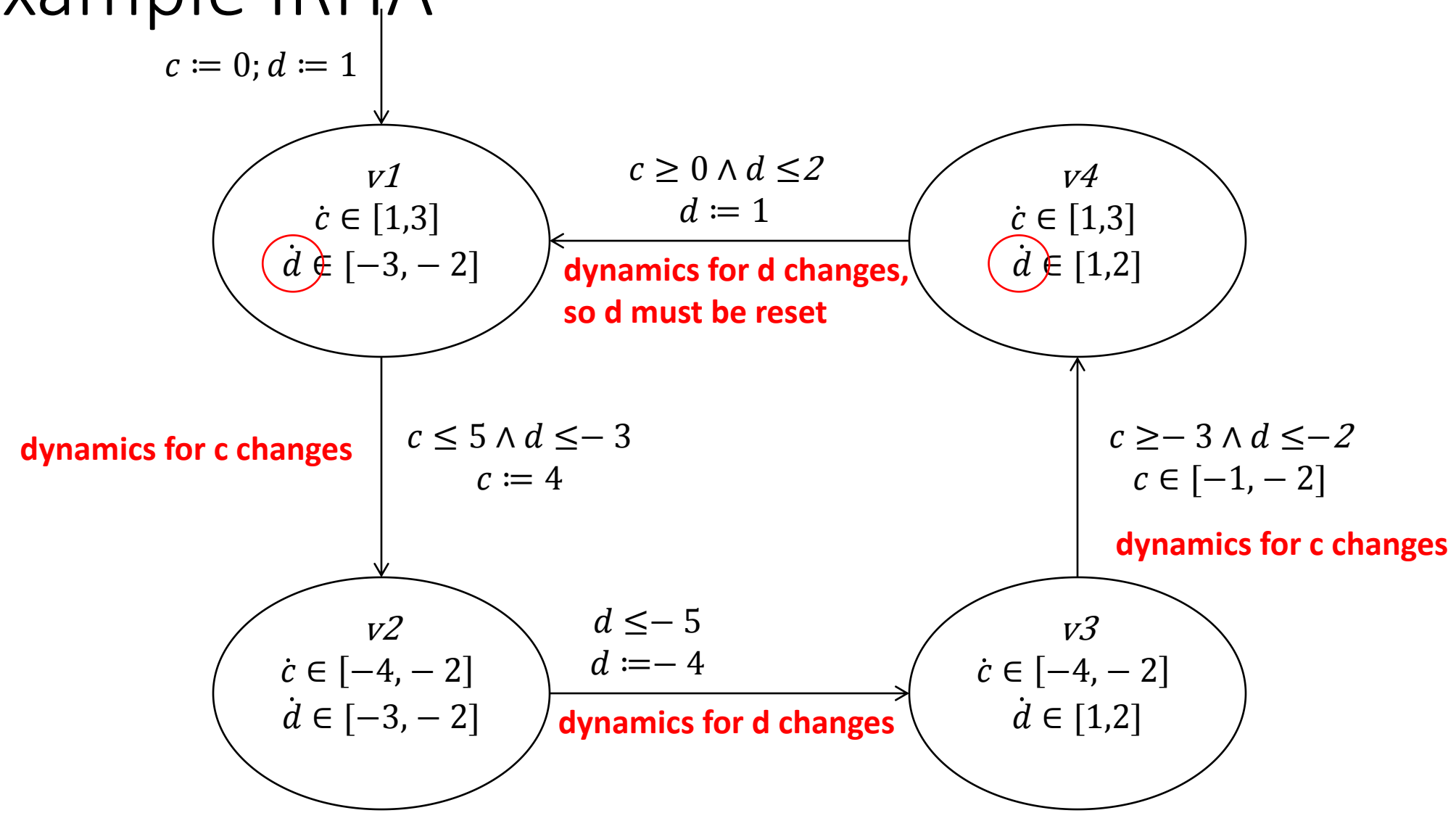
  - $\langle x, l\rangle \rightarrow_a \langle x', l'\rangle$ implies if **dynamics** $d(x)$ **changes** from ℓ to ℓ' then $x' \in [c_1, c_2]$, otherwise $x' = x$ if $d(x)$ is not changed

Example IRHA



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CSR Decidable for IRHA?

- Given an IRHA, check if a particular location is reachable from the initial states
- Is this problem decidable? Yes
- Key idea:
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  - Construct a ITA that is bisimilar to the Singular TA

From IRHA to Singular HA conversion

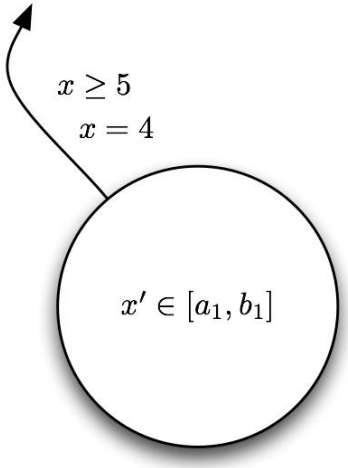
For every variable create two variables, tracking the upper and lower bounds

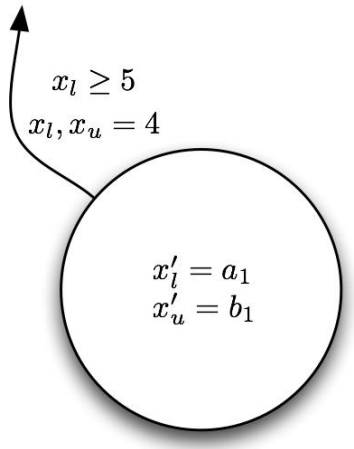
Henzinger, Thomas A., et al. "What's decidable about hybrid automata?." Proceedings of the twenty-seventh annual ACM symposium on Theory of computing. 1995.

IRHA	MRA
x	x_ℓ ; x_u
Evolve: $d(x) \in [a_1, b_1]$	Evolve: $d(x_\ell) = a_1; d(x_u) = b_1$
Eff: $x' \in [a_1, b_1]$	Eff: $x_\ell = a_1; x_u = b_1$
$x' = c$	$x_\ell = x_u = c$
Guard: $x \geq 5$	$x_l \geq 5$
	$x_l < 5 \wedge x_u \geq 5$ Eff $x_l = 5$

Two possible transitions for each guard

If the lower bound >=5, transition happens

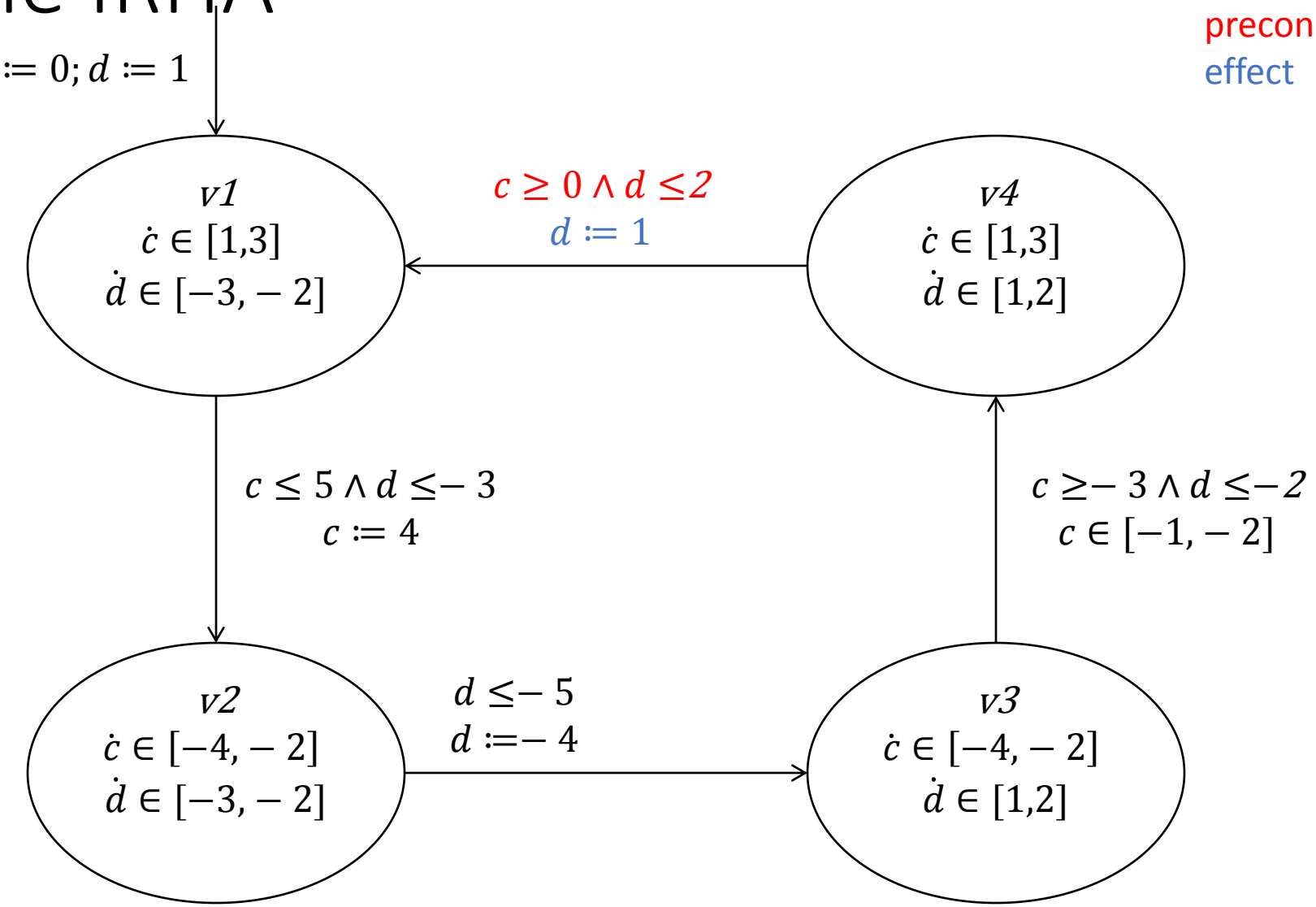




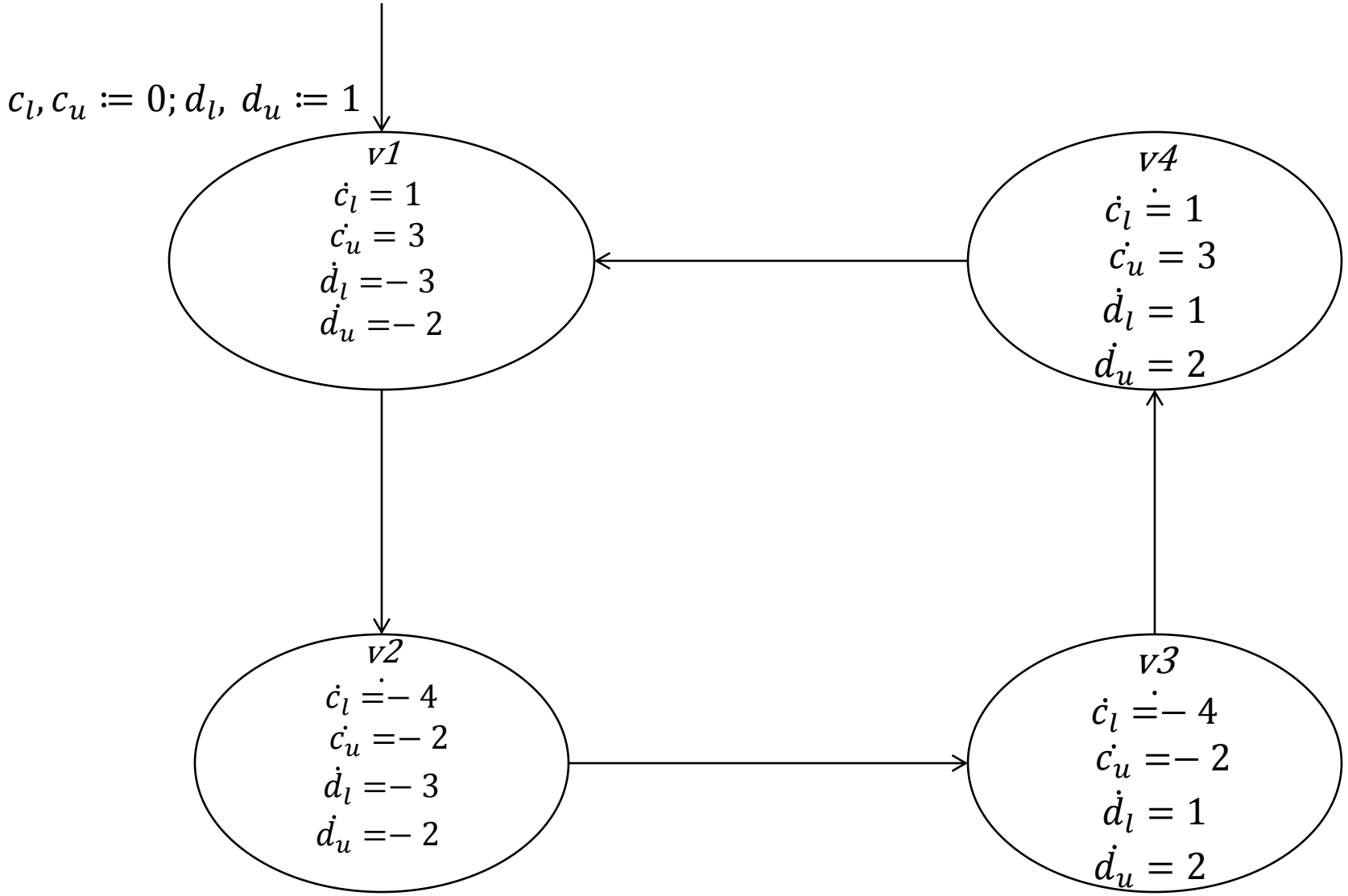
Another case: lower bound has not reached to 5 but upper bound does; in this case, xl is updated.

Example IRHA

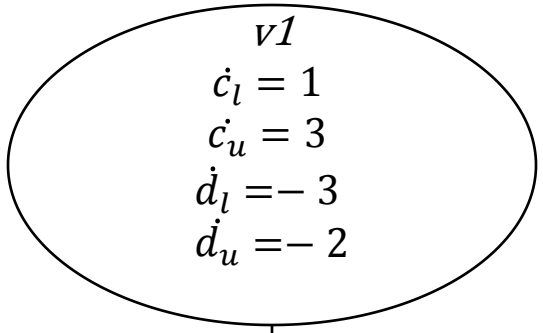
precondition
effect



Initialized Singular HA



Transitions



Transiton pre-condition from IRHA:

$c \leq 5 \wedge d \leq -3$

$c := 4$

$c \leq 5$ Converted

$c_l \leq 5$

$c_l,\ c_u := 4$

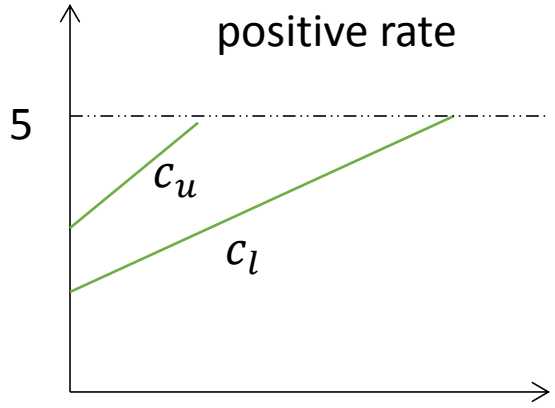
$d \leq -3$ *Converted*:

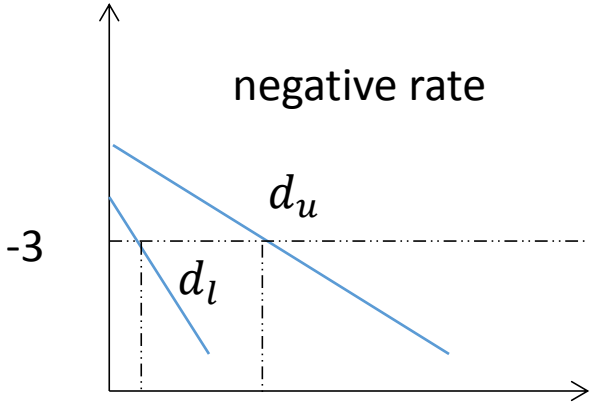
$d_u \leq -3$

no reset

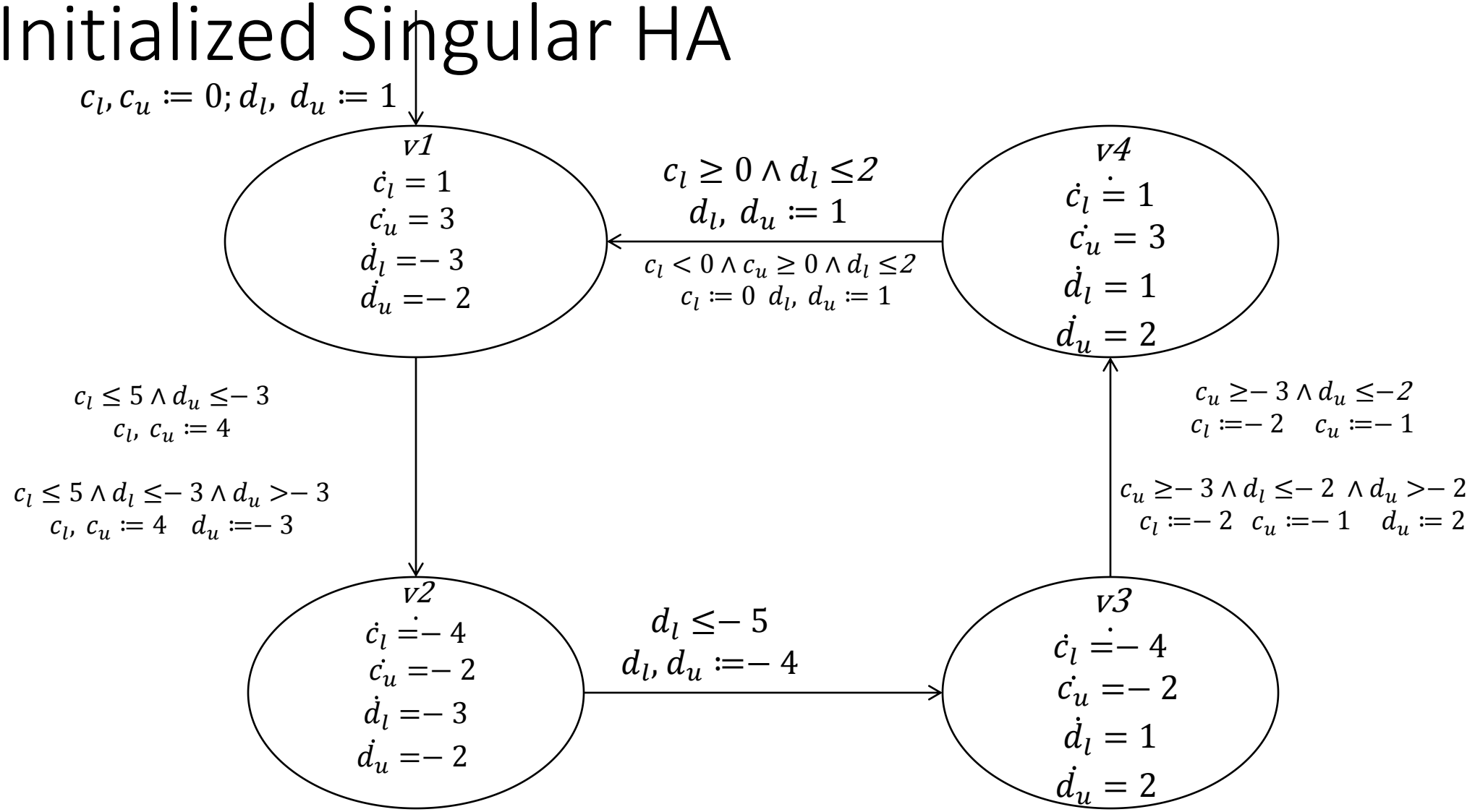
$d_u > -3 \wedge d_l \leq -3$

$d_u :=$-3





Initialized Singular HA



Practical reachability

Algorithm: BasicReach
Input: $\mathbf{A} = \langle V, \Theta, A, \mathbf{D}, \mathbf{T} \rangle$, $d > 0$
  $Rt, Reach: val(V)$
  $Rt := \Theta$;
  $Reach := \emptyset$;
  While $(Rt \not\subseteq \text{Reach})$
    $Reach := Reach \cup Rt$;
    $Rt := Rt \cup Post_{\mathbf{D}}(Rt)$;
    $Rt := Post_{\mathbf{T}(d)}(Rt)$;
Output: $Reach$

Algorithm: Post_{D}
\\ computes post of all transitions
Input: $A, \mathbf{D}, S_{in}$
  $S_{out} = \emptyset$
  For each $a \in A$
    For each $\langle g_1, g_2 \rangle \in S_{in}$
      If $[\![g_1, g_2]\!] \cap [\![g_{ga1}, g_{ga2}]\!] \neq \emptyset$
        $S_{out} := S_{out} \cup \langle g_{ra1}, g_{gra2} \rangle$
Output: S_{out}

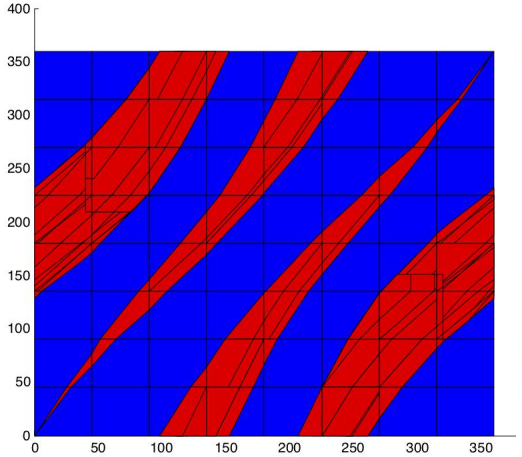
Algorithm: $\text{Post}_{\mathbf{T}(d)}$
\\ computes post of all trajectories
Input: $A, \mathbf{T}, S_{in}, d$
  $S_{out} = \emptyset$
  For each $\ell \in L$
    For each $\langle g_1, g_2 \rangle \in S_{in}$
      $P := \cup_{t \leq d} [\![g_1, g_2]\!] \oplus [\![tg_{\ell 1}, tg_{\ell 2}]\!]$
      $S_{out} := S_{out} \cup Approx(P)$
Output: S_{out}

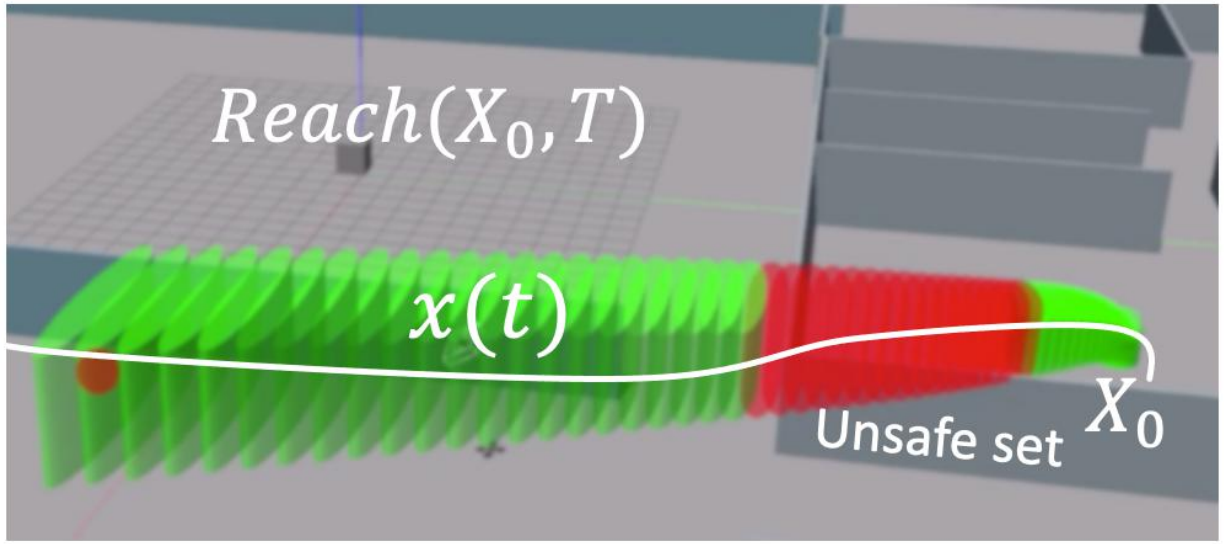
Tools:
SpaceEX
CORA
C2E2
Flow*
DryVR

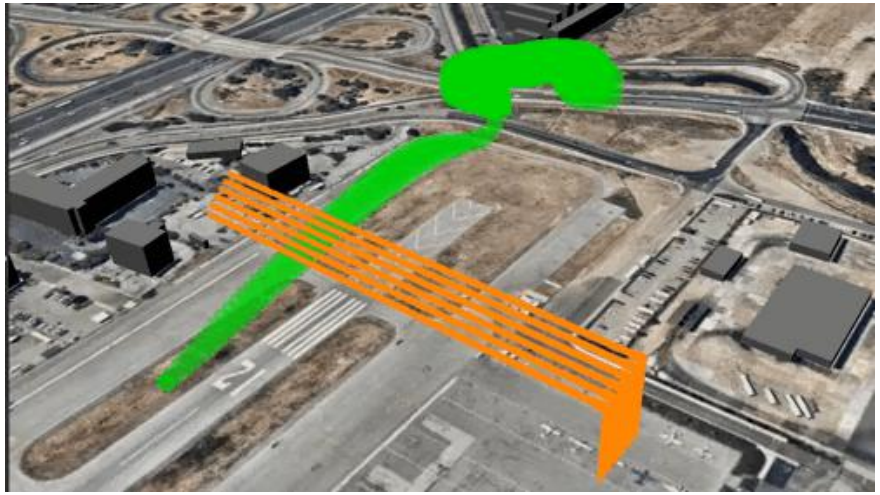
Data structures critical for reachability

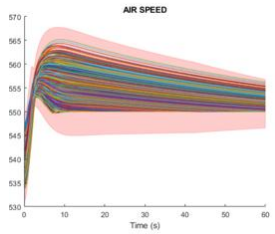
- Hyperrectangles
  - $[[g_1; g_2]] = \{x \in R^n \mid \|x - g_1\|_\infty \leq \|g_2 - g_1\|_\infty\} = \Pi_i [g_{1i}, g_{2i}]$
- Polyhedra
- Zonotopes [Girard 2005]
- Ellipsoids [Kurzhanskiy 2001]
- Support functions [Guernic et al. 2009]
- Generalized star set [Duggirala and Viswanathan 2018]

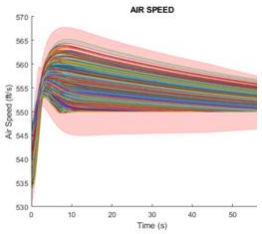
Reachability in practice



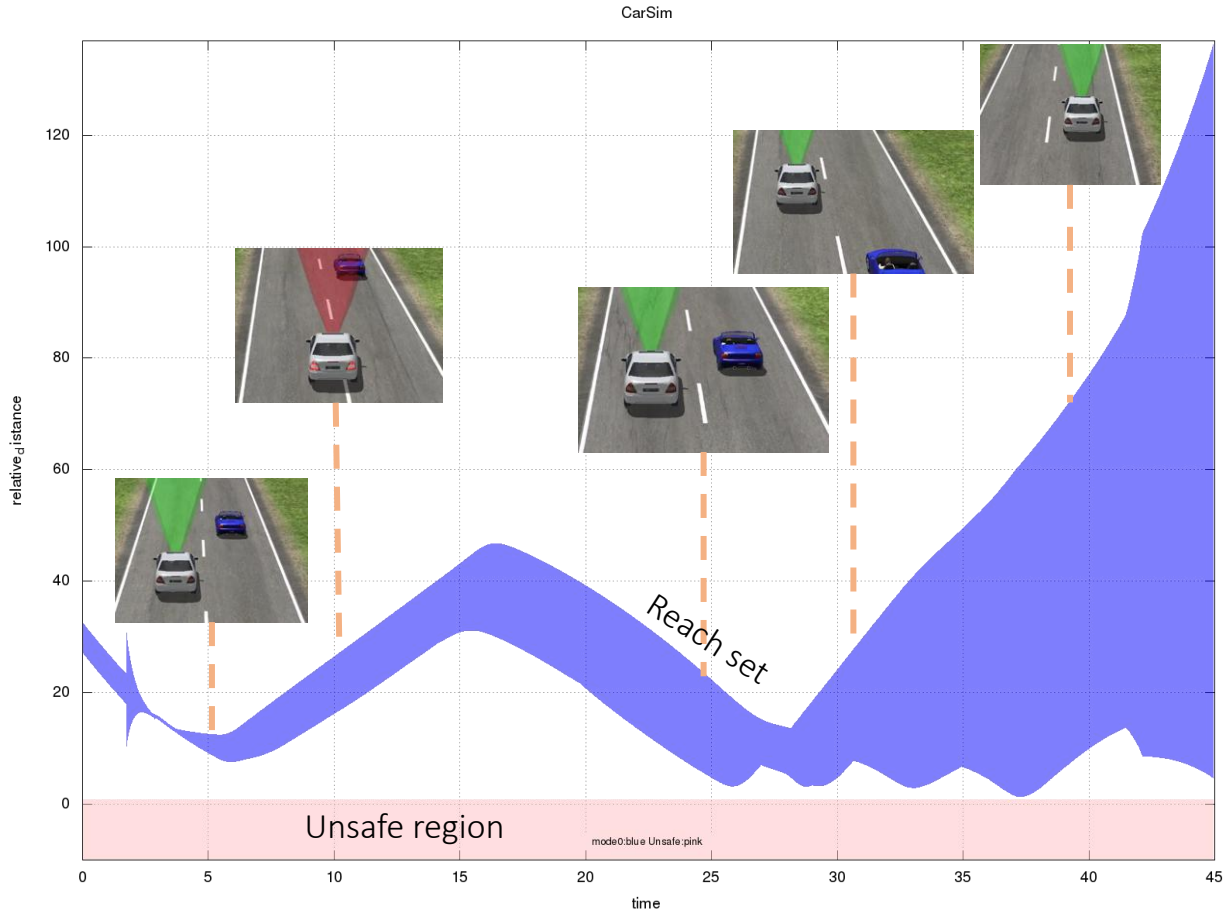


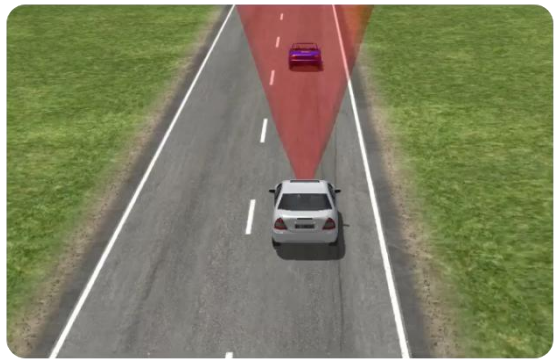






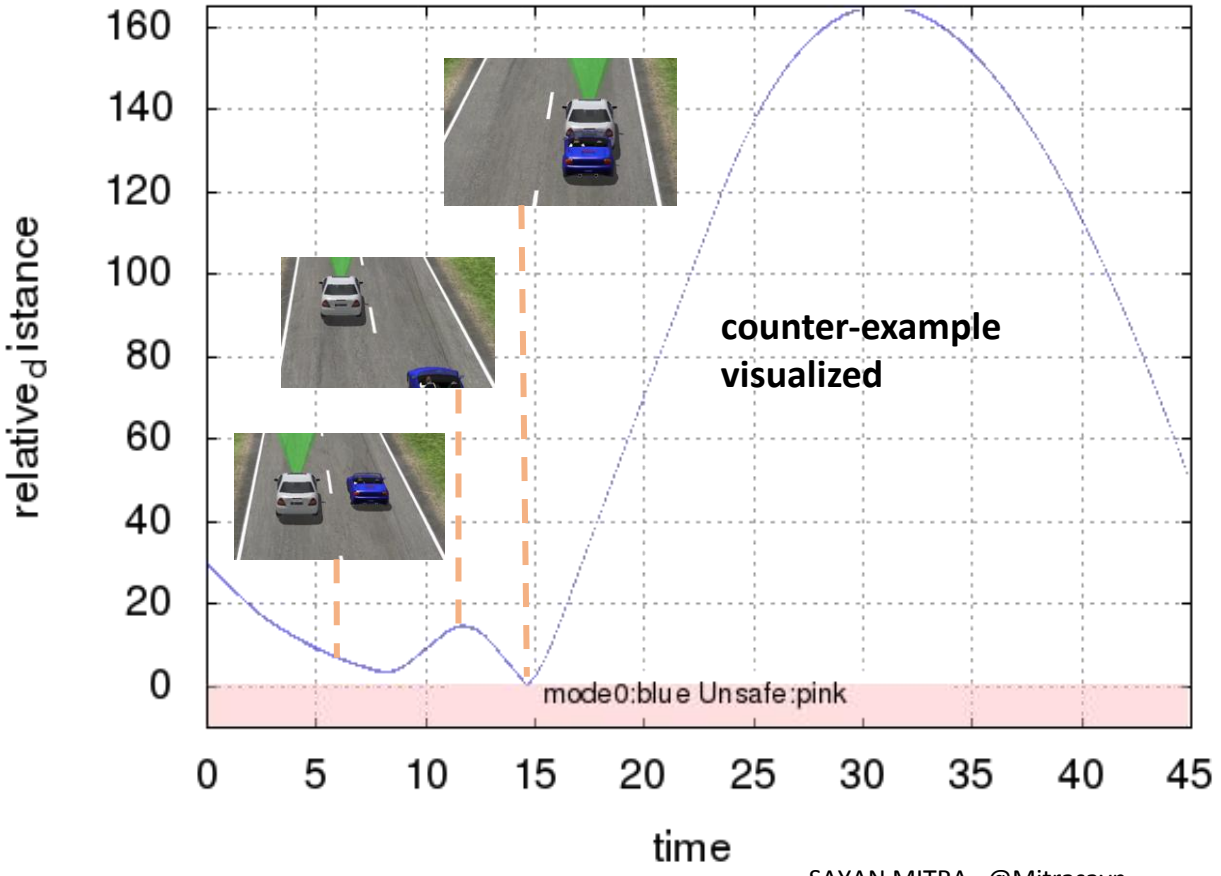
C2E2 generated safety certificate for a given user model

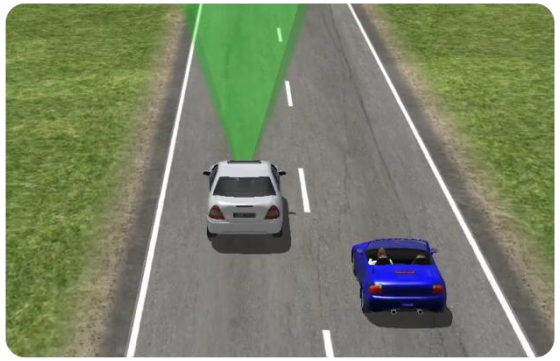




Verify no collision with **uncertainties**: speeds in [70, 85] mph and acceleration range of NPC

For a different user model C2E2 finds a corner case





Verify no collision with **uncertainties** like speeds in [70, 85] mph and **bigger** acceleration range of NPC