Lecture 14: Stability Verification of Dynamical Systems

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Slides adapted from Prof. Sayan Mitra's slides in Fall 2021

Deadlines

Project proposal is due 3/3, 11:59 pm CT

See Canvas announcement for some example project ideas on ML + verification

Homework 2 due 3/10, 11:59 pm CT

Two writing problems + two programming problems

START EARLY!

Review: Linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Define Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ \dots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$

Theorem. $\xi(t, x_0, u) = \Phi(t)x_0 + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau$

Zero input Zero state

Here $\Phi(t) := e^{At}$ is the state-transition matrix

Properties for dynamical systems

What type of properties are we interested in?

- Invariance (as in the case of automata)
- State remains bounded
- Converges to target
- Bounded input gives bounded output (BIBO)

Requirements: Stability

- We will focus on time invariant autonomous systems (closed systems, systems without inputs)

$$\dot{x}(t) = f\big(x(t)\big),\ x_0 \in \mathbb{R}^n,\ t_0 = 0 \qquad \textit{Eq. (1)}$$

- $\xi(t)$ is the solution
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume **0** to be an equilibrium point of (1) with out loss of generality

Example: Pendulum

Pendulum equation

$x_1 = \theta \;\; x_2 = \dot{\theta}$

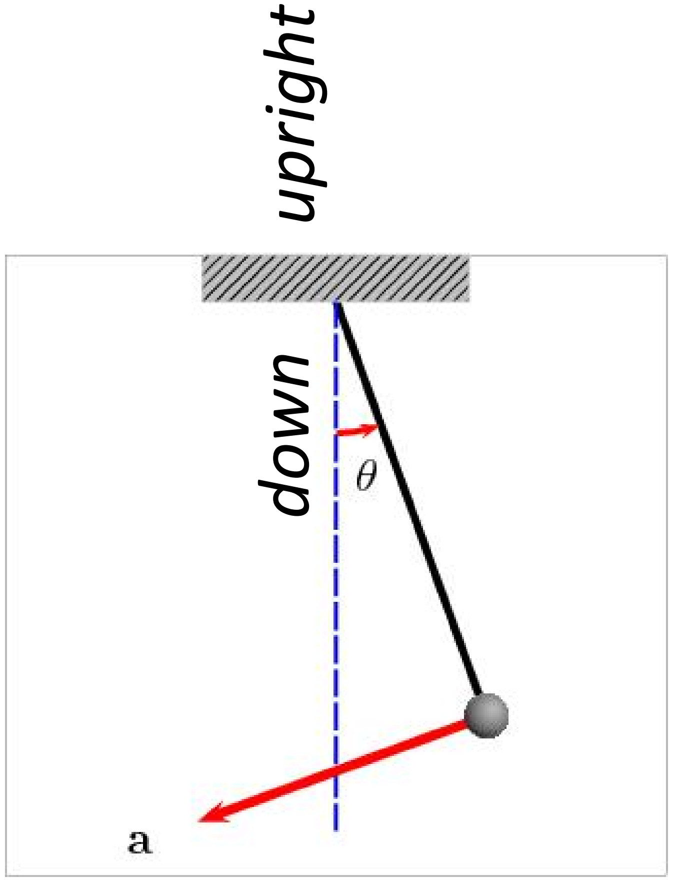
$x_2 = \dot{x}_1$

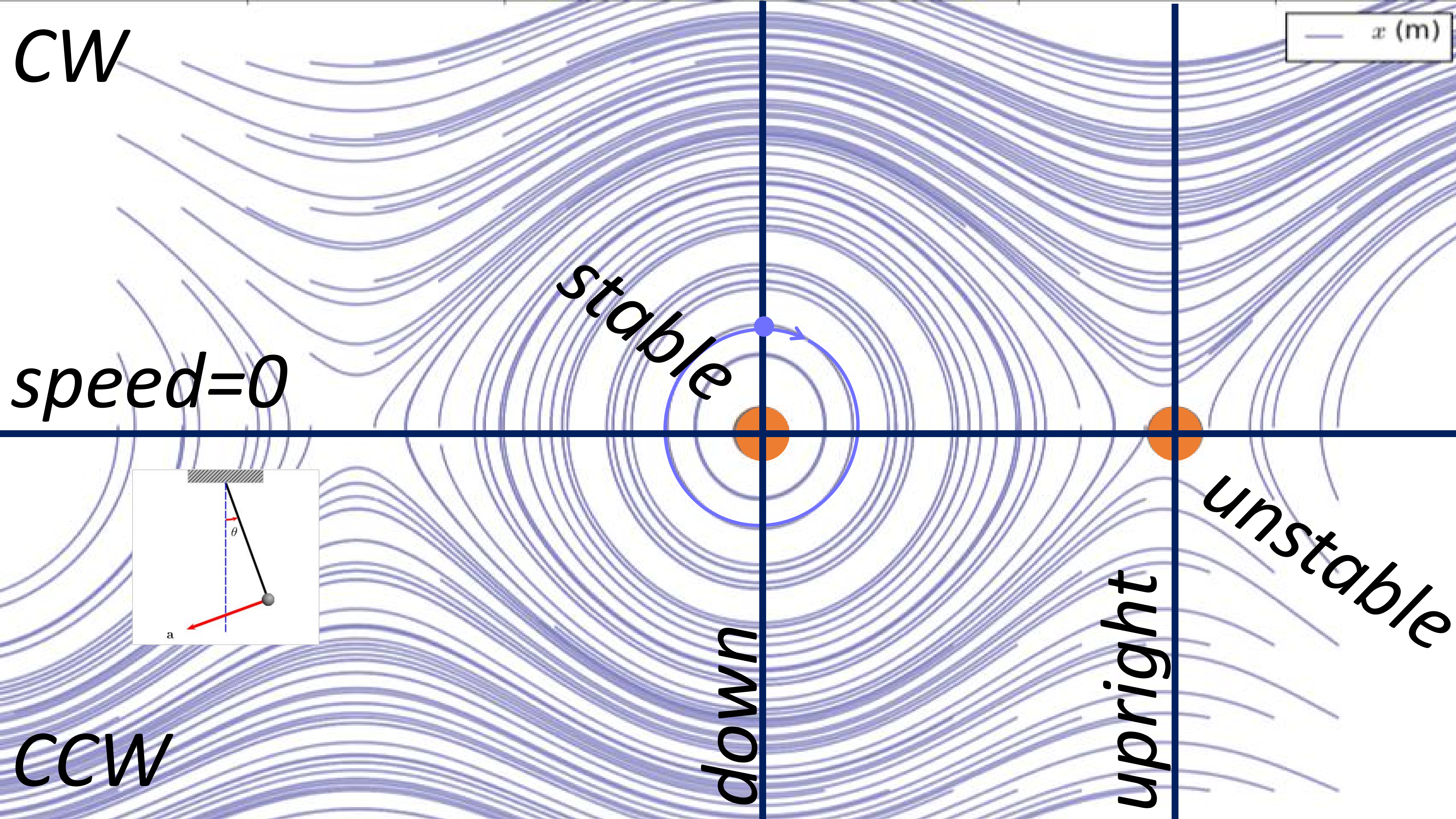
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$$

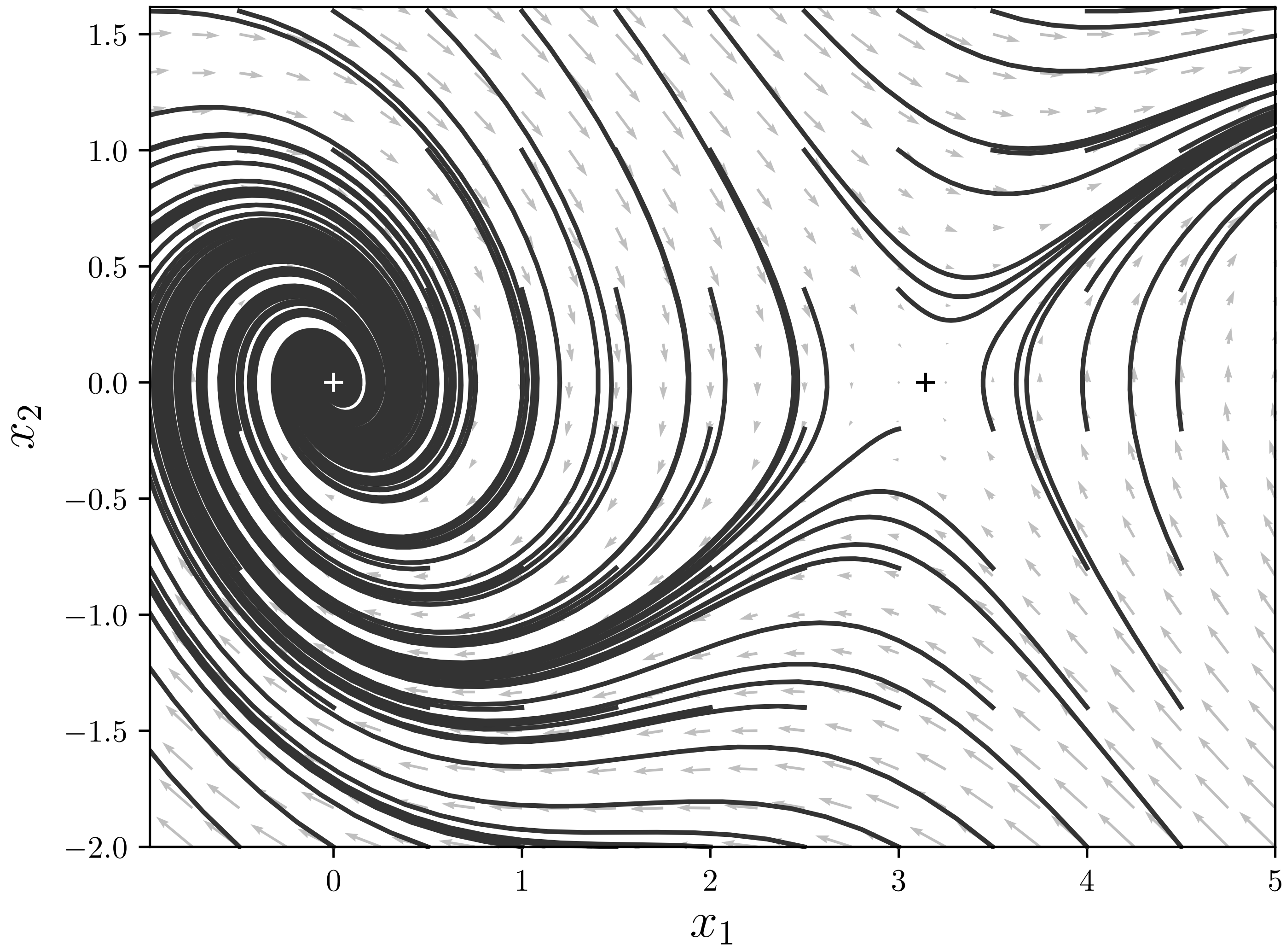
k: friction coefficient

Two equilibrium points: $(0,0)$, $(\pi, 0)$




CW
x (m)
stable
speed=0
unstable
θ
a
down
upright
CCW

Phase portrait of pendulum with friction



Aleksandr M. Lyapunov

Aleksandr Mikhailovich Lyapunov (June 6 1857–November 3, 1918) was a Russian mathematician and physicist.

His methods make it possible to define the stability of ordinary differential equations. In the theory of probability, he generalized the works of Chebyshev and Markov, and proved the Central Limit Theorem under more general conditions than his predecessors.

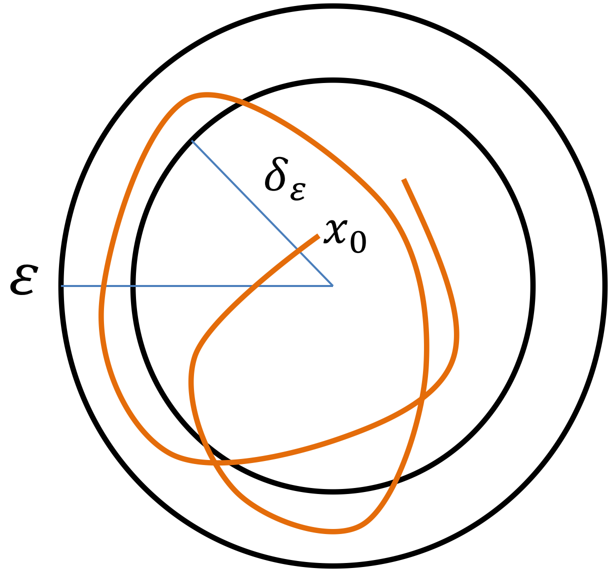
Lyapunov stability

Lyapunov stability: The system (1) is said to be ***Lyapunov stable*** (at the origin) if

$\forall \varepsilon > 0,\ \exists\ \delta_\varepsilon > 0$ such that $|x_0| \leq \delta_\varepsilon \Rightarrow \forall\ \mathrm{t} \geq 0,\ |\xi(x_0, t)| \leq \varepsilon.$

"if we start the system close enough to the equilibrium, it remains close enough"

How is this related to invariants and reachable states ?

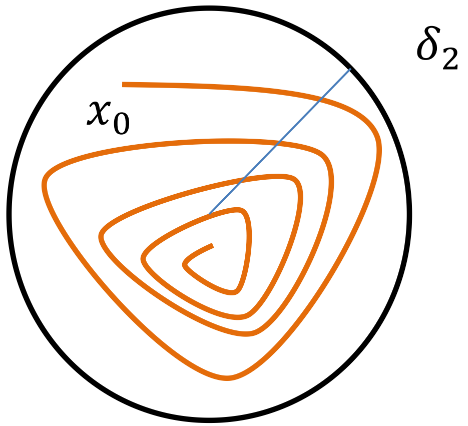


Asymptotically stability

The system (1) is said to be ***Asymptotically stable*** *(at the origin)* if it is Lyapunov stable and

$\exists\delta_2 > 0$ such that $\forall|x_0| \leq \delta_2$ as $t \rightarrow \infty$, $|\xi(x_0, t)| \rightarrow \mathbf{0}$.

If the property holds for any δ_2 then **Globally Asymptotically Stable**



Butterfly

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

All solutions converge to 0 but the equilibrium point (0,0) is **not** Lyapunov stable

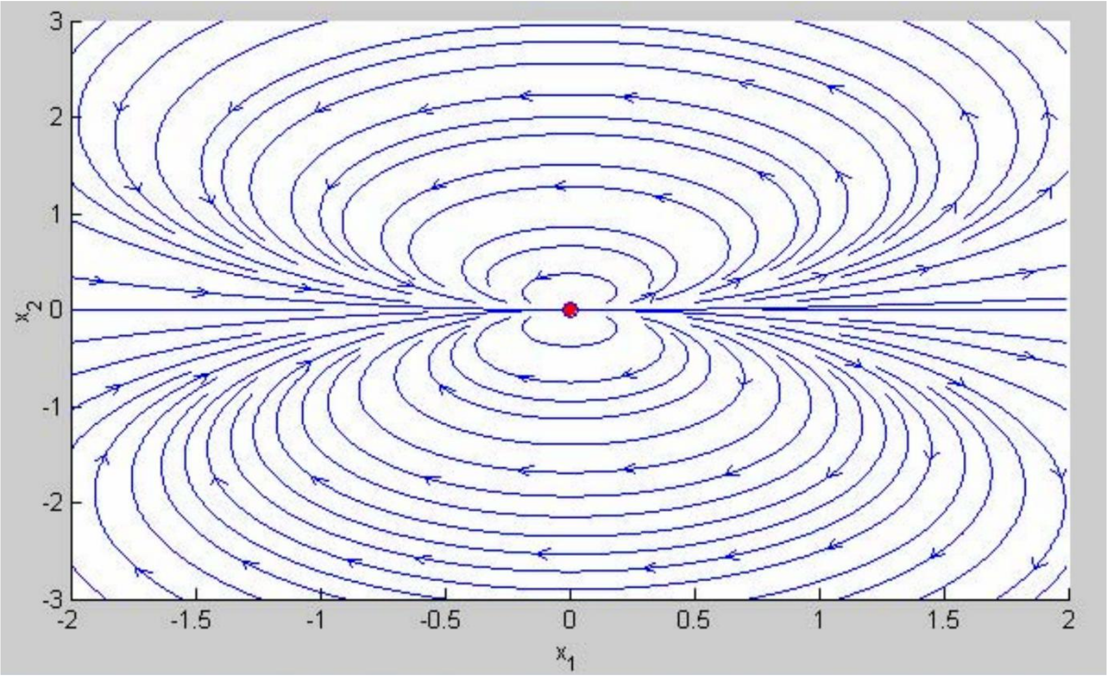


Figure from Prof. João P. Hespanha (UCSB)

Van der pol oscillator

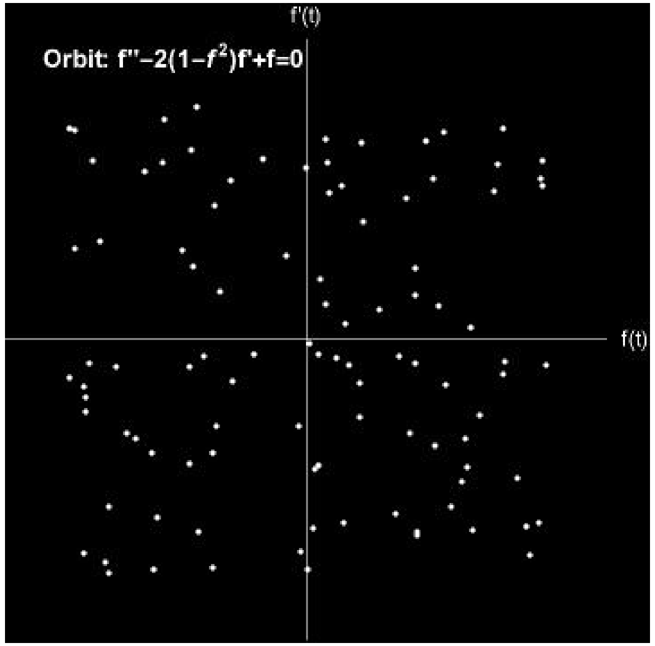
Van der pol oscillator

$$\frac{dx^2}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

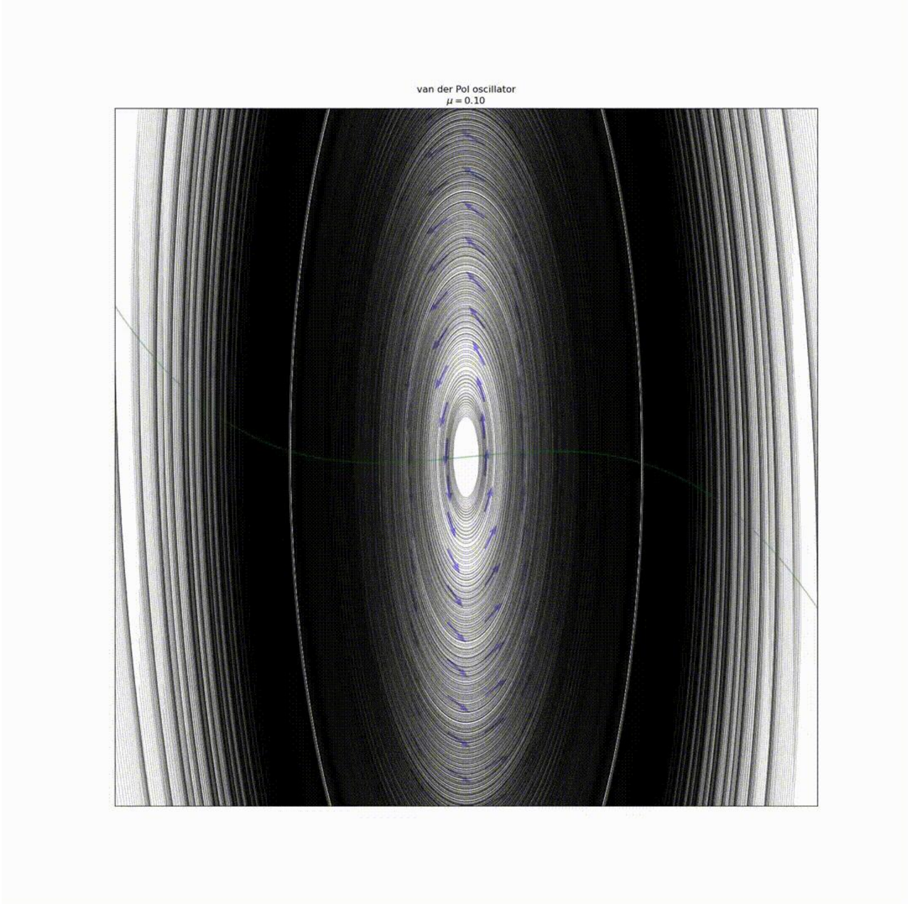
Define $x_1 = x; x_2 = \dot{x}_1;$

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} \mu(1 - x_1^2)x_2 - \ x_1 \\ x_2 \end{bmatrix}$$

stable ?



$\mu = 2$
simulation



μ from 0.01 to 4
phase portrait

Stability of solutions* (instead of points)

- For any $\xi \in \mathrm{PC}(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$ define the s-norm $\left||\xi|\right|_s = \sup_{t \in \mathbb{R}} ||\xi(t)||$

- A dynamical system can be seen as an operator that maps initial states to signals $T: \mathbb{R}^n \rightarrow PC(\mathbb{R}^{\geq 0}, \mathbb{R}^n)$

- Lyapunov stability required that this operator is continuous

- The solution ξ^* is ***Lyapunov stable*** if T is continuous as $\xi^*(0)$. $\text{ i.e.},$ for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ if $|\xi^*(0) - x_0| \leq \delta_\varepsilon$ then $\left|\left|T\left(\xi^*(t)\right) - T(x_0)\right|\right|_s \leq \varepsilon$.

*Not discussed in class

Verifying Stability for **Linear** Systems

Consider the linear system $\dot{x} = Ax$

Theorem.

1. It is asymptotically stable iff all the eigenvalues of *A* have **strictly** negative real parts (*Hurwitz*).

2. It is Lyapunov stable iff all the eigenvalues of *A* have real parts that are either zero or negative and the *Jordan blocks* corresponding to the eigenvalues with zero real parts are of size 1.

Why eigenvalues matter? (proof sketch)

$$\dot{x}(t) = Ax(t)$$

Solution is $\xi(t, x_0) = e^{At}x_0$

Assume A is diagonalizable, $A = SDS^{-1}$

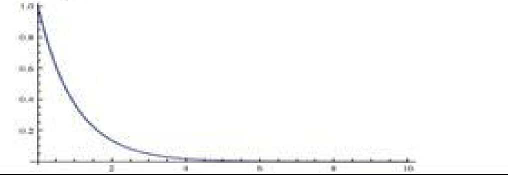
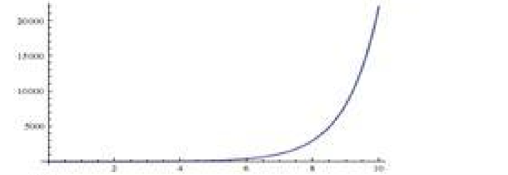
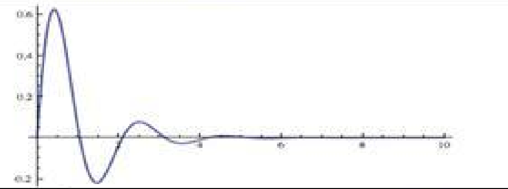
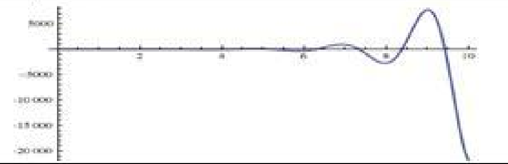
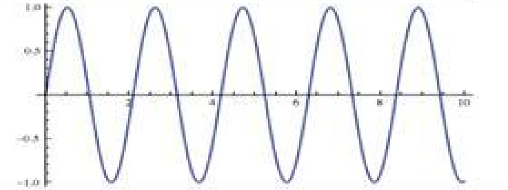
D is a diagal matrix containing all eigenvalues

$$e^{At} = S\sum_{k=0}^{\infty}\frac{(Dt)^k}{k!}S^{-1} = S\begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}S^{-1}$$

We want all e^{λ_i} to be < 1 so the exponential with t does not blow up. That means $Re(\lambda_i) < 0$

Verifying Stability for **Linear** Systems

Consider the linear system $\dot{x} = Ax$

Eigenvalues	Graph
All real and negative	
All real and one or more are positive	
All real eigenvalues are negative and there are imaginary parts	
One or more eigenvlaues have a positive real part and there are imaginary parts	
Real parts of the eigenvlaues are zero and there are imaginary parts	

Graphs from Peter Woolf's lecture from Fall'08 titled Dynamic Systems Analysis II: Evaluation Stability, Eigenvalues

Jordan decomposition

For every *n x n* matrix *A*, there exists a nonsingular *n x n* matrix *P* such that

$$PAP^{-1} = J = \begin{bmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_\ell \end{bmatrix},$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}.$$

where each J_i is a upper triangular matrix called a Jordan block

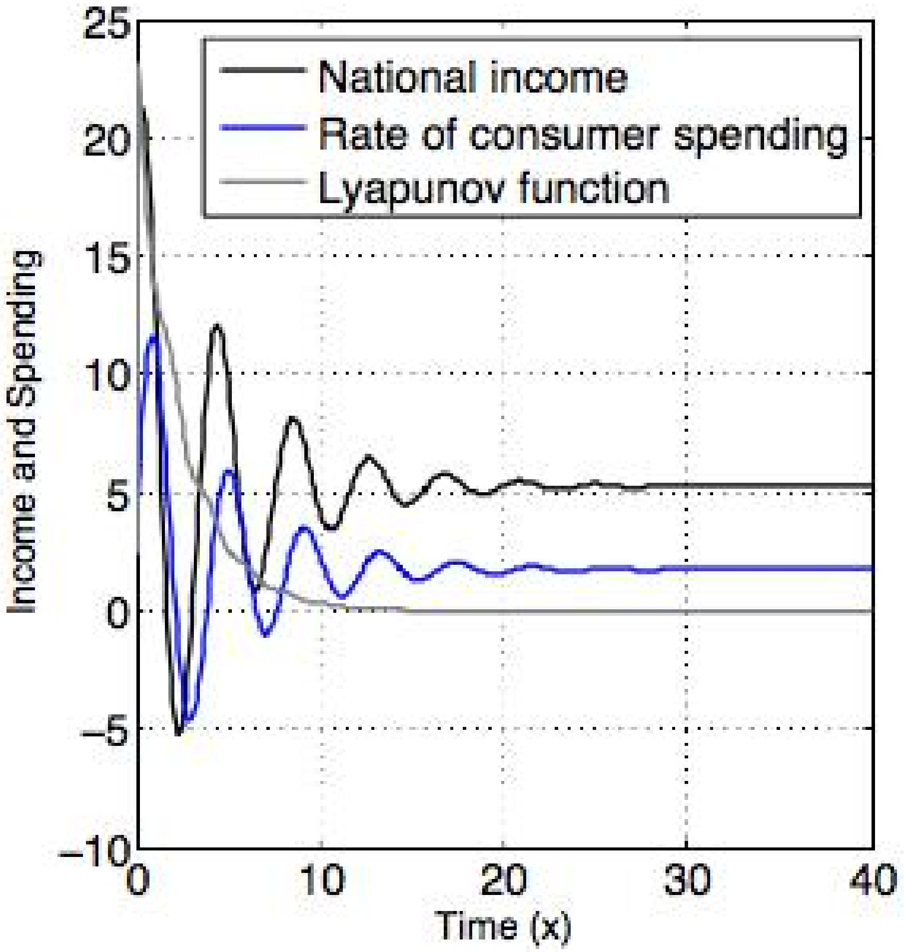
Example 1: Simple model of an economy

- x: national income y: rate of consumer spending; g: rate government expenditure
- $\dot{x} = x - \alpha y$
- $\dot{y} = \beta(x - y - g)$
- $g = g_0 + kx$ $\alpha,\ \beta,\ k$ are positive constants

- Dynamics: (this is a linear system!)
- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example: Simple linear model of an economy

- $\alpha = 3,\ \beta = 1,\ k = 0$
- $\lambda_1, \lambda_1^* = (-.25 \pm i\, 1.714)$
- Negative real parts, therefore, asymptotically stable and the national income and consumer spending rate converge to $x = 1.764\ y = 5.294$



Stability of **nonlinear** systems

- For any ***positive definite*** function of state $V: \mathbb{R}^n \to \mathbb{R}$
  that is, $V(x) \geq 0$ and $V(x) = 0$ iff $x = 0$
- ***Sublevel sets*** of $L_p := \{x \in \mathbb{R}^n \mid V(x) \leq p\}$
- Now consider $V(\xi(t))$; *V* differentiable with continuous first derivative
- $\dot{V} = d\frac{V(\xi(t))}{dt} = \ ?$

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$$\frac{\partial V}{\partial x} . \frac{d}{dt}\left(\xi(t)\right) = \frac{\partial V}{\partial x} . f(x)$$

Verifying Stability

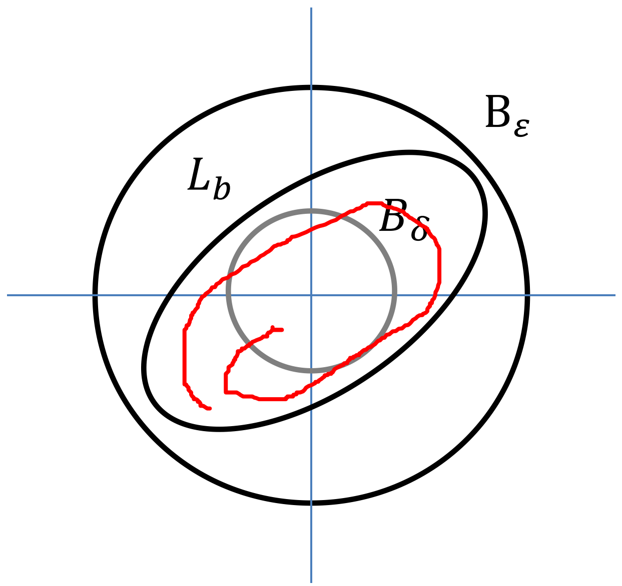
Theorem. (Lyapunov) Consider the system (1) with state space $\xi(t) \in \mathbb{R}^n$ and suppose there exists a positive definite, continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$. The system is:

1. Lyapunov stable if $\dot{V}\left(\xi(t)\right) = \frac{\partial V}{\partial x} f(x) \leq 0$, for all $x \neq 0$
2. Asymptotically stable if $\dot{V}\left(\xi(t)\right) < 0$, for all $x \neq 0$
3. It is globally AS if V is also radially unbounded.

(V is radially unbounded if $||x|| \to \infty \Rightarrow V(x) \to \infty$)

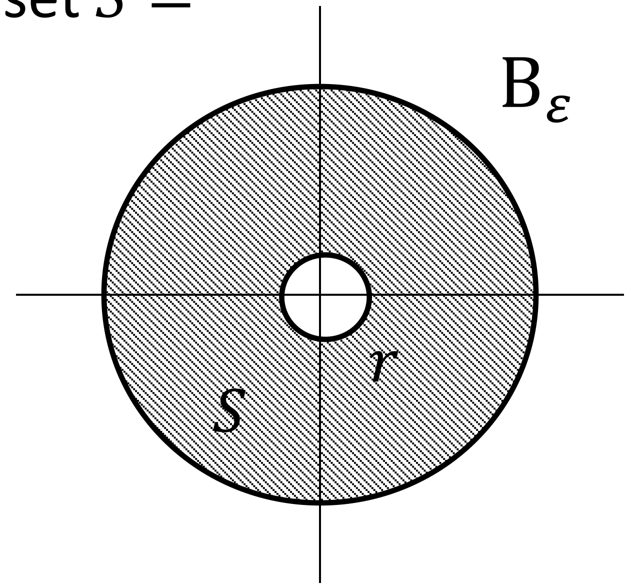
Proof sketch: Lyapunov stable if $\dot{V} \leq 0$

- Assume $\dot{V} \leq 0$
- Consider a ball B_ε around the origin of radius $\varepsilon > 0$.
- Pick a positive number $b < \min_{|x|=\varepsilon} V(x)$.
- Let δ be a radius of ball around origin which is inside $B_\delta = \{x \mid V(x) \leq b\}$
- Since along all trajectories **V is non-increasing**, starting from B_δ each solution satisfies $V(\xi(t)) \leq b$ and therefore remains in B_ε



Proof sketch: Asymptotically stable if $\dot{V} < 0$

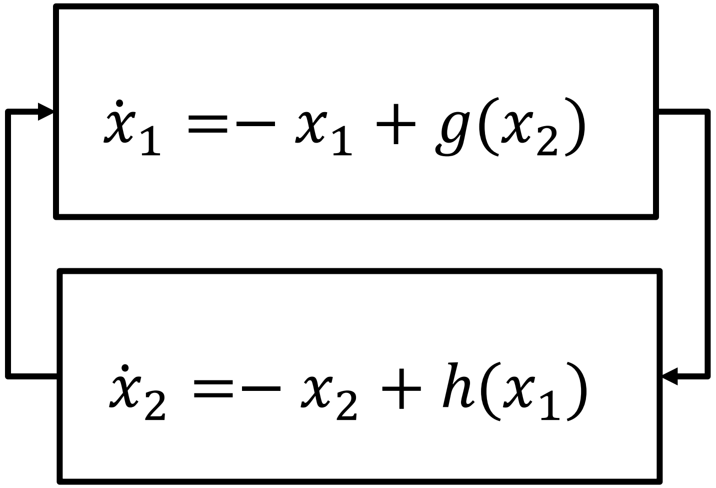
- Assume $\dot{V} < 0$ for all $x \neq 0$
- Take arbitrary initial state $|\xi(0)| \leq \delta,$ where this δ comes from some ε for Lyapunov stability
- Since $V\big(\xi(.)\big) > 0$ and decreasing along ξ it has a limit $c \geq 0$ at $t \to \infty$
- It suffices to show that this limit is actually 0, since$V(x) = 0$ iff $x = 0$
- Suppose not, c > 0 then the solution $\xi(0)$ evolves in the compact set $S = \{x \mid r \leq |x| \leq \varepsilon\}$ for some sufficiently small r
- Let $d = \max_{x \in S} \dot{V}(x)$, d is well-defined and negative
- $\dot{V}\big(\xi(t)\big) \leq d$ for all t
- $V\big(\xi(t)\big) \leq V\big(\xi(0)\big) + dt$
- But then eventually $V\big(\xi(t)\big) < c$



Example 2: Reasoning about stability without solving ODEs

$\dot{x}_1 = -x_1 + g(x_2); \dot{x}_2 = -x_2 + h(x_1)$

Given that $|g(x_2)| \leq \frac{|x_2|}{2}, |h(x_1)| \leq \frac{|x_1|}{2}$



- Use $V = \frac{1}{2}(x_1^2 + x_2^2) \geq 0$
- $\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$

$$= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1)$$

$$\leq -x_1^2 - x_2^2 + \frac{1}{2}(|x_1 x_2| + |x_2 x_1|)$$

$$\leq -\frac{1}{2}(x_1^2 + x_2^2) = -V$$

$$(|x_1| - |x_2|)^2 \geq 0$$

$$x_1^2 + x_2^2 \geq 2|x_1 x_2|$$

$$|x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$$

We conclude global asymptotic stability (in fact global exponential stability) without knowing solutions

Lyapunov function vs. Invariant

Proposition. If V is a Lyapunov function then every sublevel set of V is an invariant

Proof. $V\big(\xi(t)\big) = V\big(\xi(0)\big) + \int_0^t \dot{V}\big(\xi(\tau)\big)d\tau$

$\leq V(\xi(0))$

An aside: Checking inductive invariants

- $\boldsymbol{A} = \langle X, Q_0, T \rangle$
  - X: set of variables
  - $Q_0 \subseteq val(X)$
  - $T \subseteq val(X) \times val(X)$ written as a program $x' \subseteq T(x)$
- How do we check that $I \subseteq val(X)$ is an inductive invariant?
  - $Q_0 \Rightarrow I(X)$
  - $I(X) \Rightarrow I(T(X))$
- Implies that $Reach_{\boldsymbol{A}}(Q_0) \subseteq I$ without computing the executions or reachable states of $\boldsymbol{A}$
- The key is to find such I

Finding Lyapunov Functions

- The key to using Lyapunov theory is to *find* a Lyapunov function and verify that it has the properties
- In general, for nonlinear systems this is hard
- There are several approaches
  - Quadratic Lyapunov functions for linear systems
  - Decide the form/template of the function (e.g., quadratic, polynomial), parameterized by some parameters and find values of the parameters so that the conditions hold (Chapter 3 last section)
  - Use a neural network + neural network verification for checking the Lyapunov condition

Linear autonomous systems

- $\dot{x} = Ax,\ A \in \mathbb{R}^{n\times n}$
- The Lyapunov equation: $A^T P + PA + Q = 0$

  where $P, Q \in \mathbb{R}^{n\times n}$ are symmetric

- Interpretation: $V(x) = x^T P x$ then

$$\dot{V}(x) = (Ax)^T P x + x^T P (Ax)$$

[using chain rule $\frac{\partial u^T P v}{\partial t} = \frac{\partial u}{\partial t} P v + \frac{\partial v}{\partial t} P^T u$]

$= x^T (A^T P + PA) x$

Let $\dot{V}(x) = -\, x^T Q x$ we obtain the equation above

- If $x^T P x$ is the generalized energy then $-\, x^T Q x$ is the associated dissipation

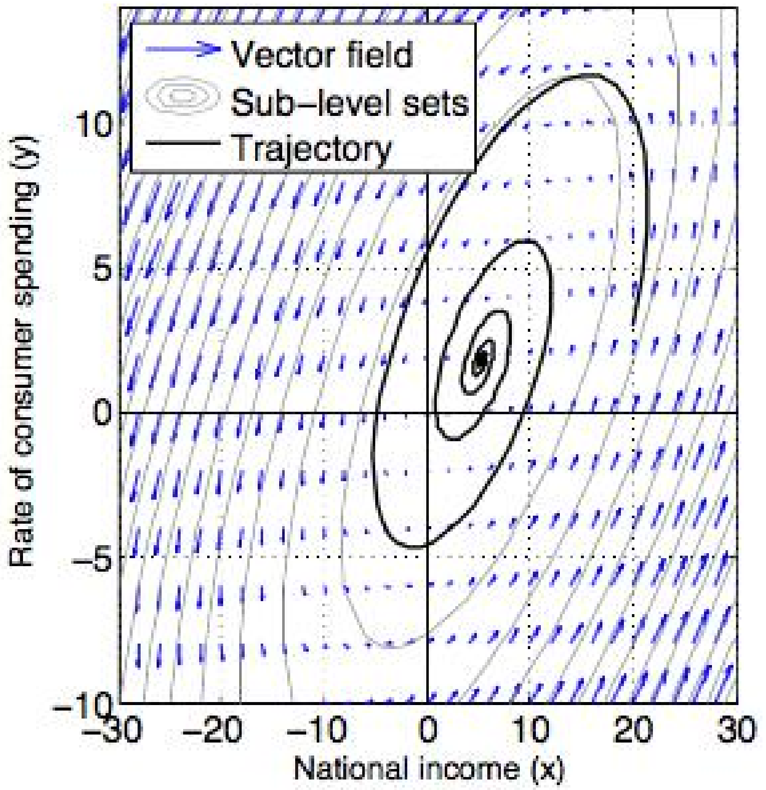
Quadratic Lyapunov Functions

- If $P > 0$ (positive definite)
- $V(x) = x^T P x = 0 \Leftrightarrow x = 0$
- The sub-level sets are ellipsoids
- If $Q > 0$ (positive definite) then the system is globally asymptotically stable

Same example

Lyapunov equations are solved as a set of $\frac{n(n+1)}{2}$ equations in $n(n+1)/2$ variables. Cost $\mathrm{O}(n^6)$

Choose $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ solving Lyapunov equations we get $P = \begin{bmatrix} 2.59 & -2.29 \\ -2.29 & 4.92 \end{bmatrix}$ and we get the quadratic Lyapunov function $(x - x^*)P(x - x^*)^T$ an a sequence of invariants



Converse Lyapunov

Converse Lyapunov theorems show that conditions of the previous theorem are also necessary. For example, if the system is asymptotically stable then there exists a positive definite, continuously differentiable function V, that satisfies the inequalities.

For example if the LTI system $\dot{x} = Ax$ is globally asymptotically stable then there is a quadratic Lyapunov function that proves it.