Lecture 22: Composition of Hybrid Automata

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Slides adapted from Prof. Sayan Mitra's slides in Fall 2021

Homework and final presentations

HW 3 due **4/27**

HW 4 due **5/11**

Final project presentation slides due **4/30, 8 am** (hard deadline, since presentations will start at 11 am)

Final project presentations:

Tuesday 11 am - 12:20 pm, ECEB 3015 (lecture time)

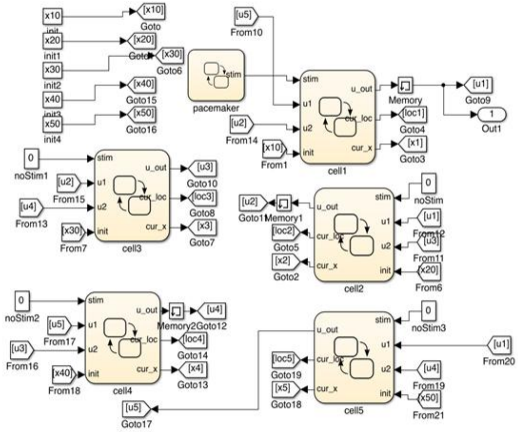
Friday 2 pm - 3:30 pm, ECEB **2015**

Schedule will be announced by the end of this week. If you cannot present on Friday, please let me and the TA Sanil (schawla7@illinois.edu) know by Thursday (4/25)

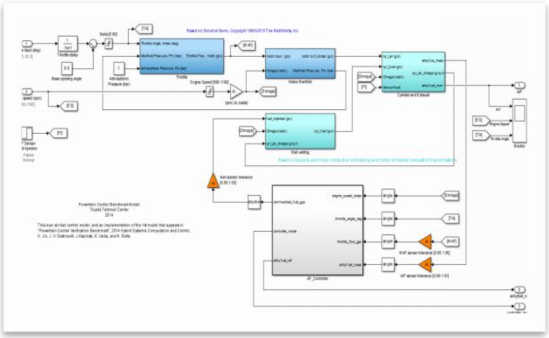
Final project report due: **5/11**

What is composition?

- Complex models and systems are built by putting together **components** or **modules**
- **Composition** is the mathematical operation of *putting together*
- Leads to precise definition of module **interfaces**
- What properties are preserved under composition?



model of a network of oscillators [Huang et. al 14]



Powertrain model from Toyota [Jin et al. 15]

- Give an example of how you’ve built something more complex from simple components

- Throughout the lecture, think if your notion of composition is captured by what we define

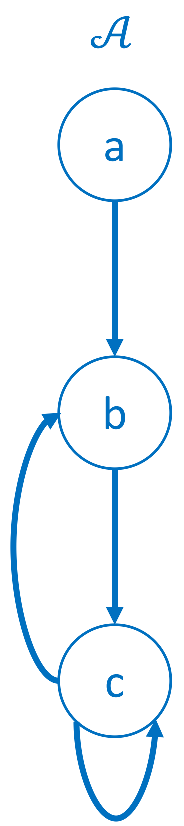
Outline

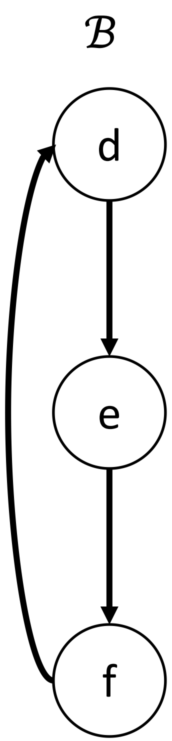
- Composition operation
  - Input/output interfaces
- I/O automata
- hybrid I/O automata
- Examples
- Properties of composition

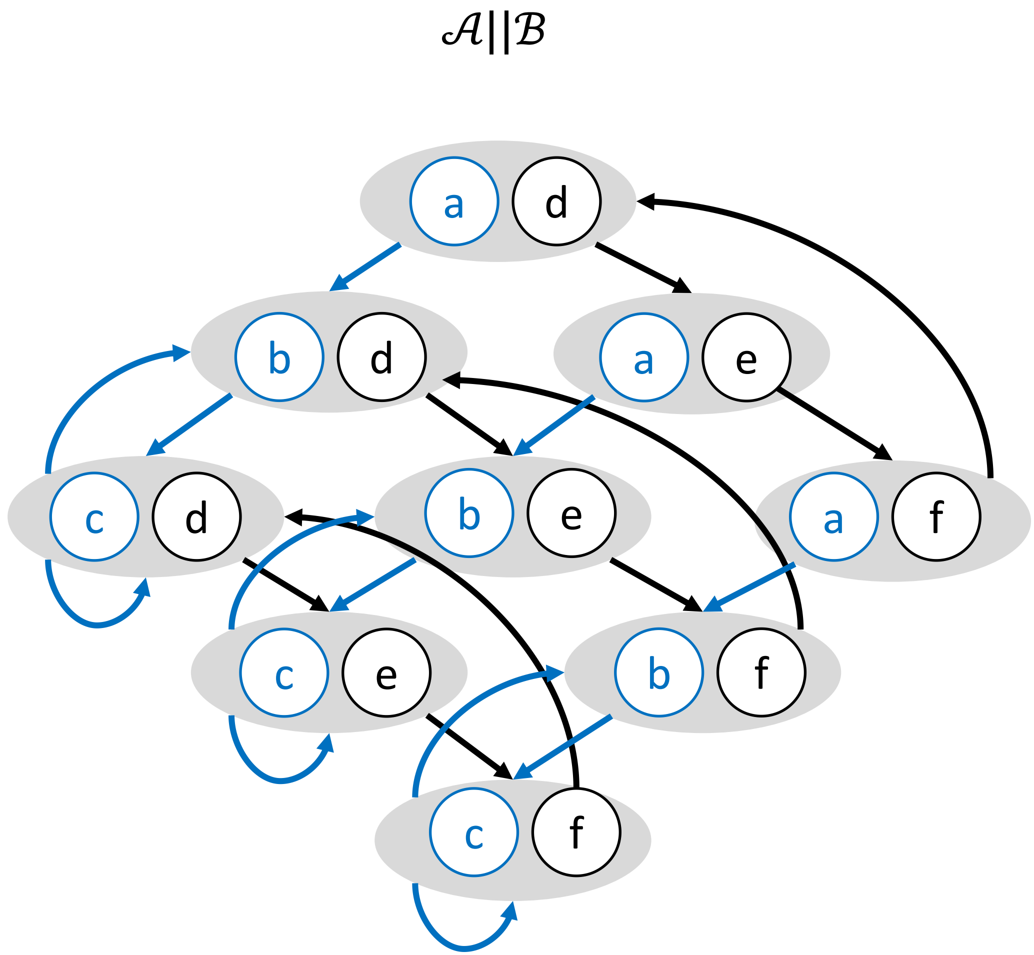
Composition of (discrete) automata

- Complex systems are built by "putting together" simpler subsystems
- Recall $\mathcal{A} = \langle X, \Theta, A, \boldsymbol{D} \rangle$
- $\mathcal{A} = \mathcal{A}_1 || \mathcal{A}_2$
  - $\mathcal{A}_1$, $\mathcal{A}_2$ are the *component automata* and
  - $\mathcal{A}$ is the *composed* automaton
  - $||$ symbol for the composition operator

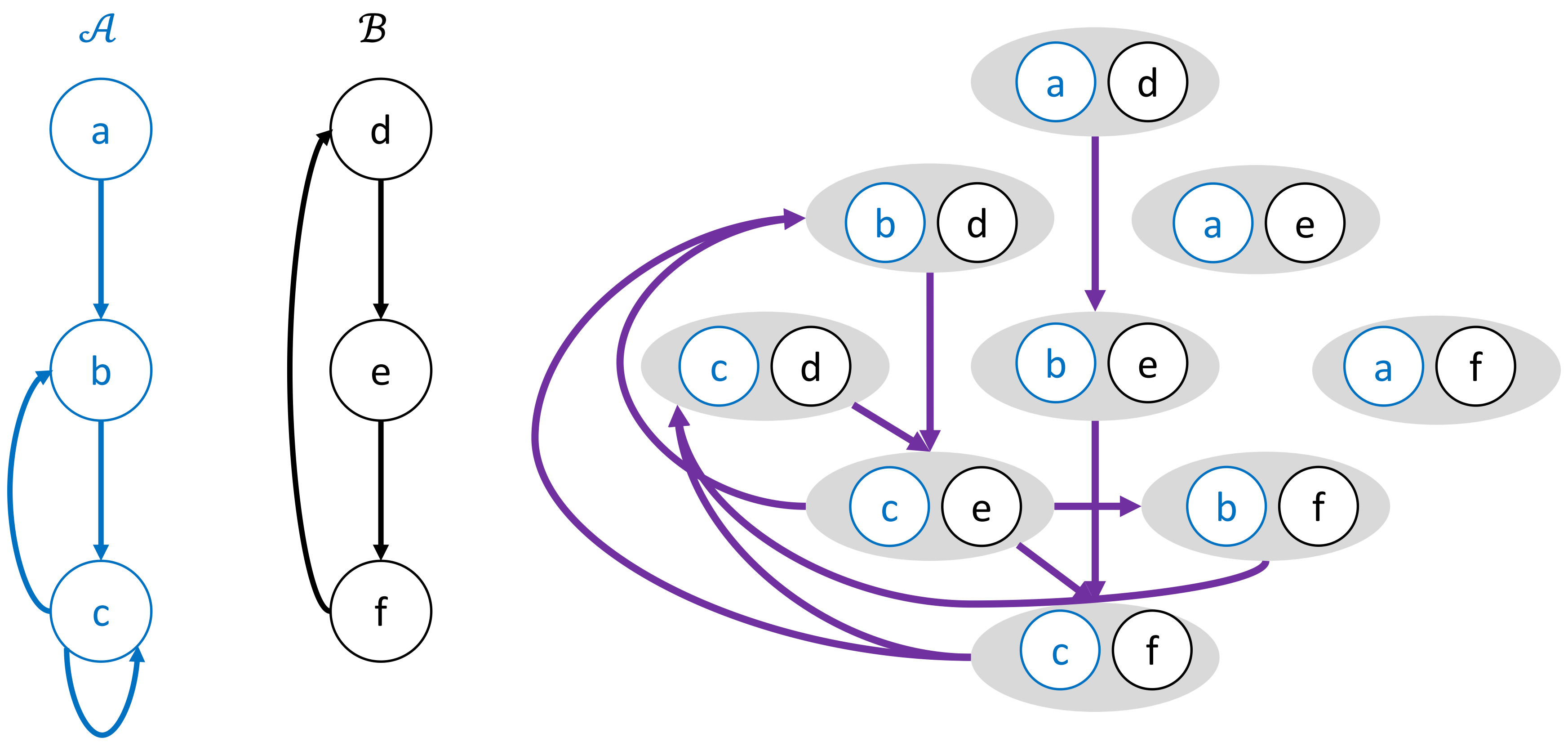
Composition: asynchronous modules







composition: modules synchronize

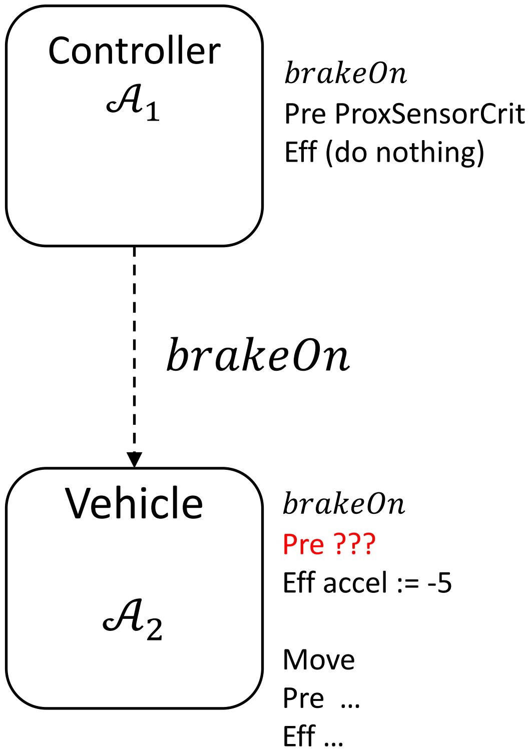


Composition of (discrete) automata

- More generally, some transitions of $\mathcal{A}$ and $\mathcal{B}$ may synchronize, while others may not synchronize
- Further, some transitions may be **controlled** by $\mathcal{A}$ which when occurs **forces** the corresponding transition of $\mathcal{B}$
- Thus, we will partition the set of actions A of $\mathcal{A} = \langle X, \Theta, A, \boldsymbol{D} \rangle$ into
  - H: **internal** (do not synchronize)
  - O: **output** (synchronized and controlled by $\mathcal{A}$)
  - I: **input** (synchronized and controlled by some other automaton)
- $A = H \cup O \cup I$
- This gives rise to **I/O automata** **[Lynch, Tuttle 1996]**

Reactivity: Input enabling

- Consider a shared action **brakeOn** controlled by $\mathcal{A}_1$ and listend-to or read by $\mathcal{A}_2$
- Input enabling ensures that when $\mathcal{A}_1$ and $\mathcal{A}_2$ are composed then $\mathcal{A}_2$ can react to **brakeOn**



Definition. An **input/output automaton** is a tuple $\mathcal{A} = \langle X, \Theta, A, \mathcal{D} \rangle$ where

- X is a set of names of variables
- $\Theta \subseteq val(X)$ is the set of initial states
- $A = I \cup O \cup H$ is a set of names of actions
- $\mathcal{D} \subseteq val(X) \times A \times val(X)$ is the set of transitions and $\mathcal{A}$ satisfies the **input enabling condition (E1):**

E1. For each $\boldsymbol{x} \in val(X)$, $a \in I$ there exists $\boldsymbol{x}' \in val(X)$ such that $\boldsymbol{x} \rightarrow_a \boldsymbol{x}'$

E1 ensures that the transition is well defined for every input action at any state

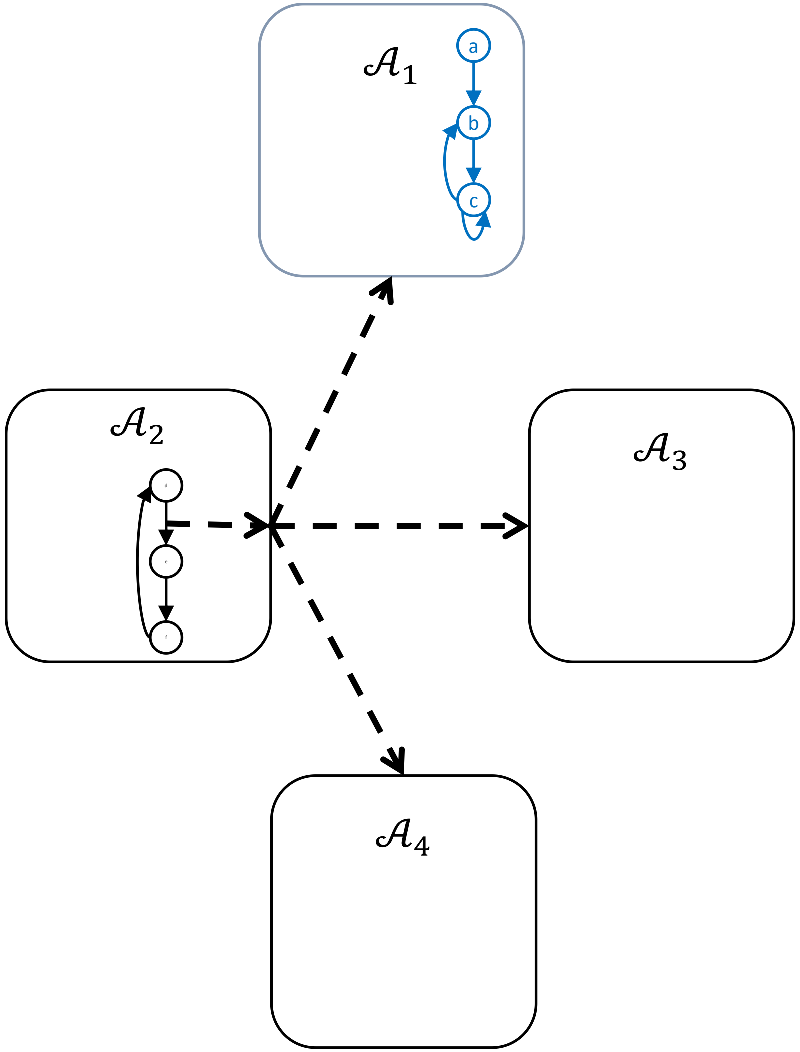
Compatibility IOA

A pair of I/O automata $\mathcal{A}_1$ and $\mathcal{A}_2$ are **compatible** if

$H_i \cap A_j = \emptyset$ no unintended interactions

$O_i \cap O_j = \emptyset$ no duplication of authority

Extended to collection of automata in the natural way



Composition of I/O automaton

Definition. For **compatible** automata $\mathcal{A}_1$ and $\mathcal{A}_2$ their composition $\mathcal{A}_1 \,||\, \mathcal{A}_2$ is the structure $\boldsymbol{\mathcal{A}}= (X, \Theta, A, \mathcal{D})$

- $X = X_1 \cup X_2$
- $\Theta = \{ \boldsymbol{x} \in val(X) |\ \forall\, i \ \in \{1,2\}\colon\ \boldsymbol{x}\lceil X_{\boldsymbol{i}} \in \Theta_i\}$

$$\left.\begin{aligned} &\bullet\ H = H_1 \ \cup H_2 \\ &\bullet\ \mathrm{O} = O_1 \ \cup O_2 \\ &\bullet\ \mathrm{I} = I_1 \ \cup I_2 \setminus O \end{aligned}\right\}\qquad A = H \cup O \cup I$$

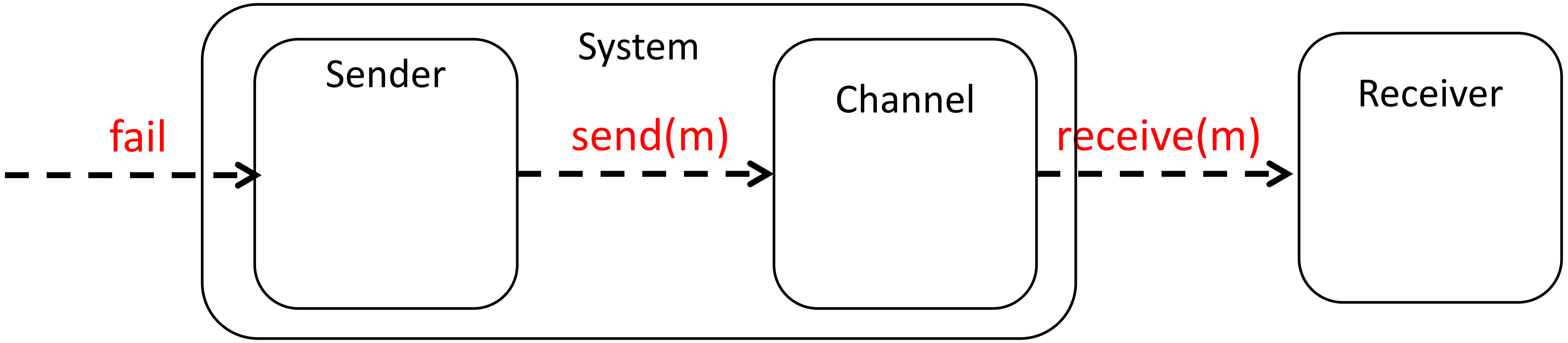
- $(\boldsymbol{x}, a, \boldsymbol{x}') \in \ \mathcal{D}$ *iff for* $i \in \{1,2\}$
  - $a \in A_i$ *and* $(\boldsymbol{x}\lceil X_i, a, \boldsymbol{x}'\lceil X_i) \in \ \mathcal{D}_i$
  - $a \notin A_i\ \boldsymbol{x}\lceil X_i = \boldsymbol{x}\lceil X_i$

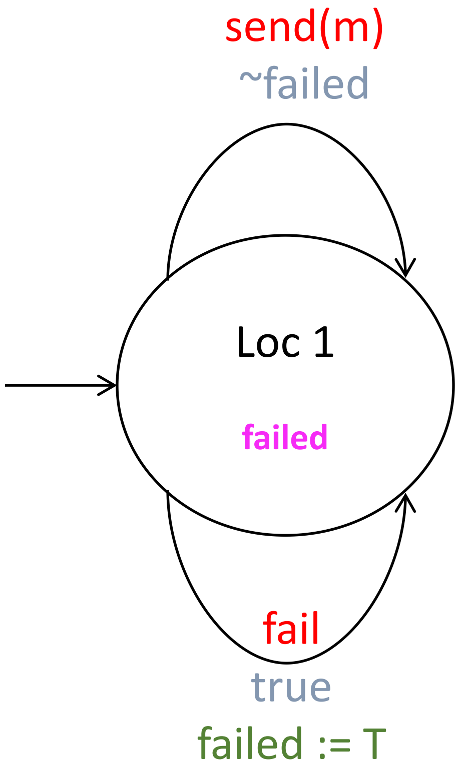
Theorem. The class of IO-automata is closed under composition. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible I/O automata then $\mathcal{A} = \mathcal{A}_1 || \mathcal{A}_2$ is also an I/O automaton.

Proof. Only 2 things to check

- (1) Input, output, and internal actions are disjoint---by construction
- (2) $\mathcal{A}$ satisfies **E1**. Consider any state $\boldsymbol{x} \in val(X_1 \cup X_2)$ and any input action a $\in I_1 \cup I_2 \setminus O$ such that a is enabled in $\boldsymbol{x}$**.**
- Suppose, w.lo.g. a $\in I_1$
- We know by **E1** of $\mathcal{A}_1$ that there exists $\boldsymbol{x}_1' \in val(X_1)$ such that $\boldsymbol{x}\lceil X_1 \rightarrow_a \boldsymbol{x}_1'$
- $a \notin O_2, I_2, H_2$ (by compatibility)
- Therefore, $\boldsymbol{x} \rightarrow_a (\boldsymbol{x}_1', x\lceil X_2)$ is a valid transition of $\mathcal{A}$ (by definition of composition)

Example: Sending process and channel





Automaton Sender(u)

variables internal

failed:Boolean := F

output send(m:M)

input fail

transitions:

output send(m)

pre ~failed

eff

input fail

pre true

eff failed := T

Does this automaton satisfy **input enabling condition (E1)?**

FIFO channel & Simple Failure Detector

Automaton Sender(u)

variables internal

failed:Boolean := F

output send(m:M)

input fail

transitions:

output send(m)

pre ~failed

eff

input fail

pre true

eff failed := T

Automaton Channel(M)

variables internal queue: **Queue**[M] := {}

actions input send(m:M)

output receive(m:M)

transitions:

input send(m)

pre true

eff queue := append(m, queue)

output receive(m)

pre $head(queue)=m$

eff queue := queue.tail

Automaton System(M)

variables queue: **Queue**[M] := {}, failed: Bool

actions input fail

output send(m:M), receive(m:M)

transitions:

output send(m)

pre ~failed

eff queue := append(m, queue)

output receive(m)

pre $head(queue)=m$

eff queue := queue.tail

input fail

pre true

eff failed := true

composing hybrid systems

Hybrid IO Automaton

In addition to interaction through shared actions hybrid input/output automata (HIOA) will allow interaction through shared variables

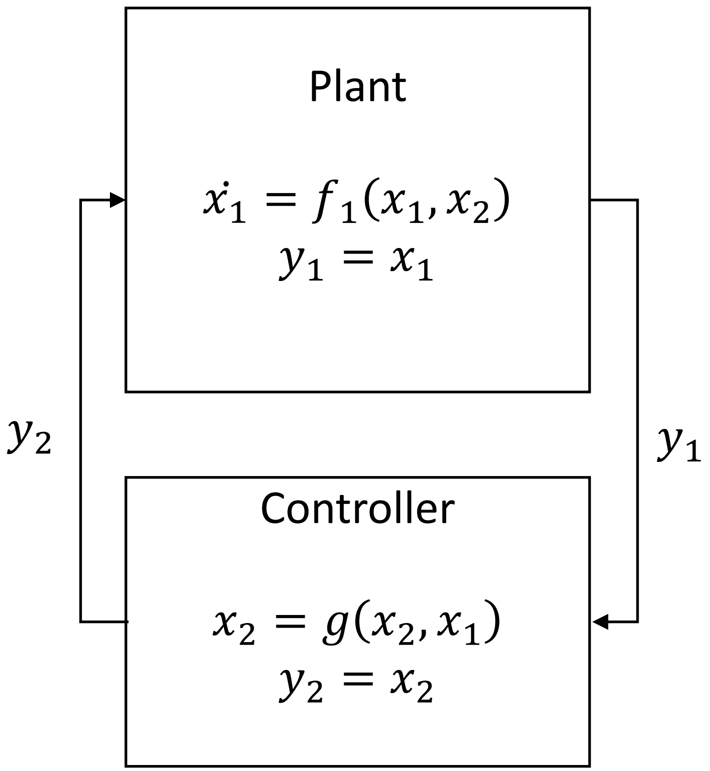
Recall a hybrid automaton $\mathcal{A} = \langle V, \Theta, A, \boldsymbol{D}, \boldsymbol{T} \rangle$

We will partition the set of variables V of $\mathcal{A}$ into

- X: **internal or state variables** (do not interact)
- Y: **output** variables
- U: **input** variables

- $V = X \cup Y \cup U$

This gives rise to **hybrid I/O automata (HIOA)** **[Lynch, Segala, Vaandrager 2002]**

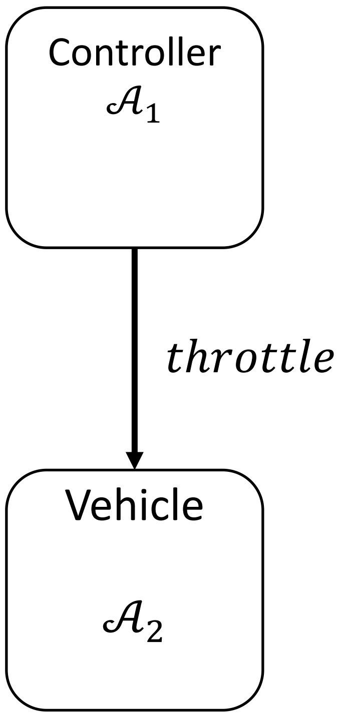


Reactivity: Input trajectory enabling

Consider a shared variable **throttle** controlled by $\mathcal{A}_1$ and listened-to or read by $\mathcal{A}_2$

Input trajectory enabling ensures that when $\mathcal{A}_1$ and $\mathcal{A}_2$ are composed then $\mathcal{A}_2$ can react to any signal generated by $\mathcal{A}_1$

If the trajectories of $\mathcal{A}_2$ are defined by ordinary differential equations, then input enabling is guaranteed if $\mathcal{A}_1$ only generates piece-wise continuous signals (throttle)



Definition. An **hybrid input/output automaton** is a tuple $\mathcal{A} = \langle V, \Theta, A, \boldsymbol{\mathcal{D}}, \boldsymbol{T} \rangle$ where

- $V = X \cup U \cup Y$ is a set of variables
- $\Theta \subseteq val(X)$ is the set of initial states
- $A = I \cup O \cup H$ is a set of actions
- $\mathcal{D} \subseteq val(X) \times A \times val(X)$ is the set of transitions
- $\boldsymbol{T}$ is a set of trajectories for V closed under prefix, suffix, and concatenation

E1. For each $\boldsymbol{x} \in val(X)$, $a \in I$ there exists $\boldsymbol{x}' \in val(X)$ such that $\boldsymbol{x} \rightarrow_a \boldsymbol{x}'$

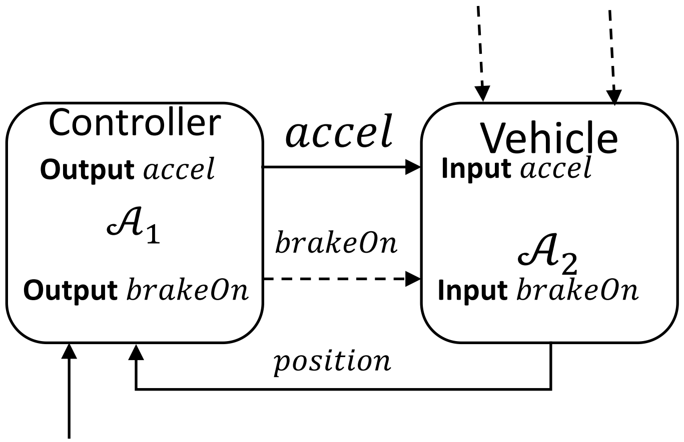
E2. For each $\boldsymbol{x} \in val(X)$, $\mathcal{A}$ should be able to react to any trajectory η of U.

i.e, $\exists\ \tau \in \boldsymbol{T}$ with $\tau.fstate = \boldsymbol{x}$ such that $\tau \downarrow U$ is a prefix of η, and either (a) $\tau \downarrow U = \eta$ or (b) τ is closed and some $a \in H \cup O$ is enabled at $\tau.lstate$. (**the HA cannot restrict its input trajectories**)

Compatibility of hybrid automata

- For the interaction of hybrid automata $\mathcal{A}_1$ and $\mathcal{A}_2$ to be well-defined we need to ensure that they have the right *interfaces*

- compatibility conditions

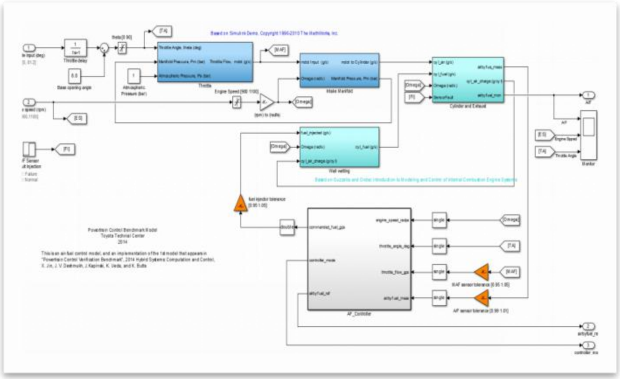


Compatibility HIOA

A pair of hybrid I/O automata $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible if

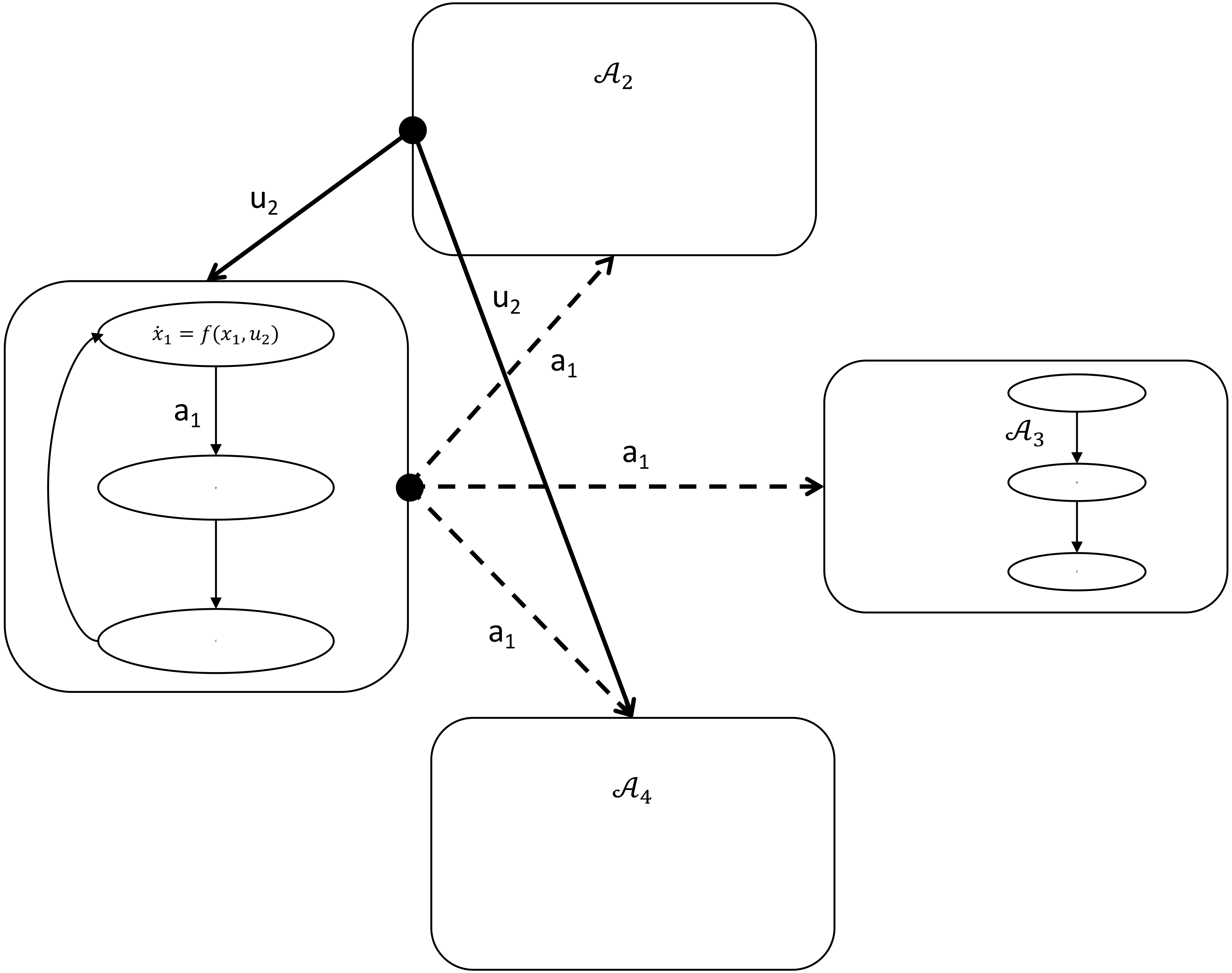
$H_i \cap A_j = \emptyset$	no unintended discrete interactions
$O_i \cap O_j = \emptyset$	no duplication of discrete authority
$X_i \cap V_j = \emptyset$	no unintended continuous interactions
$Y_i \cap Y_j = \emptyset$	no duplication of continuous authority

Extended to collection of automata in the natural way and captures most common notions of composition in, for example, Matlab/Simulink



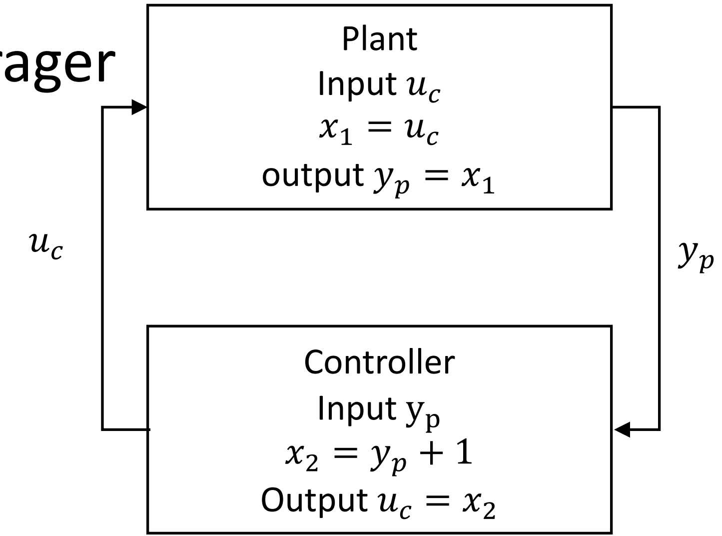
Composition

- For compatible $\mathcal{A}_1$ and $\mathcal{A}_2$ their composition $\mathcal{A}_1 \mid\mid \mathcal{A}_2$ is the structure $\boldsymbol{\mathcal{A}}= (V, \Theta, A, \mathcal{D}, \mathcal{T})$
- Variables $V = X \cup Y \cup U$
  - $X = X_1 \cup X_2, \mathrm{Y} = Y_1 \cup Y_2, U = U_1 \cup U_2 \setminus Y$
- $\Theta = \{ \boldsymbol{x} \in val(X) | \ \forall\, i \in \{1,2\}: \boldsymbol{x}\lceil X_{\boldsymbol{i}} \in \Theta_i\}$
- Actions $A = H \cup O \cup I$
  - $H = H_1 \cup H_2, \mathrm{O} = O_1 \cup O_2, \mathrm{I} = I_1 \cup I_2 \setminus O,$
- $(\boldsymbol{x}, a, \boldsymbol{x}') \in \mathcal{D}$ *iff for i* $\in \{1,2\}$
  - $a \in A_i$ *and* $(\boldsymbol{x}\lceil X_i, a, \boldsymbol{x}'\lceil X_i) \in \mathcal{D}_i$
  - $a \notin A_i\ \boldsymbol{x}\lceil X_i = \boldsymbol{x}\lceil X_i$
- $\mathcal{T}$: set of **trajectories** for V
  - $\tau \in \mathcal{T}$ iff $\forall\, i \in \{1,2\}, \tau \downarrow V_i \in \mathcal{T}_{\mathrm{i}}$

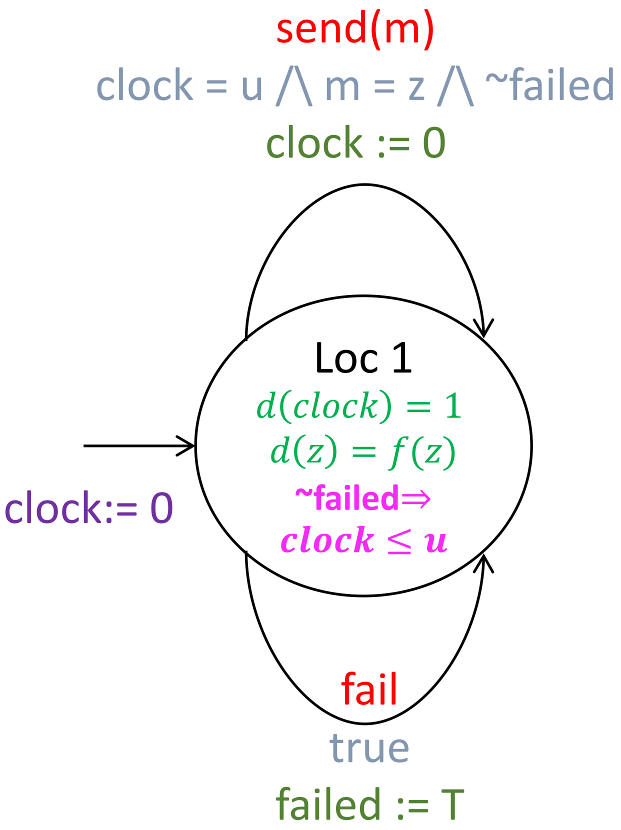

$\mathcal{A}_2$
u_2
u_2
a_1
$\dot{x}_1 = f(x_1, u_2)$
a_1
a_1
$\mathcal{A}_3$
a_1
$\mathcal{A}_4$

Closure under composition?

- Conjecture. The class of HIOA is closed under composition. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible HIOA then $\mathcal{A}_1 || \mathcal{A}_2$ is also a HIOA.
- Can we ensure that input trajectory enabled condition is satisfied in the composed automaton?
- No, in general (E2 does not always satisfy)
  - See "Hybrid I/O automata", by Nancy Lynch, Roberto Segala, Frits Vaandrager



Example 2: Periodically Sending Process



Automaton PeriodicSend(u)

variables internal

clock: Reals := 0, z:Reals, failed:Boolean := F

signature output send(m:Reals)

input fail

transitions:

output send(m)

pre clock = u /\ m = z /\ ~failed

eff clock := 0

input fail

pre true

eff failed := T

trajectories:

evolve d(clock) = 1, d(z) = f(z)

invariant failed \/ clock≤u

Time bounded channel & Simple Failure Detector

Automaton Timeout(u,M)
  variables internal suspected: Boolean := F,
    clock: Reals := 0
  signature input receive(m:M)
    output timeout
  transitions:
    input receive(m)
    pre true
    eff clock := 0; suspected := false;
    output timeout
    pre ~suspected /\ clock = u
    eff suspected := true
  trajectories:
    evolve d(clock) = 1
    invariant $clock \leq u \vee suspected$

Automaton Channel(b,M)
  variables internal queue: **Queue**[M,Reals] := {}
    clock: Reals := 0
  signature input send(m:M)
    output receive(m:M)
  transitions:
    input send(m)
    pre true
    eff queue := append(<m, clock+b>, queue)
    output receive(m)
    pre $head(queue)[1]=m \wedge$
      $head(queue)[2]=clock$
    eff queue := queue.tail
  trajectories:
    evolve d(clock) = 1
    invariant $\forall \langle m,d \rangle \in queue: d \geq clock$

Example 3: Oscillator and pulse generator

On

$d(now) = 1$

$u = 1$

$inv\ now \leq T_{on}$

pre $now \geq T_{off}$ **eff** $now := 0$

pre $now \geq T_{on}$ **eff** $now := 0$

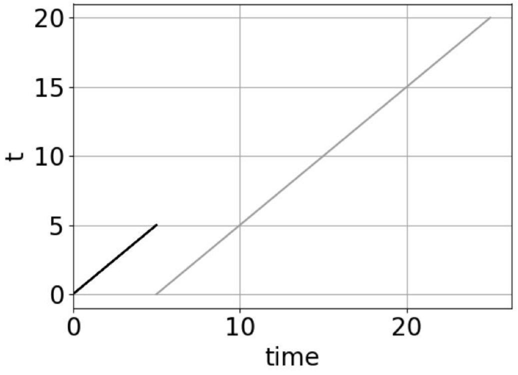
Off

$d(now) = 1$

$u = 0$

$inv\ now \leq T_{off}$

pulseGen



Mode

$$d(x_1) = -x_1(x_1^2 + 0.9x_1 + 0.9) - x_2 + \boxed{u}$$

$$d(x_2) = x_1 - 2x_2$$

Oscillator

Composed automaton

On,Mode

$d(now) = 1$

$d(x_1) = -x_1(x_1^2 + 0.9x_1 + 0.9) - x_2 + 1$

$d(x_2) = x_1 - 2x_2$

$now \leq T_{On}$

pre: $now \geq T_{Off}$ **eff**: $now := 0$

pre: $now \geq T_{On}$ **eff**: $now := 0$

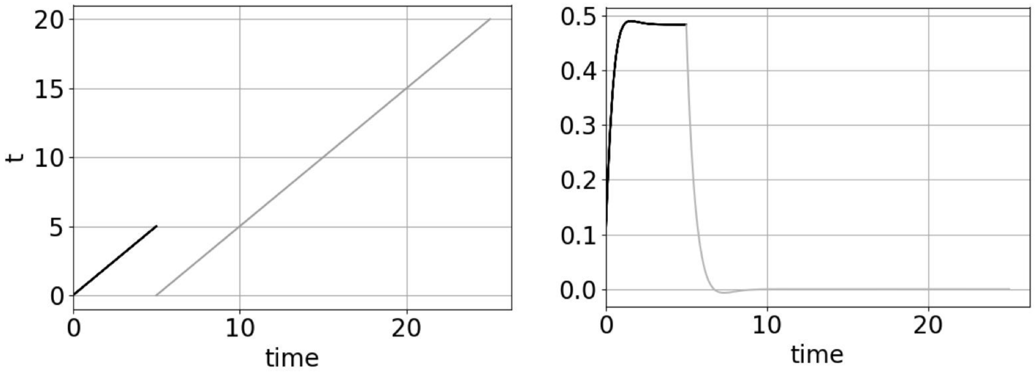
Off,Mode

$d(now) = 1$

$d(x_1) = -x_1(x_1^2 + 0.9x_1 + 0.9) - x_2 + 0$

$d(x_2) = x_1 - 2x_2$

$now \leq T_{Off}$



Cardiac oscillator network models, Grosu et al. CAV, HSCC 2007-2015

Restriction operation on exections

- Sometimes it is useful to restrict our attention to only some subset of variables and actions in an execution
- Recall the **restriction** operations $x\lceil V\ and\ \ \tau \downarrow V$
- Let $\alpha = \tau_0 a_1 \tau_1 a_2$ be an execution fragment of a hybrid automaton with set of variables V and set of actions A. Let A' be a set of actions and V' be a set of variables.

- **Restriction** of α to (A', V'), written as $\alpha\lceil(A',\ V')$ is the sequence defined inductively as:
  - $\alpha\lceil(A', V')$ = $\tau \downarrow V'$ if $\alpha = \tau$
  - $\alpha\ a\ \tau\ \lceil(A', V')$ =
    - $\alpha\ \lceil(A', V')\ a\ (\tau \downarrow V')$ if a $\in A'$
    - $\alpha\ \lceil(A', V')\ concat\ \ (\tau \downarrow V')$ if a $\notin A'$

- From the definition it follows $\alpha.lstate\ \lceil V' = \alpha\lceil(A', V').lstate$ for any A', V'

Properties of Compositions

Proposition. Let $\mathcal{A} = \mathcal{A}_1||\mathcal{A}_2$. α is an execution fragment of $\mathcal{A}$ iff

$\alpha\lceil(A_i, V_i)$, $i \in \{1,2\}$ are both execution fragments of $\mathcal{A}_i$.

- Proof of the forward direction. Fix α and i. We prove this by induction on the length of α.

- Base case: $\alpha = \tau$. $\alpha\lceil(A_i, V_i) = \tau \downarrow V_i$ by definition of composition $\tau \downarrow V_i \in T_i$. So, $\tau \downarrow V_i \in Frag_i$

- $\alpha = \alpha'\, a\, \tau\, \lceil(A_i, V_i)$ and $a \in A_i$ and by induction hypothesis $\alpha'\lceil(A_i, V_i) \in Frag_i$. Let $\alpha'\lceil(A_i, V_i).\mathrm{lstate} = v$. By the definition of composition: $\tau \downarrow V_i \in T_i$.
    - It remains to show that $v\lceil V_i \rightarrow_a (\tau \downarrow V_i).fstate$. Since $a \in A_i$, by the definition of composition: $\alpha'\lceil(A_i, V_i).\mathrm{lstate} \rightarrow_a \tau \downarrow V_i.fstate$

- $\alpha = \alpha'\, a\, \tau\, \lceil(A_i, V_i)$ and $a \notin A_i$ and by induction hypothesis $\alpha'\lceil(A_i, V_i) \in Exec_i$. Let τ' be the last trajectory in that execution.
    - Since $a \notin A_i$, by the definition of composition: $\tau'.lstate = \tau \downarrow V_i.fstate$. By concatenation closure of T_i, it follows that $\tau' concat\, \tau \downarrow V_i \in \mathrm{T}_i$. Therefore $\alpha\lceil(A_i, V_i) \in Exec_i$.

properties of executions of composed automata

- α is an *execution* iff $\alpha\lceil(A_i, V_i),\ i \in \{1,2\}$ are both executions.
- α is *time bounded* iff $\alpha\lceil(A_i, V_i),\ i \in \{1,2\}$ are both time bounded.
- α is *admissible* (infinite duration) iff $\alpha\lceil(A_i, V_i),\ i \in \{1,2\}$ are both admissible.
- α is *closed* (finite time with final trajectory) iff $\alpha\lceil(A_i, V_i),\ i \in \{1,2\}$ are both closed.
- α is *non-Zeno* iff $\alpha\lceil(A_i, V_i),\ i \in \{1,2\}$ are both time non-Zeno.

Summary

- Composition operation
  - I/O interfaces: actions and variables
  - Reactivity/input enabling
  - (non) Closure under composition
- Properties of executions preserved under composition
- Inductive invariants