Lecture 14: Modeling Cyberphysical Systems
Hybrid systems

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Slides adapted from Prof. Sayan Mitra’s slides in Fall 2021
Deadlines

Homework 2 due 3/10, 11:59 pm CT

Two writing problems + two programming problems

YOU SHOULD START TODAY! (if you haven’t started working on it)
Class project

Over 25+ project proposals received

You might be contacted by me for (optionally) forming a team of 2 if your proposed project is the same as another student.

Midterm project presentation: 3/26 and 3/28.

• **5-min** presentations for each team. (5% of final grade)

• Slides due on 3/25. We will compile all slides into a single file for fast switching.

• Presentation includes problem setting, proposed methodology, and initial results.

• I will give you some feedback after your presentation (1-min)

In addition: each person should give feedback for 5 projects that interest you most on each day. (total 10 feedbacks; count towards the **5% class participation** grades. Feedback will be submitted to Canvas and also shared to peers. Feedback template will be given.)
Review: dynamical systems

Behaviors of physical processes are described in terms of instantaneous laws.

Common notation: \[ \frac{dx(t)}{dt} = f(x(t), u(t), t) \quad \text{Eq. (1)} \]

where time \( t \in \mathbb{R} \); state \( x(t) \in \mathbb{R}^n \); input \( u(t) \in \mathbb{R}^m \); \( f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n \)

Example. \[ \frac{dx(t)}{dt} = v(t) \ ; \quad \frac{dv(t)}{dt} = -g \]

**Initial value problem:** Given system (1) and initial state \( x_0 \in \mathbb{R}^n, \ t_0 \in \mathbb{R} \), and input \( u: \mathbb{R} \rightarrow \mathbb{R}^m \), find a state trajectory or solution of (1).
Review: Linear time invariant system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

Define Matrix exponential:

\[ e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k \]

Theorem. \( \xi(t, x_0, u) = \Phi(t)x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \)

Here \( \Phi(t) = e^{At} \) is the state-transition matrix

Zero input \quad Zero state
Review Lyapunov stability

Lyapunov stability: The system \( \dot{x}(t) = f(x(t)) \) is said to be **Lyapunov stable** (at the origin) if

\[ \forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ such that } |x_0| \leq \delta_\varepsilon \Rightarrow \forall t \geq 0, \ |\xi(x_0, t)| \leq \varepsilon. \]

"if we start the system close enough to the equilibrium, it remains close enough"

How is this related to invariants and reachable states?
Review: Asymptotically stability

The system $\dot{x}(t) = f(x(t))$ is said to be **Asymptotically stable (at the origin)** if it is Lyapunov stable and

$$\exists \delta_2 > 0 \text{ such that } \forall \|x_0\| \leq \delta_2 \text{ as } t \to \infty, \ |\xi(x_0, t)| \to 0.$$ 

If the property holds for any $\delta_2$ then **Globally Asymptotically Stable**
Review: Verifying Stability

**Theorem.** (Lyapunov) Consider the system (1) with state space $x \in \mathbb{R}^n$ and suppose there exists a positive definite, continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$. The system is:

1. **Lyapunov stable** if $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0$, for all $x \neq 0$
2. **Asymptotically stable** if $\dot{V}(x) < 0$, for all $x \neq 0$
3. It is globally AS if $V$ is also radially unbounded.

   ($V$ is radially unbounded if $||x|| \to \infty \Rightarrow V(x) \to \infty$)
Today’s lecture: Hybrid systems

- Discrete transition systems, automata
- Dynamical systems
  - Differential inclusions
- Markov chains
- Probabilistic automata, Markov decision processes (MDP)
- Continuous time, continuous state MDPs
- Stochastic Hybrid systems

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Outline

• Hybrid automata
• Executions
• Special kinds of executions: admissible, Zeno
• Hybrid stability
Recall from Lecture 2. language defines an automaton

An **automaton** is a tuple $\mathcal{A} = \langle X, \Theta, A, D \rangle$ where

- $X$ is a set of names of variables; each variable $x \in X$ is associated with a type, $\text{type}(x)$
  - A valuation for $X$ maps each variable in $X$ to its type
  - Set of all valuations: $\text{val}(X)$ this is sometimes identified as the **state space** of the automaton
- $\Theta \subseteq \text{val}(X)$ is the set of **initial** or **start states**
- $A$ is a set of names of **actions** or **labels**
- $D \subseteq \text{val}(X) \times A \times \text{val}(X)$ is the set of **transitions**
  - a transition is a triple $(u, a, u')$
  - We write it as $u \xrightarrow{a} u'$

```plaintext
automaton DijkstraTR(N: Nat, K: Nat), where K > N
type ID: enumeration [0,...,N-1]
type Val: enumeration [0,...,K]
actions
  update(i:ID)
variables
  x:[ID -> Val]
transitions
  update(i:ID)
    pre i = 0 \land x[i] = x[N-1]
    eff x[i] := (x[i] + 1) % K
  update(i:ID)
    pre i >0 \land x[i] = x[i-1]
    eff x[i] := x[i-1]
```
Bouncing Ball: Hello world of CPS

**automaton** Bouncingball(c,h,g)

**variables:** \( x: \text{Reals} := h \), \( v: \text{Reals} := 0 \)

**actions:** bounce

**transitions:**
- bounce
  
  **pre** \( x = 0 \land v < 0 \)
  
  **eff** \( v := -cv \)

**trajectories:**
- freefall
  
  \( d(x) = v \)
  
  \( d(v) = -g \)
  
  \( x \geq 0 \)

Graphical Representation used in many articles

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Trajectories

Given a set of variables $X$ and a **time interval** $J$ which can be of the form $[0, T]$, $[0, T)$ or $[0, \infty)$, a **trajectory** for $X$ is a function $\tau: J \rightarrow \text{val}(X)$

We will specify $\tau$ as **solutions of differential equations**

The **first state** of a trajectory $\tau$. $\text{fstate}: = \tau(0)$

If $\tau$ is right closed then the **limit state** of a trajectory $\tau$. $\text{lstate} = \tau(T)$

If $\tau$ is finite then **duration** of $\tau$ is $\tau.\text{dur} = T$

The domain of $\tau.\text{dom} = J$

A **point trajectory** is a trajectory with $\tau.\text{dom} = [0,0]$

Operations on trajectories: prefix, suffix, concatenation

A **prefix** $\tau'$ of a trajectory $\tau$: $[0, T] \rightarrow \text{val}(X)$, is a function $\tau': [0, T'] \rightarrow \text{val}(X)$ such that $T' \leq T$ and $\tau'(t) = \tau(t)$ for all $t \in \tau'.\text{dom}$
Hybrid Automaton

\[ \mathcal{A} = (X, \Theta, A, \mathcal{D}, \mathcal{T}) \]

- \( X \): set of \textbf{state variables}
  - \( Q \subseteq \text{val}(X) \) set of \textbf{states}
- \( \Theta \subseteq Q \) set of \textbf{start states}
- set of \textbf{actions}, \( A = E \cup H \)
- \( \mathcal{D} \subseteq Q \times A \times Q \)
- \( \mathcal{T} \): set of \textbf{trajectories} for \( X \) which is closed under prefix, suffix, and concatenation

**Closed**: For every \( \tau \) in \( \mathcal{T} \), any suffix \( \tau' \) or \( \tau \) is also in \( \mathcal{T} \)
Semantics: Executions and Traces

• An execution fragment of $\mathcal{A}$ is an (possibly infinite) alternating $(A, X)$-sequence $\alpha = \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$ where
  
  $\forall i, \tau_i. \text{lstate} \xrightarrow{a_{i+1}} \tau_{i+1}. \text{fstate}$

• If $\tau_0. \text{fstate} \in \Theta$ then $\alpha$ is an execution

• $\text{Execs}_\mathcal{A}$ set of all executions

• The first state of an execution $\alpha$ is:
  
  $\alpha. \text{fstate} = \tau_0. \text{fstate}$

• If the execution $\alpha$ is finite and closed:
  
  $\tau_0 a_1 \tau_1 a_2 \tau_2 \ldots \tau_k$ then $\alpha. \text{lstate} = \tau_k. \text{lstate}$

• A state $x$ is reachable if there exists an execution $\alpha$ with $\alpha. \text{lstate} = x$
Semantics: Executions and Traces

- An **execution fragment** of $\mathcal{A}$ is an (possibly infinite) alternating $(A, X)$-sequence $\alpha = \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$ where
  - $\forall i, \tau_i. lstate \xrightarrow{a_{i+1}} \tau_{i+1}. fstate$

- If $\tau_0.fstate \in \Theta$ then $\alpha$ is an **execution**

- $\text{Execs}_{\mathcal{A}}$ set of all executions

- The first state of an execution $\alpha$ is:
  $\alpha.fstate = \tau_0.fstate$

- If the execution $\alpha$ is **finite and closed**:
  $\tau_0 a_1 \tau_1 a_2 \tau_2 \ldots \tau_k$ then $\alpha.lstate = \tau_k.lstate$

- A state $x$ is reachable if there exists an execution $\alpha$ with $\alpha.lstate = x$
Thermostat variations

**automaton** Thermostat$(u, l, K, h : \text{Real})$ where $u > l$

**type** Status enumeration [on, off]

**actions**

`turnOn; turnOff;`

**variables**

$x: \text{Real} := l \quad \text{loc: Status} := \text{on}$

**transitions**

`turnOn`  
`pre $x \leq l \land \text{loc=off}$`  
`eff loc := on`

`turnOff`  
`pre $x \geq u \land \text{loc=on}$`  
`eff loc := off`

**trajectories**

`modeOn`  
`evolve $d(x) = K(h - x)$`  
`invariant loc = on \land x \leq u`

`modeOff`  
`evolve $d(x) = -Kx$`  
`invariant loc = off \land x \geq l`

- Determinism vs nondeterminism
- mode invariants
Another Example: Periodically Sending Process

**Automaton** PeriodicSend\((u, f)\)

**variables:**
- clock: Reals := 0, z:Reals

**actions:** send\((m:\text{Reals})\)

**transitions:**
- send\((m)\)
  - **pre** \((\text{clock} = u) \land (m = z)\)
  - **eff** clock := 0

**trajectories:**
- Loc1
- evolve \(d(\text{clock}) = 1, d(z) = f(z)\)
- invariant clock \(\leq u\)
Special kinds of executions

• **Infinite**: Infinite sequence of transitions and trajectories
  \[ \tau_0, a_1, \tau_1, a_2, \tau_2, \ldots \]

• **Closed**: Finite with final trajectory with closed domain
  \[ \tau_0, a_1, \tau_1, a_2, \tau_2, \ldots, \tau_k \text{ and } \tau_k \cdot dom = [0, T] \]

• **Admissible**: Infinite duration
  • May or may not be infinite
  • \[ \tau_0, a_1, \tau_1, a_2, \tau_2, \ldots \]
  • \[ \tau_0, a_1, \tau_1, a_2, \tau_2, \ldots, \tau_k \text{ with } \tau_k \cdot dom = [0, \infty) \]

• **Zeno**: Infinite but not admissible
  • Infinite number of transitions in finite time

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Zeno’s Paradox

Achilles runs 10 times faster than the tortoise, but the turtle gets to start 1 second earlier. Can Achilles ever catch Turtle?

After $1/10^{th}$ of a second, Achilles reaches where the Turtle (T) started, and T has a head start of $1/10^{th}$ second. After another $1/100^{th}$ of a second, A catches up to where T was at $t=1/10$ sec, but T has a head start of $1/100^{th}$ ...

T is always ahead ...

Lesson: Mixing discrete transitions with continuous motion can be tricky!
Zeno’s Paradox: bouncing ball

• Infinite number of bounces to come to reset!
• Yet it needs finite time to come to reset

• The time for each bounce (e.g., $t_1 - t_0$, $t_2 - t_1$, etc) is a geometric series

Read: https://www.millersville.edu/physics/experiments/045/
Defining stability of hybrid automata

• Given an *admissible* (infinite duration) execution \( \alpha = \tau_0 \ a_1 \ \tau_1 a_2 \tau_2 \ldots \)

• To reason about stability of an execution, we would like to view an execution as \( \alpha: [0, \infty) \rightarrow val(X) \)

• But, how can we define \( \alpha(t) \)?

• define \( \alpha(t_s) = \alpha'.lstate \) where \( \alpha' \) is the longest prefix of \( \alpha \) with \( \alpha'.ltime = t_s \)
Defining stability of hybrid automata

• An hybrid automata is globally uniformly asymptotically stable if:
• For any $\varepsilon > 0$ and any state $v_0$, there is a time $T$ such that for any execution fragment $\alpha$ starting from $v_0$, for all $t \geq T$, $||\alpha(t)|| < \varepsilon$
Hybrid Instability

Can we verify the stability of a hybrid system by just verifying the stability of each mode?

Each of the modes of a walking robot are asymptotically stable.
Is it possible to switch between them to make the system unstable?
By switching between them, the system becomes unstable.
Rimless wheel: another example of hybrid system
Rimless wheel

automaton RimlessWheel(\( \alpha, \mu : \text{Real}, \ n : \text{Nat} \))
const \( \beta : \text{Real} := 2 \pi / n \)
type Spokes: enumeration [1,...,n]
actions
impact
variables
pivot: Spokes :=1
\( \theta : \text{Real} := 0 \)
\( \omega : \text{Real} := 0 \)
transitions
impact
\( \text{pre } \theta \geq \beta / 2 \)
\( \text{eff } \text{pivot} := \text{pivot} + 1 \text{ mod } n \)
\( \theta := - \beta / 2 \)
\( \omega := \mu \omega \)

trajectories
swing
evolve
\( \text{d}(\theta) = \omega \)
\( \text{d}(\omega) = \sin (\theta + \alpha) \)
invariant \( \theta \leq \frac{\beta}{2} \)
Invariants and reachability

• A state $x$ of automaton $\mathcal{A}$ is **reachable** if there exists an execution $\alpha$ with $\alpha.lstate = x$

• $\text{Reach}_\mathcal{A}(\Theta)$ is the set of all reachable state from $\Theta$

• $\text{Reach}_\mathcal{A}(\Theta, T)$ is the set of states reachable within time $T$

• $\text{Reach}_\mathcal{A}(\Theta, k)$ is the set of states reachable within $k$ transitions

• $\text{Reach}_\mathcal{A}(\Theta, T, k)$ is the set of states reachable up to time $k$ transitions and time $T$

• An invariant $I \subseteq \text{val}(X)$ is a set of states that contains $\text{Reach}_\mathcal{A}(\Theta)$