

ECE/CS 584: Verification of Embedded and Cyberphysical systems

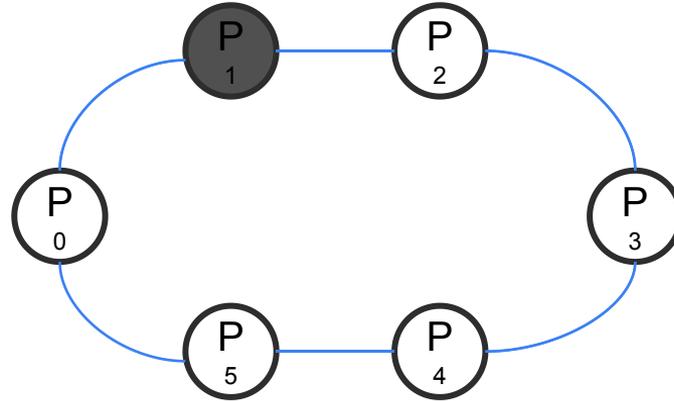
Lecture 3: Models, Satisfiability Problem

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Review: Dijkstra's mutual exclusion Algorithm



N processes: 0, 1, ..., N-1

state of each process j is a single integer variable $x[j] \in \{0, 1, 2, K-1\}$, where $K > N$

p_i has TOKEN if and only if the blue conditional below is true

The “update” action is defined differently for P0 vs. others

P_0 if $x[0] = x[N-1]$

$P_j, j > 0$ if $x[j] \neq x[j-1]$

then $x[0] := x[0] + 1 \bmod K$

then $x[j] := x[j-1]$

Review: Automaton for Dijkstra's Token Ring

automaton **DijkstraTR**($N:\text{Nat}$, $K:\text{Nat}$), where $K > N$

type **ID**: enumeration $[0, \dots, N-1]$

type **Val**: enumeration $[0, \dots, K-1]$

actions

update($i:\text{ID}$)

variables

$x:[\text{ID} \rightarrow \text{Val}]$ **initially forall** $i:\text{ID}$ $x[i] = 0$

transitions

update($i:\text{ID}$)

pre $i = 0 \wedge x[i] = x[N-1]$

eff $x[i] := (x[i] + 1) \% K$

update($i:\text{ID}$)

pre $i > 0 \wedge x[i] \neq x[i-1]$

eff $x[i] := x[i-1]$

Automaton $\mathcal{A} = \langle X, \Theta, A, \mathcal{D} \rangle$

Review: Proving invariants by induction (Chapter 7)

Theorem 7.1. Given a automaton $\mathcal{A} = \langle X, \Theta, A, \mathcal{D} \rangle$ and a set of states $I \subseteq \text{val}(X)$ if:

- (Start condition) for any $x \in \Theta$ implies $x \in I$, and
- (Transition closure) for any $x \rightarrow_a x'$ and $x \in I$ implies $x' \in I$

then I is an (inductive) invariant of \mathcal{A} . That is $\text{Reach}_{\mathcal{A}}(\Theta) \subseteq I$.

Proving invariants by induction for Dijkstra

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then I is an (inductive) invariant of \mathcal{A} . That is $\text{Reach}_{\mathcal{A}}(\Theta) \subseteq I$.

- I : “Exactly one process has the token”.

(Start condition): since $\forall i \ x[x[i]] = 0$, only P0 has token, so $x \models I$

(Transition closure): Fix a $x \rightarrow_a x'$ such that $x \in I$.

Two cases to consider.

$a = \text{update}(0)$

$a = \text{update}(i), i > 0$

automaton `DijkstraTR(N: Nat, K: Nat)`, where $K > N$

type ID: enumeration $[0, \dots, N-1]$

type Val: enumeration $[0, \dots, K-1]$

actions

`update(i: ID)`

variables

`x: [ID -> Val]` **initially forall i: ID x[i] = 0**

transitions

`update(i: ID)`

pre $i = 0 \wedge x[i] = x[(N-1)]$

eff $x[i] := (x[i] + 1) \% K$

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pre $i > 0 \wedge x[i] \sim x[i-1]$

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- I : “Exactly one process has the token”.

(Transition closure): Fix a $x \xrightarrow{a} x'$ such that $x \in I$.

Two cases to consider.

1. If $a = \text{update}(0)$ then

(a) since $x \models \text{Pre}(\text{update}(0))$ it follows that $x[x[0] = x[x[N-1]]$

(b) since $x \models I$ it follows that $\forall i > 0, x[x[i] = x[x[i-1]]$

(c) $x'[x[0] \neq x'[x[N-1]]$ by applying (a) and $\text{Eff}(\text{update}(0))$ to x

(d) $x'[x[1] \neq x'[x[0]]$ by applying (b) and $\text{Eff}(\text{update}(0))$ to x

(e) $\forall i > 1, x'[x[i] = x'[x[i-1]]$ by applying (b) and $\text{Eff}(\text{update}(0))$ to x

Now there is only one token held by P1. Therefore $x' \models I$.

2. If $a = \text{update}(i), i > 0$ then fix arbitrary $i > 0 \dots$ (do it as an exercise)

automaton `DijkstraTR(N: Nat, K: Nat)`, where $K > N$

type ID: enumeration $[0, \dots, N-1]$

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`update(i: ID)`

variables

`x: [ID -> Val]` initially forall `i: ID` `x[i] = 0`

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From above **Theorem** it follows that I is an invariant of `DijkstraTR`

Proving invariants by induction for Dijkstra

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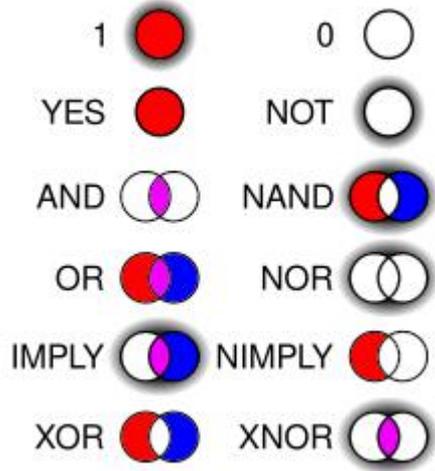
`update(i: ID)`

pre `i > 0` \wedge `x[i] ~ = x[i-1]`

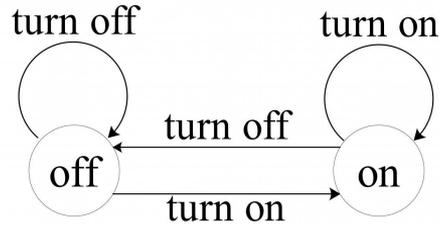
eff `x[i] := x[i-1]`

Can we prove this part automatically?
Yes! Use a *satisfiability* solver! (HW1)

Many different types of models...



Boolean logic,
propositional logic



Automaton



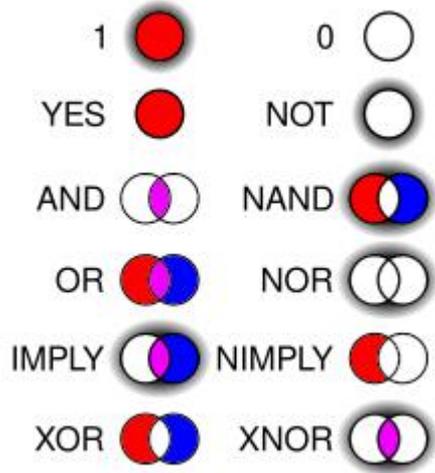
"Whenever the nth floor's call button is pressed, the cabin will eventually stop at the nth floor and open the door"

Temporal logic



Hybrid automaton

Today: boolean logic and the satisfiability problem

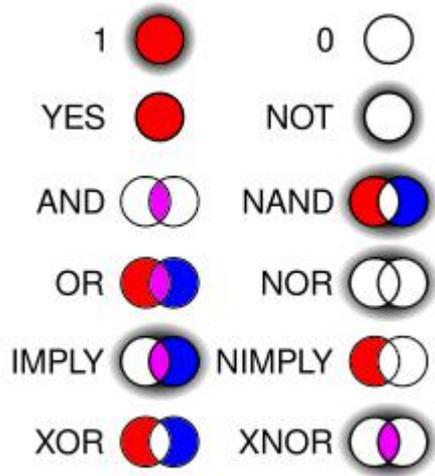


Set of **variables**: $X = \{x_1, x_2, \dots, x_n\}$

Each variable is Boolean $\{0, 1\}$ or $\{T, F\}$

Formula α built from propositional operators (“and” \wedge , “or” \vee , “not” \neg , “implies” \rightarrow , “exclusive or” \oplus , “iff” \leftrightarrow)

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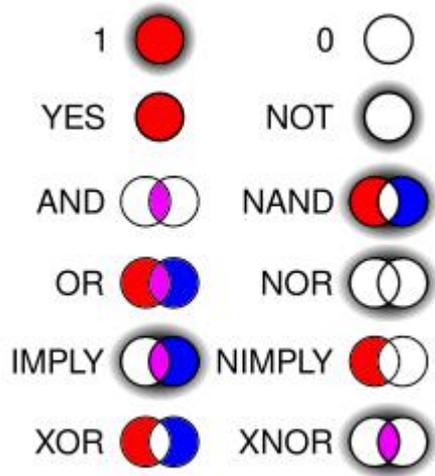
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Formula α built from propositional operators (“and” \wedge , “or” \vee , “not” \neg , “implies” \rightarrow , “exclusive or” \oplus , “iff” \leftrightarrow)

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

Today: boolean logic and the satisfiability problem



A “**valuation**” \mathbf{x} of X maps each x_i to a value 0 or 1

$val(X)$ denotes the set of all possible valuations

A valuation \mathbf{x} of X *satisfies* α when each x_i in α replaced by the corresponding value in \mathbf{x} evaluates to *true*.

We write this as $\mathbf{x} \models \alpha$; otherwise, we write $\mathbf{x} \not\models \alpha$

Example:

$$\alpha(x_1, x_2, \dots, x_n) \equiv (x_1 \wedge x_2 \vee x_3) \wedge (x_1 \wedge \neg x_3 \vee x_2)$$

$$\mathbf{x} \equiv \langle x_1 \mapsto 1, x_2 \mapsto 1, x_3 \mapsto 0 \rangle$$

$$\mathbf{x} \models \alpha$$

Example: model a problem using boolean logic

Suppose you need to schedule 6 exams (E1, E2, ..., E6) on a single day. The day is divided into 4 time slots (S1, S2, S3, S4).

The following constraints must be satisfied:

1. Each exam must be scheduled exactly once.
2. The student conflicts are:
 - E1 and E2 cannot be at the same time
 - E3 and E4 cannot be at the same time
 - E3 and E6 cannot be at the same time
3. Due to prerequisites, E1 must be scheduled in the morning (first 2 slots) and E4 must be in the afternoon (last 2 slots)

Can you use boolean logic to represent all the constraints?

Example: model a problem using boolean logic

Suppose you need to schedule 6 exams (E1, E2, ..., E6) on a single day. The day is divided into 4 time slots (S1, S2, S3, S4).

First step: define variables x_{ij} which indicates example i is schedule to slot j

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1. Each exam i must be scheduled exactly once.

Ensure each example is scheduled in some slot: $(x_{i1} \vee x_{i2} \vee x_{i3} \vee x_{i4})$ for all $i \in \{1,2,3,4,5,6\}$

Ensure each pair of slots j and k , ensure the example is not scheduled to more than one slot:

$(\neg x_{1j} \vee \neg x_{1k})$ for all pairs of $j \neq k; j, k \in \{1,2,3,4\}$

Example: model a problem using boolean logic

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- E1 and E2 cannot be at the same time
- E3 and E4 cannot be at the same time
- E3 and E6 cannot be at the same time

$$(\neg x_{1j} \vee \neg x_{2j})$$

$$(\neg x_{3j} \vee \neg x_{4j})$$

$$(\neg x_{3j} \vee \neg x_{6j})$$

for all j

Example: model a problem using boolean logic

Suppose you need to schedule 6 exams (E1, E2, ..., E6) on a single day. The day is divided into 4 time slots (S1, S2, S3, S4).

3. Due to prerequisites, E1 must be scheduled in the morning (first 2 slots) and E4 must be in the afternoon (last 2 slots)

$$(x_{11} \vee x_{12})$$

$$(x_{43} \vee x_{44})$$

The satisfiability problem: can we find an exam schedule?

Suppose you need to schedule 6 exams (E1, E2, ..., E6) on a single day. The day is divided into 4 time slots (S1, S2, S3, S4).

$$(x_{i1} \vee x_{i2} \vee x_{i3} \vee x_{i4}) \quad \text{for all } i \in \{1,2,3,4,5,6\}$$

$$(\neg x_{1j} \vee \neg x_{1k}) \quad \text{for all pairs of } j \neq k; j, k \in \{1,2,3,4\}$$

$$(\neg x_{1j} \vee \neg x_{2j}) \quad (\neg x_{3j} \vee \neg x_{4j}) \quad (\neg x_{3j} \vee \neg x_{6j}) \quad \text{for all } j \in \{1,2,3,4\}$$

$$(x_{11} \vee x_{12}) \quad (x_{43} \vee x_{44})$$

The satisfiability problem (SAT)

Given a well-formed formula in boolean logic, determine whether there exists a satisfying solution:

$$\exists x \in \text{val}(X): x \models \alpha?$$

If the answer is “No” then α is said to be *unsatisfiable*

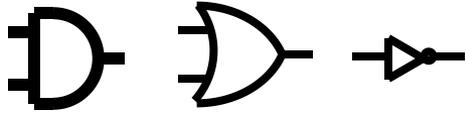
If $\forall x \in \text{val}(X): x \models \alpha$ then α is said to be *valid* or *a tautology*

If α is valid then $\neg\alpha$ is unsatisfiable

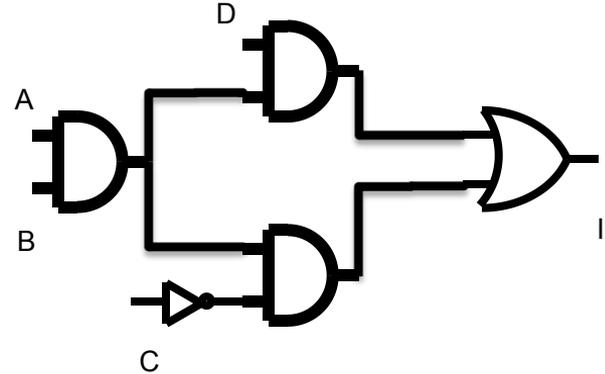
α and α' are *tautologically equivalent* if they have the same truth tables

$$\forall x \in \text{val}(X): x \models \alpha \leftrightarrow x \models \alpha'$$

Equisatisfiability



$$I \equiv (D \wedge (A \wedge B)) \vee (\neg C \wedge (A \wedge B))$$



Repeated subexpression is inefficient

How about: rename $(A \wedge B) \leftrightarrow E$, reducing circuit complexity

$$I' \equiv (D \wedge E) \vee (\neg C \wedge E) \wedge ((A \wedge B) \leftrightarrow E)$$

Recall that:

$$\begin{aligned} A \leftrightarrow B \\ (A \rightarrow B) \wedge (B \rightarrow A) \\ (\neg A \vee B) \wedge (\neg B \vee A) \end{aligned}$$

I and I' are **not tautologically equivalent**

$C = 0, A = B = 1, E = 0$ satisfies I since I does not mention E

But they are **equisatisfiable**, i.e., I is satisfiable iff I' is also satisfiable
(there could be two different valuations to make them satisfiable)

The CNF Form

We will assume α to be in *conjunctive normal form (CNF)*

literals: variable or its negation, e.g., x_3 , $\neg x_3$

clause: disjunction (or) of literals, e.g., $(x_1 \vee x_2 \vee \neg x_3)$

CNF formula: conjunction (and) of clauses,

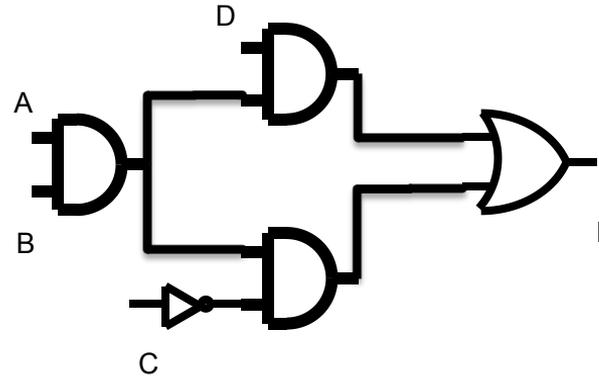
e.g., $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_1)$

A variable may appear *positively* or *negatively* in a clause

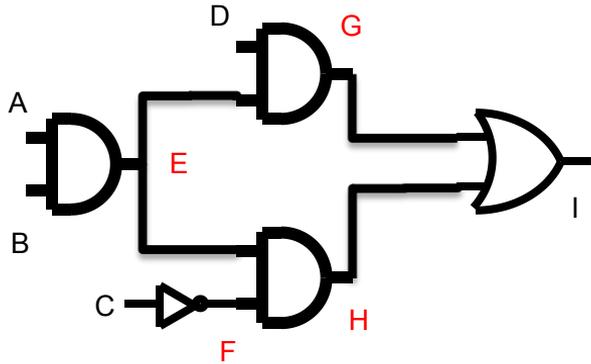
Any boolean formula can be converted to CNF

One approach:

- View the formula as a graph
- Give new names (variables) to non-leaves
- Relate the inputs and the outputs of the nonleaves and add this as a new clause
- Take conjunction of all of this



Converting to CNF



$$I \equiv (D \wedge (A \wedge B)) \vee (\neg C \wedge (A \wedge B))$$

- $F \leftrightarrow \neg C$
 - $F \rightarrow \neg C \wedge \neg C \rightarrow F$
 - $(\neg F \vee \neg C) \wedge (C \vee F)$
- $(A \wedge B) \leftrightarrow E$
 - $((A \wedge B) \rightarrow E) \wedge (E \rightarrow (A \wedge B))$
 - $(\neg(A \wedge B) \vee E) \wedge (\neg E \vee (A \wedge B))$
 - $(\neg A \vee \neg B \vee E) \wedge ((\neg E \vee A) \wedge (\neg E \vee B))$
- $(D \wedge E) \leftrightarrow G$
 - $(\neg D \vee \neg E \vee G) \wedge (\neg G \vee D) \wedge (\neg G \vee E)$
- $(F \wedge E) \leftrightarrow H$
 - $(\neg F \vee \neg E \vee H) \wedge (\neg H \vee F) \wedge (\neg H \vee E)$
- $(G \vee H) \leftrightarrow I$
 - $((G \vee H) \rightarrow I) \wedge (I \rightarrow (G \vee H))$
 - $(\neg G \wedge \neg H \vee I) \wedge (\neg I \vee G \vee H)$
 - $(\neg G \vee I) \wedge (\neg H \vee I) \wedge (\neg I \vee G \vee H)$

Challenge: how to solve the SAT problem?

Try this one:

$$(x \vee y) \wedge (\neg x \vee z) \wedge (y \vee \neg z) \wedge (\neg y \vee \neg z)$$

If satisfiable, what is the valuation of x , y , z ?

If not satisfiable, why?

2-SAT

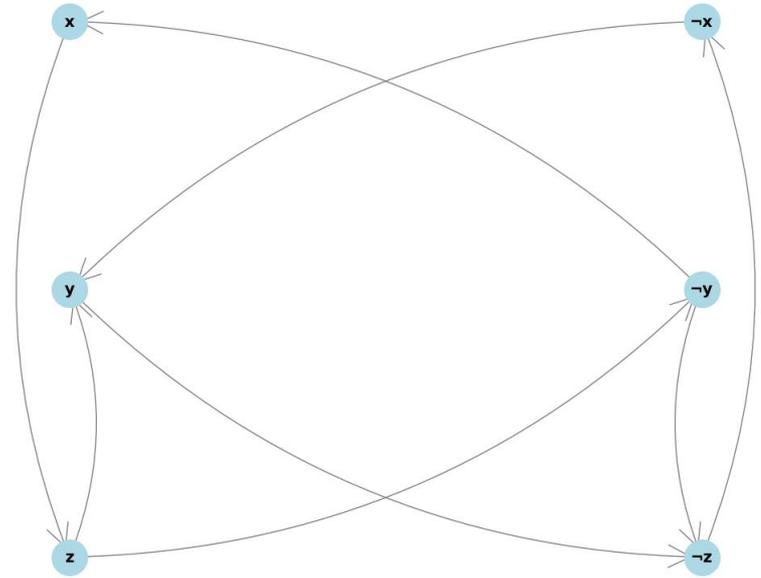
$$(x \vee y) \wedge (\neg x \vee z) \wedge (y \vee \neg z) \wedge (\neg y \vee \neg z)$$

2-SAT can be solved in polynomial time

Idea is to build the **implication graph**:

e.g., $(x \vee y)$ becomes $(\neg x \rightarrow y) \wedge (\neg y \rightarrow x)$

(following the definition of implication)



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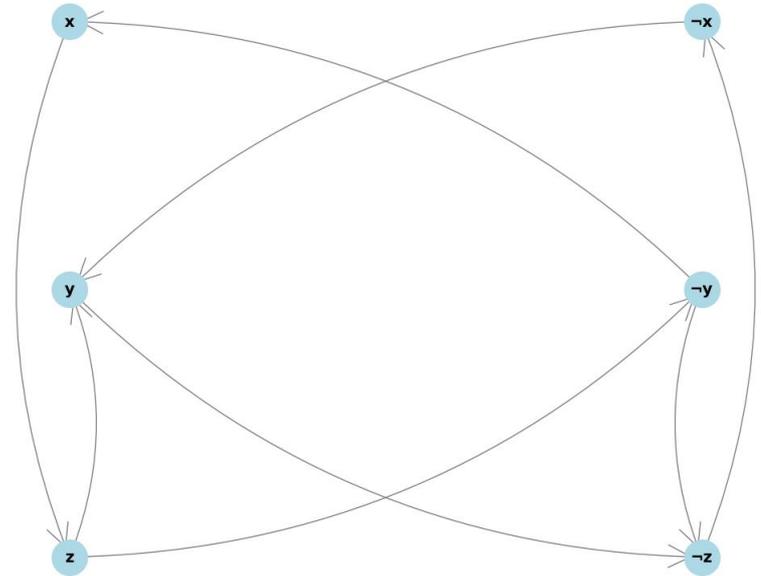
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Follow the graph we have

$$x \rightarrow \neg x, \neg y \rightarrow y, z \rightarrow \neg z$$

How do you assign x, y, z to make them true?

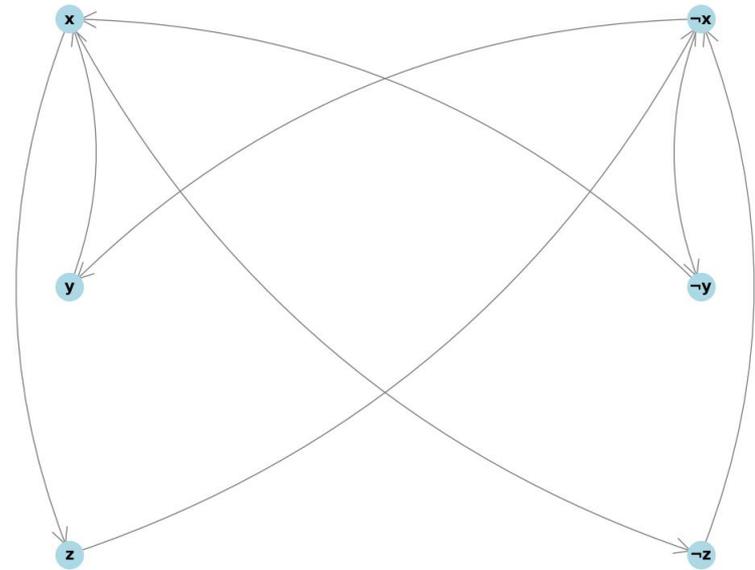


2-SAT another example

$$(x \vee y) \wedge (x \vee \neg y) \wedge (\neg x \vee z) \wedge (\neg x \vee \neg z)$$

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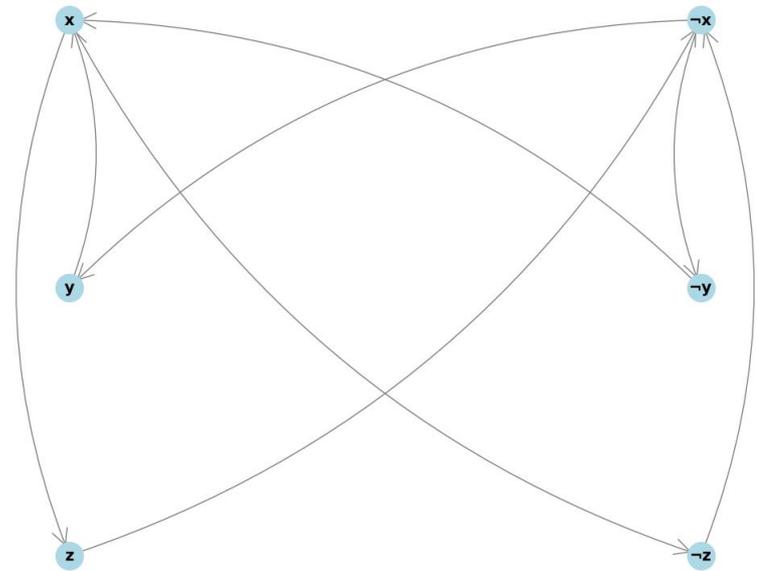
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Follow the graph we have

$$x \rightarrow \neg x, \neg x \rightarrow x$$

No possible assignment for x, UNSAT



3-SAT

$$(x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (y \vee \neg z) \wedge (\neg y \vee \neg z)$$

The implication graph idea cannot be applied anymore.

Any other ideas to solve this problem?

SAT is NP-complete

SAT was the first problem shown to be NP-complete [Cook 71]

2-SAT can be solved in polynomial (actually, linear) time

This has real implications

1. Essentially we don't know better than the naïve algorithm
2. A solver for SAT can be used to solve any other problem in the NP class with only polytime slowdown. i.e., makes a lot of sense to build SAT solvers
3. SAT/SMT solving is the cornerstone of *many* verification procedures

Stephen Cook, The complexity of theorem-proving procedures. In Proceedings of the third annual ACM symposium on theory of computing. STOC '71.

Online SAT solvers [edit]

- BoolSAT – Solves formulas in the DIMACS-CNF format or in a more
- Logictools – Provides different solvers in javascript for learning, co
- minisat-in-your-browser – Solves formulas in the DIMACS-CNF for
- SATRennesPA – Solves formulas written in a user-friendly way. Ru
- somerby.net/mack/logic – Solves formulas written in symbolic logic

Offline SAT solvers [edit]

- MiniSAT – DIMACS-CNF format and OPB format for its companion
- Lingeling – won a gold medal in a 2011 SAT competition.
 - PicoSAT – an earlier solver from the Lingeling group.
- Sat4j – DIMACS-CNF format. Java source code available.
- Glucose – DIMACS-CNF format.
- RSat – won a gold medal in a 2007 SAT competition.
- UBCSAT. Supports unweighted and weighted clauses, both in the
- CryptoMiniSat – won a gold medal in a 2011 SAT competition. C++ MiniSat 2.0 core, PrecoSat ver 236, and Glucose into one package, .
- Spear – Supports bit-vector arithmetic. Can use the DIMACS-CNF
 - HyperSAT – Written to experiment with B-cubing search space solver from the developers of Spear.
- BASolver
- ArgoSAT
- Fast SAT Solver – based on genetic algorithms.
- zChaff – not supported anymore.

thousands variables
millions of clauses
are solvable

[The international SAT Competitions web page](#)

Current Competition	
SAT 2019 Race	
Organizers	Marjin Heule, Matt Järvisalo, Martin Suda
Past Competitions	
SAT 2018 Competition	
Organizers	Marjin Heule, Matt Järvisalo, Martin Suda
Organizers	Marjin Heule, Matt Järvisalo, Tomáš Balyo
Stats used at SAT 2017	
Descriptions of the solvers and benchmarks	
Benchmarks	Available here
Solvers	Available here
SAT 2017 Competition	
Organizers	Marjin Heule, Matt Järvisalo, Tomáš Balyo
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SAT 2016 Competition	
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SAT 2015 Competition	
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