Lecture 23: Progress Verification

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Slides adapted from Prof. Sayan Mitra's slides in Fall 2021

Homework and final presentations

HW 3 due **4/27**

HW 4 due **5/11**

Final project presentation slides due **4/30, 8 am** (hard deadline, since presentations will start at 11 am)

Final project presentations:

Tuesday 11 am - 12:20 pm, ECEB 3015 (lecture time)

Friday 2 pm - 3:30 pm, ECEB **2015**

Schedule will be announced by the end of this week. If you cannot present on Friday, please let me and the TA Sanil (schawla7@illinois.edu) know by Thursday (4/25)

Final project report due: **5/11**

Visualizing CTL semantics

Path quantifiers

E: Exists some path

A: All paths

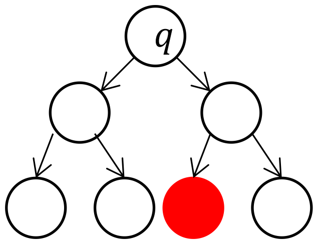
Temporal operators

X: Next state

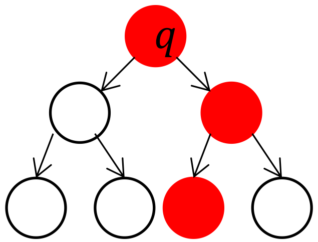
U: Until

F: Eventually

G: Globally (Always)



$q \vDash EF\, red$



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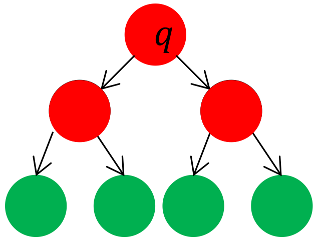
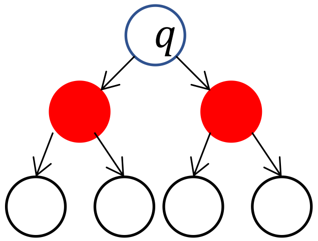
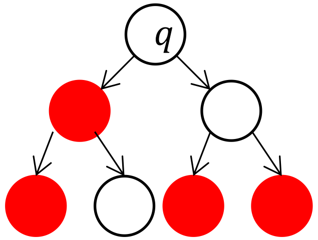


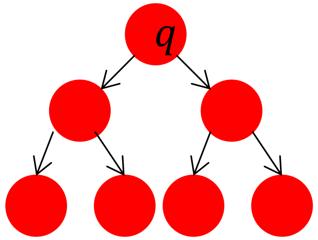
$q \vDash A\, [red\ U\ green]$



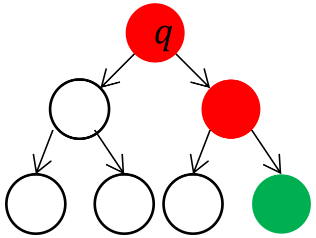
$q \vDash AX\, red$



$q \vDash AF\, red$



$q \vDash AG\, red$



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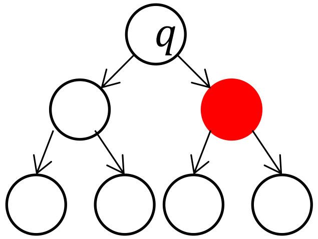

$q \vDash EX\, red$

Progress properties

- Every behavior system $\mathcal{A}$ will *eventually* reach a goal **goal**
- CTL: **AF goal**
- Dijkstra: From any state, (possibly >1 tokens) all executions get to a state with 1 token

Invariance/safety

- No behavior of **A** goes outside of **unsafe**
- CTL: **AG unsafe**
- Dijkstra: Starting a state with a 1 token, all executions have 1 token
- Finding a counterexample to safety does <u>not</u> prove progress

Proving termination for automata

- Automaton $\mathcal{A} = (V, \Theta,\ \boldsymbol{D})$
- Recall $\boldsymbol{D} \subseteq val(V) \times val(V)$
- Automaton terminates if it does not have any infinite executions

- **Definition:** A **well-founded relation** < on a set S is a binary relation $< \ \subseteq S\ \times S$ such that every subset $S' \subseteq S$ has a **least** element.
- In other words, there are no infinite decreasing chains of elements s_0, s_1, ..., with $s_{i+1} < s_i$.
- Example: totally order set, e.g., {1, 2, 3, ...} with the usual order
- Example: $S = \mathbb{Z}^+$ a < b iff a divides b and a ≠ b
- Example: $S = \{0,1\}^*$ a < b iff a is a proper substring of b
- Example: $S = \{-1,\ -2,\ -3, \ldots\}$, < is the usual order, then < is **not** a well-founded relation

Proving termination for automata

Theorem. Automaton $\mathcal{A} = (V, \Theta,\ \boldsymbol{D})$ terminates iff there exists a well-founded relation R such that $\boldsymbol{D} \cap Reach_{\mathcal{A}} \times Reach_{\mathcal{A}} \subseteq R$.

Proof. If there exists R and automaton does not terminate.

Then there exists an infinite sequence of states $s_0, s_1, \ldots,$ with $s_i\ \boldsymbol{D}\ s_{i+1}$. Since these are reachable states, $s_i\ R\ s_{i+1}$. This violates the definition of a well-founded relation.

Suppose $\mathcal{A}$ is terminating, we define

$R = \ \boldsymbol{D} \cap Reach_{\mathcal{A}} \times Reach_{\mathcal{A}}$

check that R is indeed well-founded (because $\boldsymbol{D}$ does not permit infinite sequences)

Ranking functions

Often the well-founded relation is defined in terms of a ***ranking function*** $f\colon \mathrm{val}(V) \to \mathbb{N}$ such that for any reachable $\boldsymbol{v} \in val(V)$ and $\boldsymbol{v}'\ such\ that\ (\boldsymbol{v}, \boldsymbol{v}') \in D,\ f(\boldsymbol{v}') < f(\boldsymbol{v})$

Here < is a the usual comparison on integers

Instead of $\mathbb{N}$, the ranking function could use any other range set with a lower bound

Example

```
automaton UpDown
  signature
    internal up(d:Nat), down

  variables
    internal x, y : Int

transitions
  internal up(d) where d = 1
    pre x > 0 ∧ y > 0
    eff x := x -1
      y := y + d

  internal down
    pre y > 0
    eff y := y -1
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Consider the ranking function $f(x, y) = 2x + y$

Check that for any transition $(x, y) \to (x', y')$

Up(1) $2x' + y' = 2(x - 1) + y + 1 = 2x + y - 1 = f(x, y) - 1 < f(x, y)$

Down: $2x' + y' = 2x + y - 1 = f(x, y) - 1 < f(x, y)$

Hence, the automaton terminates

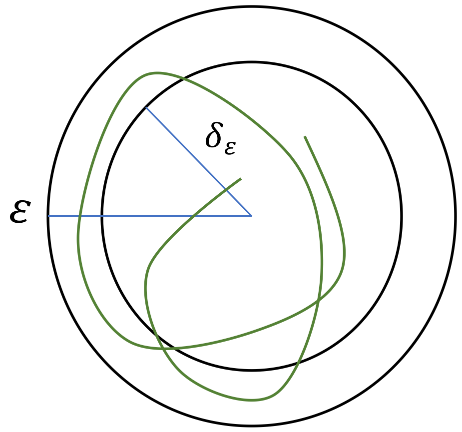
What if d > 1 ?

Recall Stability

- Time invariant autonomous systems (closed systems, systems without inputs)
- $\dot{x}(t) = f\big(x(t)\big),\ x_0 \in \mathbb{R}^n,\ t_0 = 0$ *(Eq. 1)*
- $\xi(t)$ is the solution
- $|\xi(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume **0** to be an equilibrium point of (1) with out loss of generality

Lyapunov stability

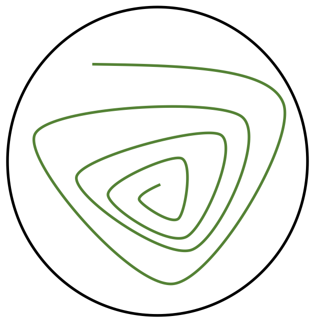
Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every if $|\xi(0)| \leq \delta_\varepsilon$ then for all t ≥ 0, $|\xi(t)| \leq \varepsilon$.



Asymptotically stability

The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and there exists $\delta_2 > 0$ such that for every if $|\xi(0)| \leq \delta_2$ then t $\to \infty$, $|\xi(t)| \to \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



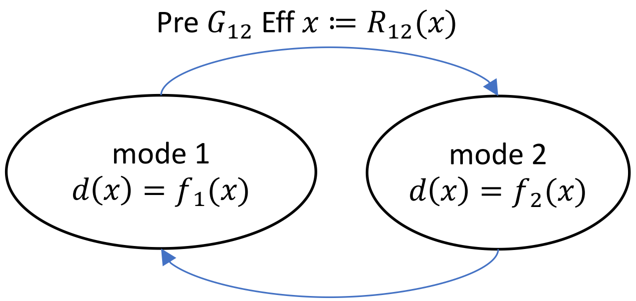
Verifying Stability for one dynamical system

Theorem. (Lyapunov) Consider the system (1) with state space $\xi(t) \in \mathbb{R}^n$ and suppose there exists a positive definite, continuously differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}$. The system is:

1. Lyapunov stable if $\dot{V}\big(\xi(t)\big) := \frac{\partial V}{\partial x} f(x) \leq 0$, for all $x \neq 0$
2. Asymptotically stable if $\dot{V}\big(\xi(t)\big) < 0$, for all $x \neq 0$
3. It is globally AS if *V* is also radially unbounded.

(V is radially unbounded if $\big||x|\big| \to \infty \Rightarrow V(x) \to \infty$)

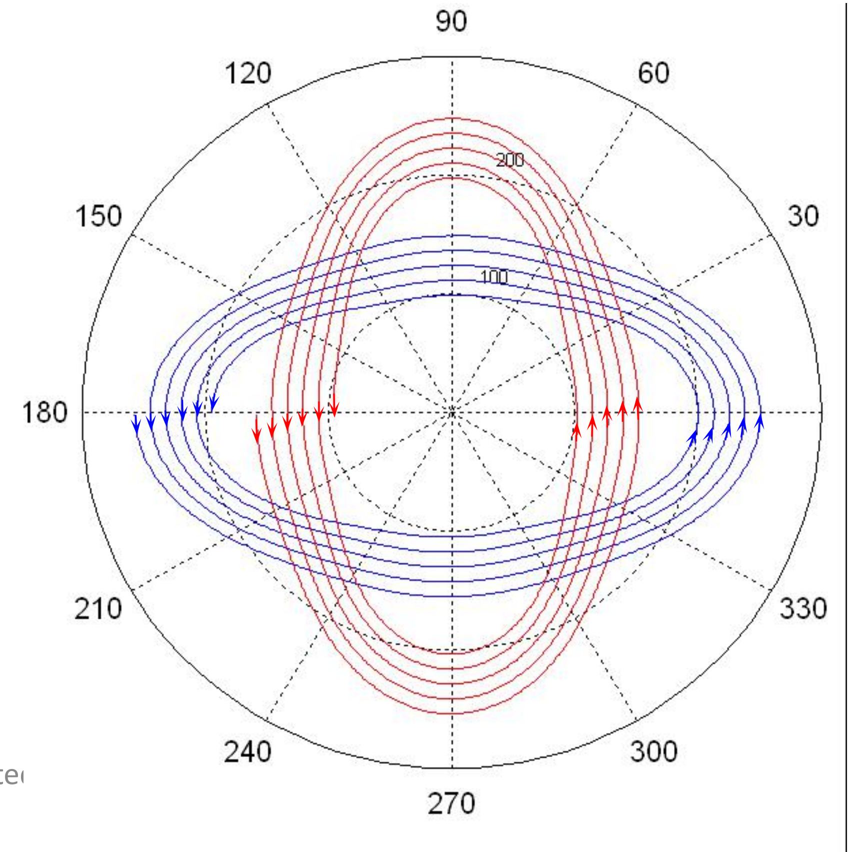
Defining stability of hybrid systems



- Hybrid automaton: $\mathbf{A} = \langle V, A, D, \mathrm{T}\rangle$
  - $V = X \cup \{\ell\}$
- Execution $\alpha = \tau_0 a_1 \tau_1 a_2 \ldots$
- Notation $\alpha(t)$: denotes the valuation $\beta.lstate$ where β is the longest prefix with $\beta.\mathrm{ltime} = t$
- $|\alpha(t)|$: norm of the continuous state X
- ***A*** is ***Lyapunov stable*** (at the origin) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every if $|\alpha(0)| \leq \delta_\varepsilon$ then for all t ≥ 0, $|\alpha(t)| \leq \varepsilon$.
- ***Asymptotically stable*** if it is Lyapunov stable and there exists $\delta_2 > 0$ such that for every if $|\alpha(0)| \leq \delta_2$ then t $\rightarrow \infty$, $|\alpha(t)| \rightarrow \mathbf{0}$.

Question:Stability Verification

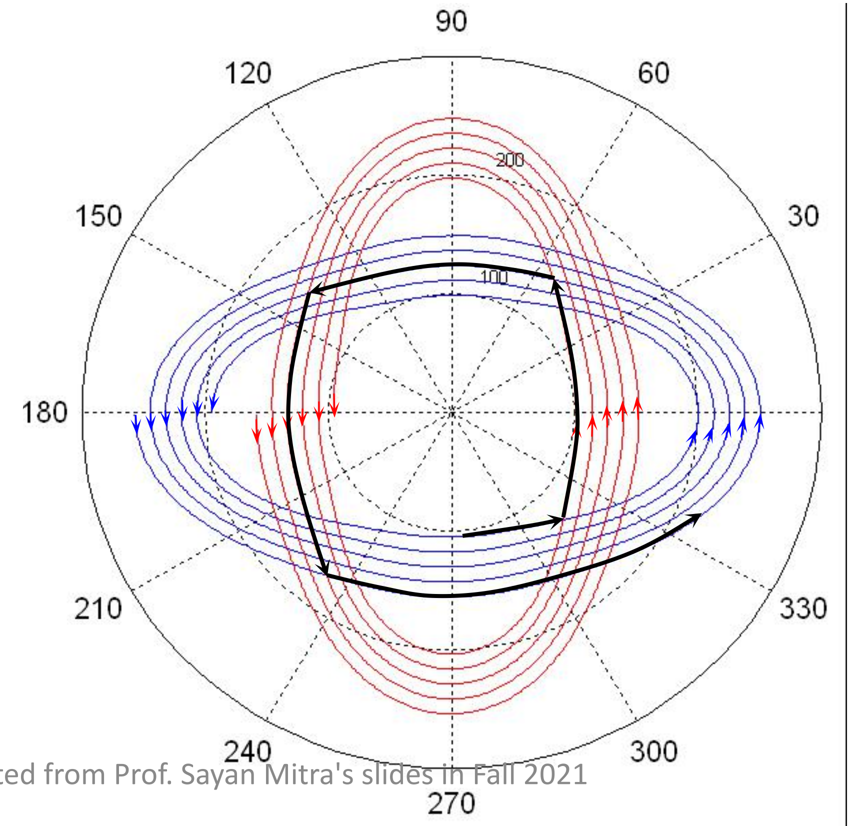
- If each mode is asymptotically stable then is ***A*** *also asymptotically stable?*



Slides adapte

Question:Stability Verification

- If each mode is asymptotically stable then is ***A*** *also asymptotically stable?*
- ***No***



Common Lyapunov Function

- If there exists positive definite continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$ and a positive definite function W: $\mathbb{R}^n \to \mathbb{R}$ such that for each mode $i, \frac{\partial V}{\partial t} f_i(x) <- W(x)$ for all $x \neq 0$ then V is called a common Lyapunov function for A.

- V is called a common Lyapunov function

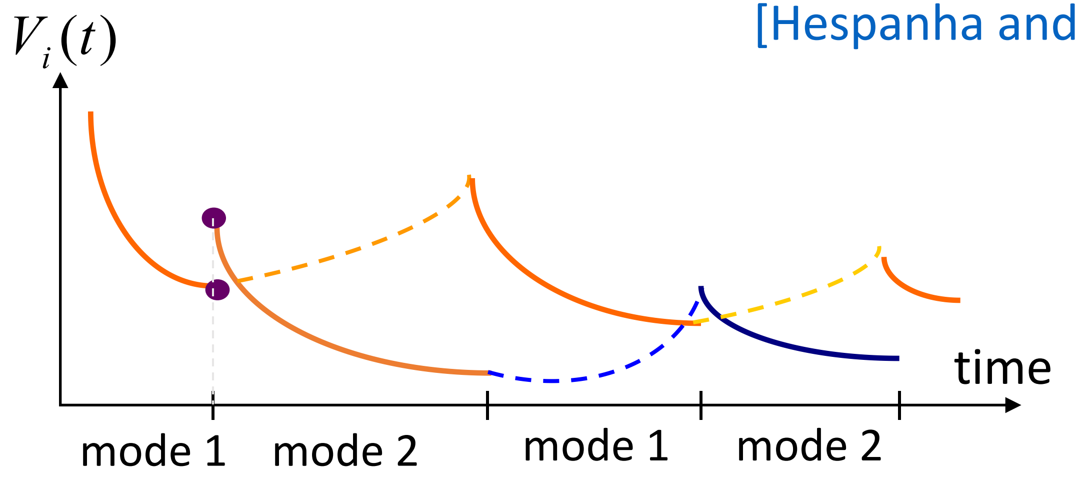
- **Theorem.** ***A*** is globally asymptotically stable if there exists a common Lyapunov function.

Multiple Lyapunov Functions

- In the absence of a common lyapunov function the stability verification has to rely of the discrete transitions.
- The following theorem gives such a stability in terms of *multiple Lyapunov function*.
- **Theorem** [Branicky] If there exists a family of positive definite continuously differentiable **Lyapunov** functions $V_i: \mathbb{R}^n \to \mathbb{R}$ and a positive definite function $\mathrm{W_i}: \mathbb{R}^n \to \mathbb{R}$ such that for any execution α and for any time t_1 t_2 $\alpha(t_1).\ell = \alpha(t_2).\ell = i$ and for all time $t \in (t_1, t_2), \alpha(t).\ell \neq i$
  - $V_i(\alpha(t_2).x) - V_i(\alpha(t_1).x) \leq - W_i(\alpha(t_1).x)$

Stability Under Slow Switching

[Hespanha and Morse`99]



- Average Dwell Time (ADT) characterizes rate of mode switches
- Definition: H has ADT T if there exists a constant N_0 such that for every execution α, the number of mode switches in α:

$$N(\alpha) \leq N_0 + \text{duration}(\alpha)/T.$$

Stability Under Slow Switching

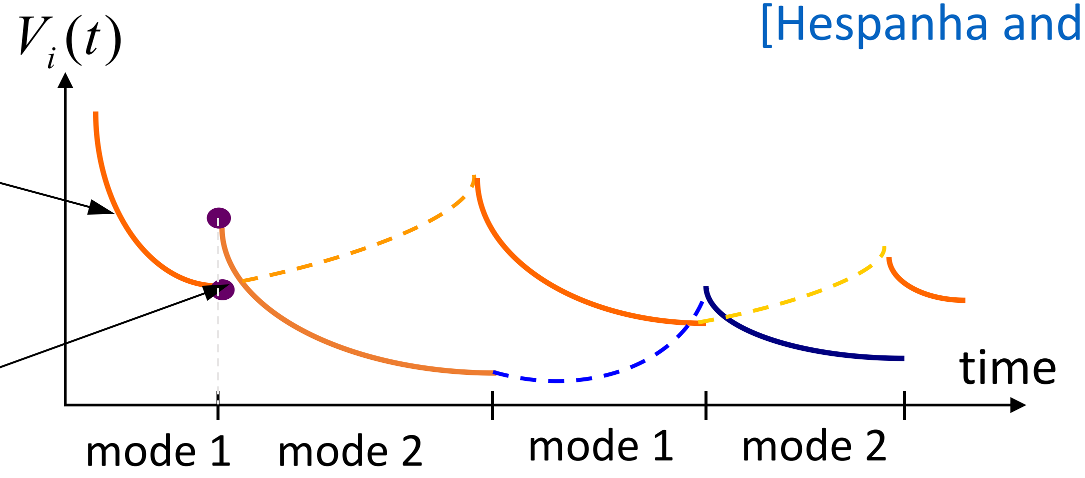
[Hespanha and Morse`99]

Stable in each mode

$$\frac{\partial V_i}{\partial x} \leq -2\lambda_0 V_i(x)$$

“Energy” may increase when switch but up to a factor of μ

$$V_2 \leq \mu\ V_1$$



- Average Dwell Time (ADT) characterizes rate of mode switches
- Definition: H has ADT T if there exists a constant N_0 such that for every execution α, the number of mode switches in α: N(α) ≤ N_0 + duration(α)/T.

N(α): **Theorem** [HM`99] H is asymptotically stable if its modes have a set of Lyapunov functions (μ, λ_0) and $\boxed{\text{ADT(H)} > \log \mu/\lambda_0}$.

Dwell time is long enough so energy can decrease sufficiently in each mode

Remarks about ADT theorem assumptions

1. If f_i is globally asymptotically stable, then there exists a Lyapunov function V_i that satisfies $\frac{\partial V_i}{\partial x} \leq -2\lambda_i V_i(x)$ for appropriately chosen $\lambda_i > 0$
2. If the set of modes is finite, choose λ_0 independent of i
3. The other assumption restricts the maximum increase in the value of the current Lyapunov functions over any mode switch, by a factor of μ.
4. We will also assume that there exist strictly increasing functions $\beta_1$and β_2 such that $\beta_1(|x|) \leq V_i(x) \leq \beta_2(|x|)$

Proof sketch

Suppose α is any execution of A.

Let $T = \alpha.ltime$ and $t_1, \ldots, t_{N(\alpha)}$ be instants of mode switches in α.

We will find an upper-bound on the value of $V_{\alpha(T).l}(\alpha(T).x)$

Define $W(t) = e^{2\lambda_0 t} V_{\alpha(t).l}(\alpha(t).x)$

W is non-increasing between mode switches $\left[\frac{\partial V_i}{\partial x} \leq -2\lambda_0 V_i(x)\right]$

That is, $W(t_{i+1}^-) \leq W\left(t_i\right)$

$W\left(t_{i+1}\right) \leq \mu W(t_{i+1}^-) \leq \mu W(t_i)$

Iterating this $N(\alpha)$ times: $W(T) \leq \mu^{N(\alpha)} W(0)$

$$e^{2\lambda_0 T} V_{\alpha(T).l}(\alpha(T).x) \leq \mu^{N(\alpha)} V_{\alpha(0).l}(\alpha(0).x)$$

$$V_{\alpha(T).l}(\alpha(T).x) \leq \mu^{N(\alpha)} e^{-2\lambda_0 T} V_{\alpha(0).l}(\alpha(0).x) = e^{-2\lambda_0 T + N(\alpha)\log\mu} V_{\alpha(0).l}(\alpha(0).x)$$

If α has ADT τ_a then, recall, $N(\alpha) \leq N_0 + T/\tau_a$ and $V_{\alpha(T).l}(\alpha(T).x) \leq e^{-2\lambda_0 T + (N_0 + T/\tau_a)\log\mu} V_{\alpha(0).l}(\alpha(0).x) \leq C\, e^{T(-2\lambda_0 + \log\mu/\tau_a)}$

If $\tau_a > \log\mu/2\lambda_0$ then second term converges to 0 as $T \to \infty$ then from assumption 4 it follows that α converges to 0.

Our goals in this course

Write programs (tools) that prove correctness

- *Understand fundamental limits of creating such tools*
- *Learn models of CPS at different levels of abstractions*
- *Gain research experience*

What we have learned in this course

- Satisifiability problems:
  - SAT (DPLL)
  - SMT (DPLL-T)
  - Neural network verification (CROWN bound propagation, branch-and-bound)
- Computation Tree Logic
  - CTL model checking
- Dynamical systems (reachability & invariance):
  - Linear/nonlinear systems, LTI systems
  - stability verification, Lyapunov functions
- Verification of hybrid automata and timed automata
  - Abstractions
  - Composition
  - Progress Analysis
  - Common/Multiple Lyapunov functions