Lecture 6: Satisfiability modulo theories Part 2: DPLL(T) and Simplex Algorithm

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Slides adapted from Prof. Sayan Mitra's slides in Fall 2021
Some of the slides for this lecture are adapted from slides by Clark Barrett

George B. Dantzig

Today

- SMT (cont)
- Decision procedure for Linear Real Arithmetic
  Simplex Algorithm [Dantzig 1947]

- Next 2 - 3 weeks: Verification of Neural Networks and Machine Learning

Review: theories in SMT

- Linear (real) arithmetic
  - $4x - 3y + 6z \leq 10, x + y - z \leq 1;$
- Nonlinear real arithmetic
  - $4x^2 + 6y - 9z^3 \leq 5$
- Bit vectors
- Arrays
  - $x'[i] = x[i] + 1$
- Uninterpreted functions (UF) $\Sigma_F := \{f, g, \ldots\}$, $\Sigma_P := \{=\}$, $V := \{x_i\}$
  - $x_1 = x_2 \ \wedge \ x_3 \neq x_2 \ \wedge \ f(x_3) \neq f(x_2)$
- Difference logic $\Sigma_F := \{0,1,2,3, \ldots, - \}, \Sigma_P := \{<, \leq, =, >, \geq\}$, $V := \{x_i\}$
  - $x_1 - x_2 \gtreqless k$, where $\gtreqless \in \{<, \leq, =, >, \geq\}$

Review: Uninterpreted functions

Useful for abstractly reasoning about programs

- $\Sigma_F := \{f, g, \ldots\}$, $\Sigma_P := \{=\}$, $V := \{x_i\}$

Literals are of the form $x_1 = x_2 \ \wedge \ x_3 \neq x_2 \ \wedge \ f(x_3) \neq f(x_2)$

We know nothing about *f, g*, ... except for its name and arity

Review: Difference Logic

A useful fragment of linear arithmetic
$\Sigma_F := \{0,1,2,..,-\ \}$
$\Sigma_P := \{< , \leq , \ = , \neq , > , \geq\}$

Literals are of the form $x_1 - x_2 \gtreqless k$, where $\gtreqless \in \{< , \leq , \ = , > , \geq\}$

x_1, x_2 are Integers

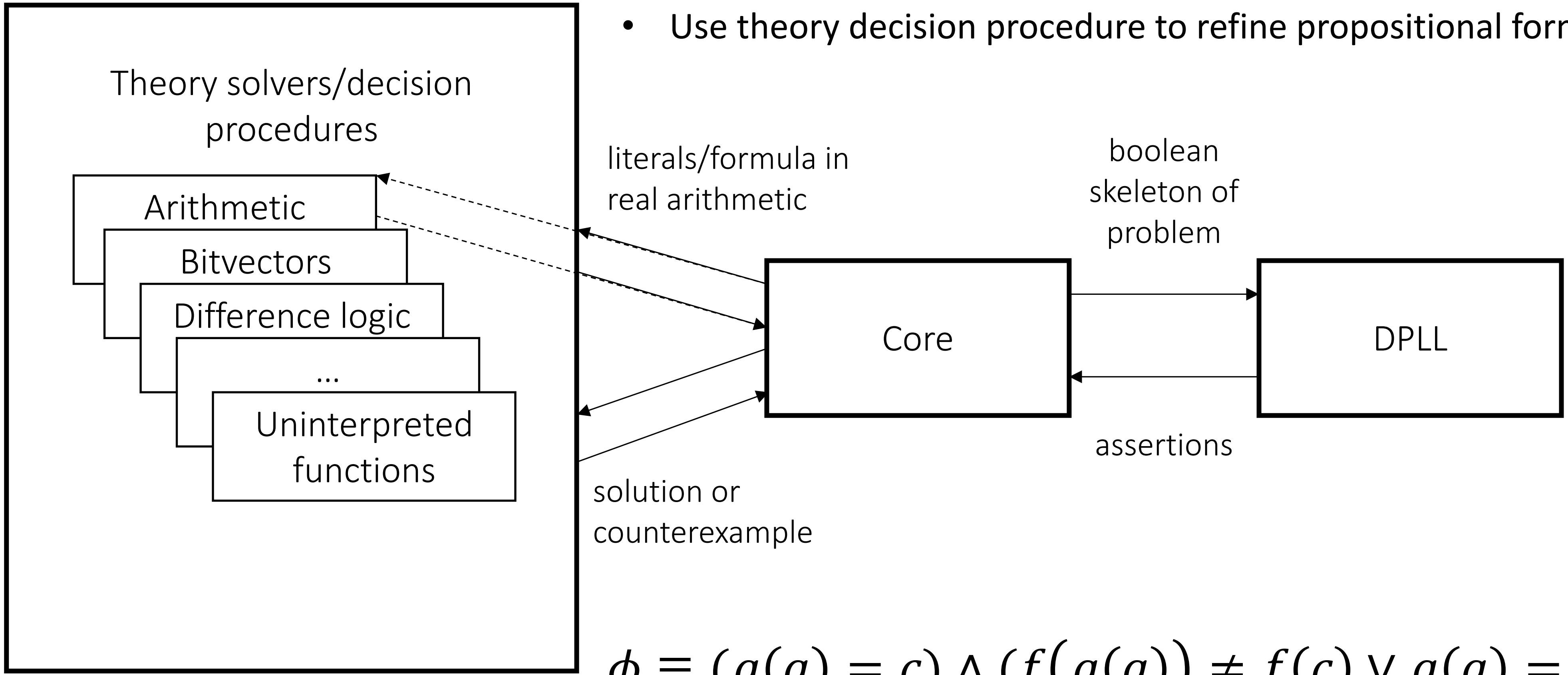
Example: $\phi = \ (x - y = 5) \wedge (z \ - y \geq 2) \wedge (z \ - x > 2) \wedge (w - x = 2) \wedge \ \ (z \ - w < 0)$

Satisfiability problem: checking whether this formula is consistent

How to solve SMT

Several approaches, lazy approach:

- Abstract ϕ to propositional form
- Feed to DPLL
- Use theory decision procedure to refine propositional formula a guide SAT



$$\phi \equiv (g(a) = c) \wedge (f(g(a)) \neq f(c) \vee g(a) = d) \wedge c \neq d$$

$$\text{abstract } \phi \equiv x_1 \wedge (\neg x_2 \vee x_3) \wedge \neg x_4$$

DPLL$^{T:}$ DPLL modulo theories

How can we extend DPLL to handle formulas over other theories like

- Difference Logic (DL)
- Uninterpreted functions (UF)
- Linear Real Arithmetic (LRA)

Idea: Start with a *Boolean abstraction* (or skeleton) and
incrementally add more *theory* information
until we can conclusively say SAT or UNSAT

Example: DPLL^{LRA}

$F \equiv (x \leq 0 \vee x \leq 10) \wedge (\neg x \leq 0)$

Boolean abstraction: replace every unique linear inequality with a Boolean variable
$F^B \equiv (p \vee q) \wedge (\neg p)$

where p *abstracts* $x \leq 0$ and q *abstracts* $x \leq 10$

Abstraction because information is lost

The relationship $x > 10 \Rightarrow x > 0$, i.e., $\neg q \Rightarrow \neg p$ is lost in F_B

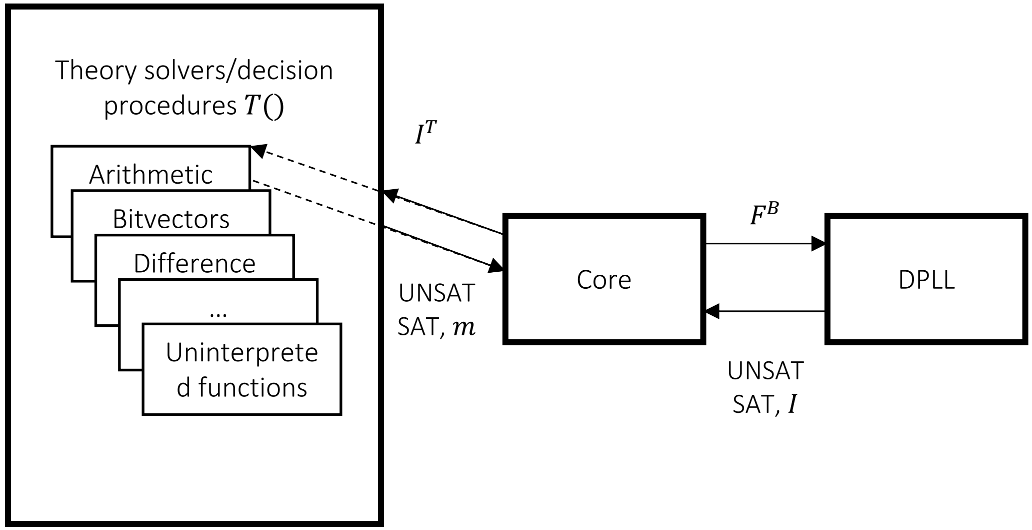
Notation. $(F^B)^T$ maps F^B back to theory T, i.e., $(F^B)^T = F$.

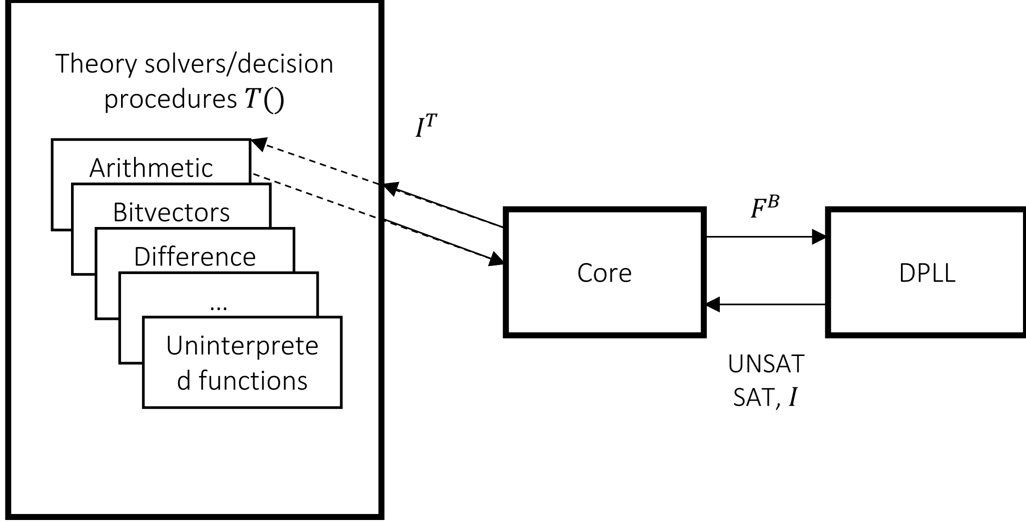
Proposition. If F^B is UNSAT then F is UNSAT, but the converse does not hold, i.e., F^B is SAT does not mean that F is SAT.

Example. $F_1 \equiv (x \leq 0 \wedge x \geq 10)$ is clearly UNSAT, however $F_1^B \equiv p \wedge q$ is SAT.

Lazy DPLL$^{\mathrm{T}}$ Algorithm using a Decision Procedure $T()$

Input: A formula F in CNF form over theory T
Output: $I \vDash F$ or UNSAT
Let F^B be the abstraction of F
while true **do**
  if $\mathrm{DPLL}(F^B)$ is unsat then **return** UNSAT
else
  Let I be the model returned by $DPLL$
  Assume I is represented as a formula
  if $T(I^T)$ is sat **then return** SAT and the model returned by $T()$
  else $F^B := F^B \wedge \neg I$





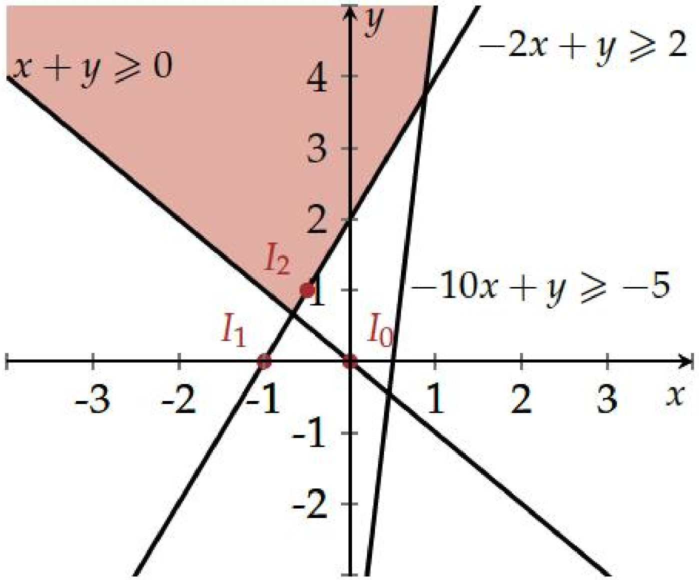
- $\phi \equiv \underbrace{g(a) = c}_{1} \wedge (\underbrace{f\big(g(a)\big) \neq f(c)}_{\bar{2}} \vee \underbrace{g(a) = d}_{3}) \wedge \underbrace{c \neq d}_{\bar{4}}$
- abstract $\phi \equiv x_1 \wedge (\neg x_2 \vee x_3) \wedge \neg x_4$
- send $\phi^B \equiv \{1, \bar{2} \vee 3, \bar{4}\}$ to DPLL
- DPLL returns SAT with model I:$\{1, \bar{2}, \bar{4}\}$
- UF solver concretizes $I^{UF} \equiv g(a) = c, f\big(g(a)\big) \neq f(c), c \neq d$
- UF checks I^{UF} as UNSAT
- send $\phi^B \wedge \neg I$: $\{1, \bar{2} \vee 3, \bar{4}, \bar{1} \vee 2 \vee 4\}$ to DPLL; this is a new fact learned by DPLL
- DPLL returns model I': $\{1, 2, 3, \bar{4}\}$
- UF solver concretizes I'^{UF} and finds this to be UNSAT
- send $\phi^B \wedge \neg I \wedge \neg I'$: $\{1, \bar{2} \vee 3, \bar{4}, \bar{1} \vee 2 \vee 4, \bar{1} \vee \bar{2} \vee \bar{3} \vee 4\}$ to DPLL; another fact
- returns UNSAT

DPLL^T: DPLL modulo theories

How can we extend DPLL to handle formulas over other theories like

- Difference Logic (DL)
- Uninterpreted functions (UFs)
- **Today: Linear Real Arithmetic (LRA)**

$(x + y \geq 0) \wedge (-2x + y \geq 2) \wedge (-10x + y \geq -5)$



Decision Procedure for Linear Real Arithmetic

Input: $F \equiv \wedge_{i=1}^{n} \Sigma_{j=1}^{m} c_{ij} x_j \leq b_i$ where $c_{ij}, b_i \in \mathbb{R}$

Output: $\exists$ a model $\boldsymbol{x} \in \mathbb{R}^m$ such that $\boldsymbol{x} \vDash F$?

Solution based on Simplex Algorithm [Dantzig 1947]

Simplex solves

$\max \Sigma_{j=1}^{m} a_j x_j$ subject to

$\wedge_{i=1}^{n} \Sigma_{j=1}^{m} c_{ij} x_j \leq b_i$

Our focus will be on finding any solution $\boldsymbol{x} \in \mathbb{R}^m$ that satisfies F

Decision Procedure for Linear Real Arithmetic

Input: $F \equiv \wedge_{i=1}^{n} \Sigma_{j=1}^{m} c_{ij} x_j \leq b_i$ where $c_{ij}, b_i \in \mathbb{R}$

Output: $\exists$ a model $\boldsymbol{x} \in \mathbb{R}^m$ such that $\boldsymbol{x} \models F$?

Simplex expects F to be expressed in the Simplex form, which is a conjunction of

- Linear equalities: $\Sigma_{i=1}^{m} c_i x_i = 0$
- Bounds: $\mathrm{l_i} \leq x_i \leq u_i$

Transforming to Simplex Form

Consider the i^{th} inequality in F: $\Sigma_{j=1}^{m} c_{ij} x_j \leq b_i$

Rewrite this as:
$s_i = \Sigma_{j=1}^{m} c_{ij} x_j \wedge s_i \leq b_i$

s_i is called a *slack variable*

Putting together all the rewritten conjuncts we get F_S

Proposition.

1. Any model of F_S is a model of F, disregarding the assignments to the slack variables.
2. If F_S is UNSAT then F is UNSAT.

Simplex (Informal)

Idea. Simultaneously try to find a model or a proof of UNSAT

Start with some *model (or valuation)* that satisfies all linear equalities (say, $x_i = 0,\ \forall i$)

In each iteration, pick a bound that is not satisfied and modify the model to satisfy the bound
OR
discover that the formula is UNSAT

Variable naming and ordering for Simplex

The input formula F_S (after rewriting) has two types of variables

- **Basic variables** appear on the **LHS** of **one** equality; initially these are the *slack variables*
- **Non-basic variables** all others

$$
\begin{aligned}
s_1 &= x + y \\
s_2 &= -2x + y \\
s_3 &= -10x + y \\
s_1 &\geq 0 \\
s_2 &\geq 2 \\
s_3 &\geq -5 \\
-\infty &\leq x \leq \infty
\end{aligned}
$$

$$
\begin{aligned}
x &= -0.5s_2 + 0.5y \\
s_1 &= -0.5s_2 + 1.5y \\
s_3 &= 5s_2 - 4y \\
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We fix an *arbitrary total ordering* on variables $x_1, \ldots, x_n$

For a basic variable x_i and non-basic variable x_j we denote by c_{ij} the coefficient of x_j in the definition of x_i, i.e.,
$x_i = \ \ldots + c_{ij}\, x_j + \ \ldots$

The upper and lower bounds of x_i are called u_i and l_i (possibly $\infty, \ -\infty$)

Pivoting: switch basic and non-basic variables

The pivoting operation change one non-basic variable to a basic variable (we say this variable is “entering”), while one other basic variable is changed to non-basic (we say this variable is “leaving”)

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\begin{aligned}
&s_1 = x + y \\
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&s_1 \geq 0 \\
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s_3 &\geq -5 \\
-\infty &\leq x \leq \infty
\end{aligned}
\quad \Rightarrow \quad
\boxed{x = -0.5s_2 + 0.5y}
$$

$$
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x &= -0.5s_2 + 0.5y \\
s_1 &= -0.5s_2 + 1.5y \\
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-\infty &\leq x \leq \infty
\end{aligned}
$$

$x_i = \Sigma_{k \in N}^{m} c_{ik} x_k, j \in N$

Pivoting x_i and x_j rewrites x_j as basic variable

$x_i = c_{ij} x_j + \Sigma_{k \in N \backslash \{j\}}^{m} c_{ik} x_k$

$$\boxed{x_j = \frac{x_i}{c_{ij}} - \Sigma_{k \in N \backslash \{j\}}^{m} \frac{c_{ik}}{c_{ij}} x_k}$$

Simplex (Formal) 1

The algorithm maintains two invariants

1. The model $\boldsymbol{x}$ always satisfies the equalities; bounds may be violated.

   Why is this invariant satisfied by our initialization of all 0s?

2. The bounds of all non-basic variables are all satisfied.

   Why is this invariant satisfied by our initialization?

$F \equiv \wedge_{i=1}^{n} \Sigma_{j=1}^{m} c_{ij} x_j \leq b_i$ where $c_{ij}, b_i \in \mathbb{R}$

Linear equalities: $\Sigma_{i=1}^{m} c_i x_i = 0$
Bounds: $\mathrm{l_i} \leq x_i \leq u_i$ (only for slack variables)

Simplex Algorithm: DP for LRA

Input: A formula F_S in Simplex form
Output: $\boldsymbol{x} \vDash F_S$ or UNSAT
$\boldsymbol{x} \coloneqq \langle x_i \mapsto 0 \rangle$
while true **do**
if $\boldsymbol{x} \vDash F_S$ **then return** $\boldsymbol{x}$
Let x_i be the first basic variable s.t. $x_i < \mathrm{l_i}$ or $x_i > \mathrm{u_i}$
if $x_i < \mathrm{l_i}$ **then**

Let x_j be the first non-basic variable s.t.
$(x_j < u_j \wedge c_{ij} > 0) \vee (x_j > l_j \wedge c_{ij} < 0)$
If no such x_j exists **then return** UNSAT
$x_j \coloneqq x_j + \frac{l_i - x_i}{c_{ij}}$; $x_i \coloneqq \mathrm{l}_i$

else Let x_j be the first non-basic variable s.t.
$(x_j > l_j \wedge c_{ij} > 0) \vee (x_j < u_j \wedge c_{ij} < 0)$
If no such x_j exists **then return** UNSAT
$x_j \coloneqq x_j + \frac{u_i - x_i}{c_{ij}}$; $x_i \coloneqq \mathrm{u}_i$

Pivot x_i and x_j; update x_i, x_j, and all basic variables

$x_i = \Sigma_{k \in N}^{m} c_{ik} x_k, j \in N$
Pivoting x_i and x_j rewrites x_j as basic variable
$x_i = c_{ij} x_j + \Sigma_{k \in N \setminus \{j\}}^{m} c_{ik} x_k$
$x_j = \frac{x_i}{c_{ij}} - \Sigma_{k \in N \setminus \{j\}}^{m} \frac{c_{ik}}{c_{ij}} x_k$

Example

$x + y \geq 0$
$-2x + y \geq 2$
$-10x + y \geq -5$

Rewritten in Simplex form

$s_1 = x + y$
$s_2 = -2x + y$
$s_3 = -10x + y$
$s_1 \geq 0$
$s_2 \geq 2$
$s_3 \geq -5$

Example continued

$s_1 = x + y$
$s_2 = -2x + y$
$s_3 = -10x + y$
$s_1 \geq 0$
$s_2 \geq 2$
$s_3 \geq -5$
$-\infty \leq x \leq \infty$

Variable ordering
x, y, s_1, s_2, s_3

Initialization $\boldsymbol{x_0} = \langle x \mapsto 0, y \mapsto 0, s_1 \mapsto 0, s_2 \mapsto 0, s_3 \mapsto 0 \rangle$

$\boldsymbol{x_0}$ satisfies equalities, bounds of s_1 s_3 are satisfied

Pick the first variable x to fix the bound of s_2

Since upper and lower bounds of x are ∞ and $-\infty$ it easily satisfies the blue condition

To increase s_2 to 2 and satisfy its lowerbound we decrease x to -1

$$\boldsymbol{x_1} = \langle x \mapsto 0 + \frac{2-0}{-2} = -1, y \mapsto 0, s_1 \mapsto -1, s_2 \mapsto 2, s_3 \mapsto 10 \rangle$$

$x = -0.5s_2 + 0.5y$
$s_1 = -0.5s_2 + 1.5y$
$s_3 = 5s_2 - 4y$
$s_1 \geq 0$
$s_2 \geq 2$
$s_3 \geq -5$
$-\infty \leq x \leq \infty$

Pivot s_2 with x

Now x becomes a basic variable, s_2 non-basic

Example continued

$x = -0.5s_2 + 0.5y$
$s_1 = -0.5s_2 + 1.5y$
$s_3 = 5s_2 - 4y$
$s_1 \geq 0$
$s_2 \geq 2$
$s_3 \geq -5$
$-\infty \leq x \leq \infty$

$\boldsymbol{x_1} = \langle x \mapsto -1, y \mapsto 0, s_1 \mapsto -1, s_2 \mapsto 2, s_3 \mapsto 10 \rangle$
All equalities are still satisfied (invariant)
The only basic variable not satisfying its bounds is now s_1
The first non-basic variable we can tweak is y

Setting $s_1 \mapsto 0$ to satisfy the lowerbound of s1 we get

$\boldsymbol{x_2} = \langle x \mapsto -2/3, y \mapsto 0 + \dfrac{0-(-1)}{1.5} = 2/3,$
$s_1 \mapsto 0, s_2 \mapsto 2, s_3 \mapsto 22/3 \rangle$
Pivot s_1 with y

$$s_1 = -0.5s_2 + 1.5y \longrightarrow y = \frac{2}{3}s_1 + \frac{1}{3}s_2$$

$\boldsymbol{x_2} \vDash F_S$

$y = \frac{2}{3}s_1 + \frac{1}{3}s_2$
$x = \frac{1}{3}s_1 - \frac{1}{3}s_2$
$s_3 = -\frac{8}{3}s_1 + \frac{11}{3}s_2$
$s_2 \geq 2$
$s_1 \geq 0$
$s_3 \geq -5$
$-\infty \leq x \leq \infty$

Why is simplex correct?

- Why does it terminate?

  Because we always looks for the first variable violating the bounds. There is a property (Bland's rule) that ensures that we never revisit the same set of basic and non-basic variables.

- Why does it give the right answer (sound)?
  - If it returns $\boldsymbol{x}$ does it satisfy $\boldsymbol{x} \vDash F$?

    This follows from the condition before **return** $\boldsymbol{x}$
  - If it returns UNSAT is F really unsatisfiable?

For proofs, check Dutertre, B., de Moura, L.: Integrating Simplex with DPLL(T). Technical report, CSL-06-01, SRI International (2006)

Unsatisfiable example

$s_1 = x + y$
$s_2 = -x - 2y$
$s_3 = -x + y$
$s_1 \geq 0$
$s_2 \geq 2$
$s_3 \geq 1$

Consider a Simplex execution in which there are two pivots:

Pivot 1: s_1 with x , $s_1$set to 0.
$x = s_1 - y$
$s_2 = -s_1 - y$
$s_3 = -s_1 + 2y$

Pivot 2: s_2 with y , $s_2$set to 2
$x = 2s_1 + s_2$
$y = -s_1 - s_2$
$s_3 = -3s_1 - 2s_2$

Non-basic variables satisfy their bounds (invariant), and so
$s_1 \geq 0, s_2 \geq 2$
If s_3 violates the bound then
$s_3 = -3s_1 - 2s_2 < 1$

We can make s_3 bigger by decreasing s_1 and s_2 but even at the extreme (smallest possible) s_1 and s_2
$s_3 = -3 \times 0 - 2 \times 2 = -4$

which is still less than 1 and Simplex concludes that the formula is UNSAT.

The blue conditions for choosing x_j encodes this condition.

Simplex Algorithm: DP for LRA

$x_i = \Sigma_{k \in N}^{m} c_{ik} x_k, j \in N$

Pivoting x_i and x_j rewrites x_j as basic variable

$x_i = c_{ij} x_j + \Sigma_{k \in N \setminus \{j\}}^{m} c_{ik} x_k$

$x_j = \frac{x_i}{c_{ij}} - \Sigma_{k \in N \setminus \{j\}}^{m} \frac{c_{ik}}{c_{ij}} x_k$

Input: A formula F_S in Simplex form
Output: $\boldsymbol{x} \vDash F_S$ or UNSAT
$\boldsymbol{x} := \langle x_i \mapsto 0 \rangle$
while true **do**
if $\boldsymbol{x} \vDash F_S$ **then return** $\boldsymbol{x}$
Let x_i be the first basic variable s.t. $x_i < \mathrm{l_i}$ or $x_i > \mathrm{u_i}$
if $x_i < \mathrm{l_i}$ **then**
    Let x_j be the first non-basic variable s.t.
    $(x_j < u_j \wedge c_{ij} > 0) \vee (x_j > l_j \wedge c_{ij} < 0)$
    If no such x_j exists **then return** UNSAT
    $x_j := x_j + \frac{l_i - x_i}{c_{ij}}$; $x_i := \mathrm{l}_i$
else Let x_j be the first non-basic variable s.t.
    $(x_j > l_j \wedge c_{ij} > 0) \vee (x_j < u_j \wedge c_{ij} < 0)$
    If no such x_j exists **then return** UNSAT
    $x_j := x_j + \frac{u_i - x_i}{c_{ij}}$; $x_i := \mathrm{u}_i$
Pivot x_i and x_j; update x_i, x_j, and all basic variables

i.e., find a non-basic variable that is **not** set to their extreme value u_j or l_j according to the sign of c_{ij}

If the non-basic variable x_j has been set to its extreme, we cannot further change it to fix x_i

Summary and Takeaways

- Satisfiability modulo theory solvers use theory solvers and DPLL to check satisfiability of formulas in other theories
  - DPLL takes care of disjunctions
  - Theory solvers take care of conjunctions

- Simplex or more generally Linear programming (LP) solvers is a theory solver for linear real arithmetic
  - Simplex algorithm solves LP by incrementally fixing the bounds of basic variables