Lecture 13: Modeling Physics (Part II: Linear systems) Stability Verification

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Slides adapted from Prof. Sayan Mitra’s slides in Fall 2021

Deadlines

Project proposal is due 3/3, 11:59 pm CT

See Canvas announcement for some example project ideas on ML + verification

Homework 2 due 3/10, 11:59 pm CT

Two writing problems + two programming problems

Some project ideas

https://canvas.illinois.edu/courses/44138/discussion_topics/604292

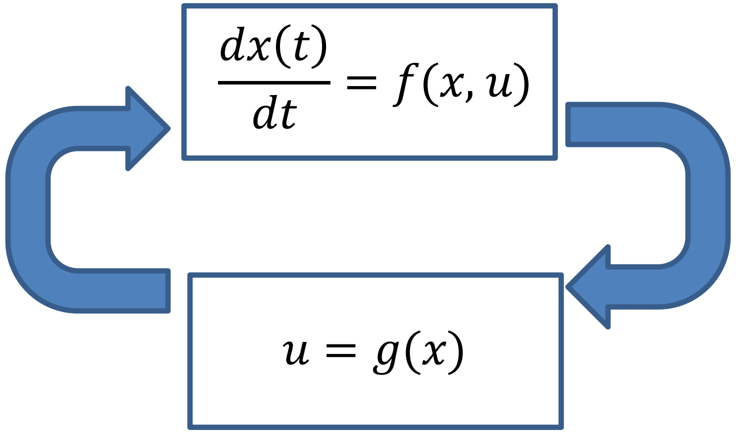
Review: dynamical systems

Behaviors of physical processes are described in terms of instantaneous laws

Common notation: $\frac{dx(t)}{dt} = f(x(t), u(t), t) \; - \; Eq. (1)$

where time $t \in \mathbb{R}$; **state** $x(t) \in \mathbb{R}^n$; $\mathbf{input}$ $u(t) \in \mathbb{R}^m$; $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$

Example. $\frac{dx(t)}{dt} = v(t)$; $\frac{dv(t)}{dt} = a$



Review: dynamical systems

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Example. $\frac{dx(t)}{dt} = v(t)\ ; \frac{dv(t)}{dt} =- g$

Initial value problem: Given system (1) and initial state $x_0 \in \mathbb{R}^n,\ t_0 \in \mathbb{R}$, and input $\mathrm{u}: \mathbb{R} \rightarrow \mathbb{R}^m$, find a state trajectory or *solution* of (1).

Review: Existence and uniqueness of solutions

Theorem. If $f(x(t), u(t), t)$ is Lipschitz continuous in the first argument, and $u(t)$ is piece-wise continous then (1) has unique solutions.

In general, for nonlinear dynamical systems we do not have closed form solutions, but there are numerical solvers

Linear system and solutions

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ (Linear time varying)

$\dot{x}(t) = Ax(t) + Bu(t)$ (Linear time invariant)

For a given initial state $x_0 \in \mathbb{R}^n, u(t) \in PC(\mathbb{R}, \mathbb{R}^n)$ the *solution* is a function

$$\xi(t, x_0, u): \mathbb{R} \to \mathbb{R}^n$$

Note that *t* is the variable.

We studied several properties of ξ: continuity with respect to first and third argument, linearity, decomposition

Linear system and solutions

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$u(t)$ continuous everywhere except D_x

Theorem. Let $\xi(t, x_0, u)$ be the solution for (2) with points of discontinuity , D_x

1. $\forall x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(\cdot\,, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable $\forall\ t \in \mathbb{R} \setminus D_x$
2. $\forall t \in \mathbb{R},\ u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, \cdot\,, u): \mathbb{R}^n \to \mathbb{R}^n$ is continuous
3. linearity: $\forall t \in \mathbb{R}, x_{01}, x_{02} \in \mathbb{R}^n, u_{1,} u_2 \in PC(\mathbb{R}, \mathbb{R}^m), a_1, a_2 \in \mathbb{R},\ \xi(t, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 \xi(t, x_{01},\ u_1) + a_2 \xi(t, x_{02},\ u_2)$
4. decomposition: $\forall t \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t,\ x_0,\ u) = \xi(t,\ x_0,\ \mathbf{0}) + \xi(t, 0,\ u)$

Linear system and solutions

- $\xi(.,x_0,u):\mathbb{R}\to\mathbb{R}^n$ is a linear function of the initial state x_0 and input u, (linearity property). Let us first focus on the linear function $\xi(.,x_0,0)$ about the initial state x_0

- Define $\Phi(.)x_0=\xi(.,x_0,0)$
- This $\Phi(.)\colon\mathbb{R}\to\mathbb{R}^{n\times n}$ is called the state transition matrix

Linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

A and B are not function of t.

Solution of the system ξ can be explicitly derived. How to do that?

Consider the decomposition property, we solve two problems:

$$\forall t \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t,\ x_0,\ u) = \xi(t,\ x_0,\ \mathbf{0}) + \xi(t, 0,\ u)$$

Linear time invariant system

First set input u(t) to 0 (we do this due to the decomposition)

$$\frac{dx(t)}{dt} = A\,x(t),\; x(t=0) = x_0$$

Due to linearity, the solution is in this form:

$$x(t) = \phi(t)x_0 = (E + \phi_1 t + \phi_2 t^2 + \ldots + \phi_n t^n + \ldots)x_0$$

Taylor expansion of Φ(t)

Substitute into the differential equation:

$$\frac{d}{dt}(\phi(t)x_0) = A\,\phi(t)\,x_0$$

$$(\phi_1 + 2\,\phi_2\,t + \ldots + n\,\phi_n\,t^{n-1} + \ldots)x_0 = (A + A\phi_1 t + A\phi_2 t^2 + \ldots + A\phi_n t^n + \ldots)\,x_0$$

Linear time invariant system

$$\frac{d}{dt}(\phi(t)x_0) = A\,\phi(t)\,x_0$$
$$(\phi_1 + 2\,\phi_2\,t + \ldots + n\,\phi_n\,t^{n-1} + \ldots)x_0 = (A + A\phi_1 t + A\phi_2 t^2 + \ldots + A\phi_n t^n + \ldots)\,x_0$$

ve want to solve Φ(t), by comparing the terms:

$$\begin{aligned}
\phi_1 &= A \\
\phi_2 &= \frac{1}{2}A\,\phi_1 = \frac{1}{2!}A^2 \\
\phi_3 &= \frac{1}{3}A\,\phi_2 = \frac{1}{3!}A^3 \\
&\cdots \\
\phi_n &= \frac{1}{n!}A^n.
\end{aligned}$$

Linear time invariant system

$$\phi_1 = A$$
$$\phi_2 = \frac{1}{2} A \phi_1 = \frac{1}{2!} A^2$$
$$\phi_3 = \frac{1}{3} A \phi_2 = \frac{1}{3!} A^3$$
$$\cdots$$
$$\phi_n = \frac{1}{n!} A^n.$$

$$\phi(t) = e^{At} = E + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \ldots + \frac{1}{n!} A^n t^n + \ldots$$

$$\forall t \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t,\ x_0,\ u) = \boxed{\xi(t,\ x_0,\ \mathbf{0})} + \xi(t, 0,\ u)$$

This part done

Linear time invariant system

Consider the decomposition property, we solve two problems:

$$\forall t \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t,\ x_0,\ u) = \xi(t,\ x_0,\ \mathbf{0}) + \xi(t, 0,\ u)$$

Now the second part

Now for $\xi(t, 0,\ u)$, assume x_0 = 0, solve $\frac{dx(t)}{dt} = Ax(t)\ +\ Bu(t)$

Rearrange: $\frac{dx(t)}{dt} - Ax(t) =\ \ Bu(t)$

Multiply a common factory: $e^{-At}\frac{dx(t)}{dt} - e^{-At}Ax(t) =\ e^{-At}\,Bu(t)$

Note the perfect differential: $\frac{d}{dt}(e^{-At}x(t)) =\ e^{-At}\,Bu(t)$

Linear time invariant system

$$\frac{d}{dt}(e^{-At}x(t)) = e^{-At}\,Bu(t)$$

Integration on both sides:

$$\int_0^t \frac{d}{d\tau}(e^{-A\tau}x(\tau)) = \int_0^t e^{-A\tau}\,Bu(\tau)\,d\tau$$

$$e^{-At}x(t) - e^{A0}x(0) = \int_0^t e^{-A\tau}\,Bu(\tau)\,d\tau$$

Since x(0) = 0:

$$x(t) = e^{At}\int_0^t e^{-A\tau}\,Bu(\tau)\,d\tau$$

$$x(t) = \int_0^t e^{A(t-\tau)}\,Bu(\tau)\,d\tau$$

Linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Define Matrix exponential:

$$e^{At} = 1 + At + \frac{1}{2!}(At)^2 + \ \ldots = \sum_{0}^{\infty} \frac{1}{k!}(At)^k$$

Theorem. $\xi(t, x_0, u) = \Phi(t)x_0 + \int_0^t \mathrm{e}^{A(t-\tau)} Bu(\tau)d\tau$

Zero input　　　　Zero state

Here $\Phi(t) := e^{At}$ is the state-transition matrix

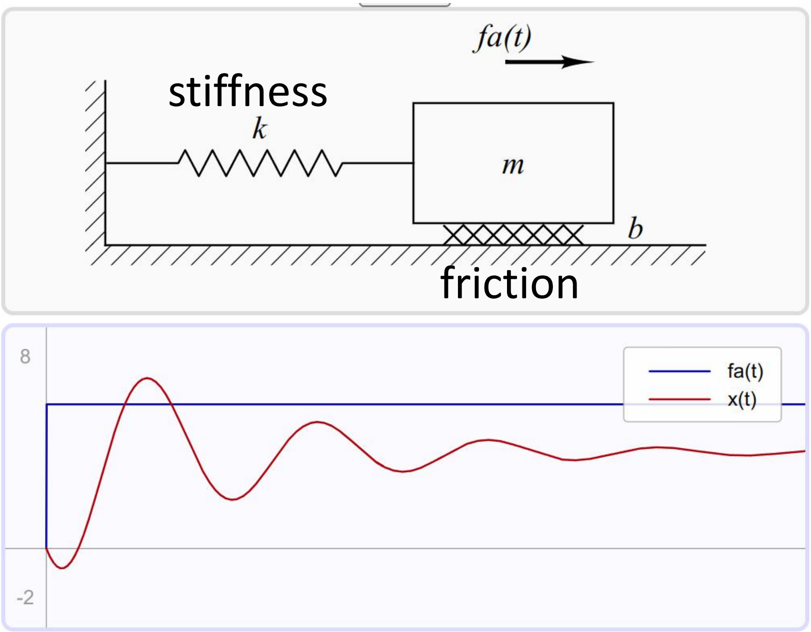
Example

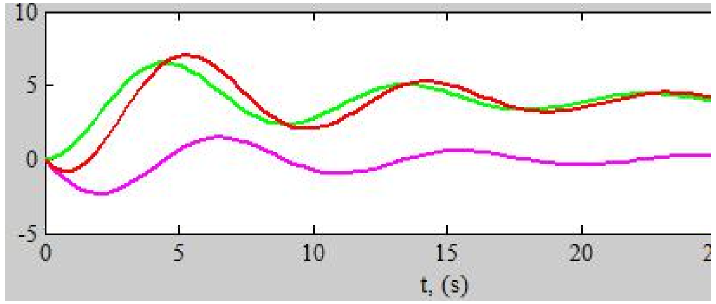
$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

States x: postion (0m), velocity (-2m/s),
Input u(t): Force $f_a(t)$ = 6 Newtons

$$m\frac{dx_2(t)}{dt} = u(t) - b\frac{dx_1(t)}{dt} - kx_1(t)$$

$$x_2(t) = \frac{dx_1(t)}{dt}$$





Zero state
Complete
Zero input

Source: https://lpsa.swarthmore.edu/Transient/TransZIZS.html

Solution of linear time-varying systems in Φ

More generally, for time varying systems we have

Theorem.

$$\xi(t, t_0, x_0, u) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

Note that $\Phi(t, t_0)$ here also includes t_0 as an parameter

Discrete time models / discrete transition systems

- $x(t+1) = f\big(x(t),\ u(t)\big)$
- $x(t+1) = f(x(t))$ No input (autonomous)
- Execution defined as: $x_0, f(x_0),\ f^2(x_0), \ldots$
- Can be define as an automaton $\boldsymbol{A} = \langle Q, Q_0, T \rangle$
  - $Q = \mathbb{R}^n,\ Q_0 = \{x_0\}$
  - $T: \mathbb{R}^n \to \mathbb{R}^n; \mathrm{T}(x) = f(x)$
- Deterministic

Discretized or sampled-time model

- $\dot{x}(t) = f\big(x(t),\ u(t)\big)$
- Assume: $u \in PC(\mathbb{R}, U)\ where\ U \subseteq \mathbb{R}^m$ is a finite set
- Given the solution $\xi(t, x_0, u)$
- Fix a sampling period $\delta > 0$
- $\boldsymbol{A_\delta} = \langle Q, Q_0, U, T \rangle$
  - $Q = \mathbb{R}^n,\ Q_0 = \{x_0\},\ Act = U,$
  - $T \subseteq \mathbb{R}^n \times U \times \mathbb{R}^n; (x, u, x') \in \mathrm{T}\ iff\ x' = \xi(\delta, x, u)$

Properties for dynamical systems

What type of properties are we interested in?

- Invariance (as in the case of automata)
- State remains bounded
- Converges to target
- Bounded input gives bounded output (BIBO)

Requirements: Stability

- We will focus on time invariant autonomous systems (closed systems, systems without inputs)
- $\dot{x}(t) = f\big(x(t)\big),\ x_0 \in \mathbb{R}^n,\ t_0 = 0$
- $\xi(t)$ is the solution
- $|\xi(t)|$ norm
- $x^* \in \mathbb{R}^n$ is an **equilibrium point** if $f(x^*) = 0$.
- For analysis we will assume **0** to be an equilibrium point with out loss of generality

Example: Pendulum

Pendulum equation

$x_1 = \theta \;\; x_2 = \dot{\theta}$

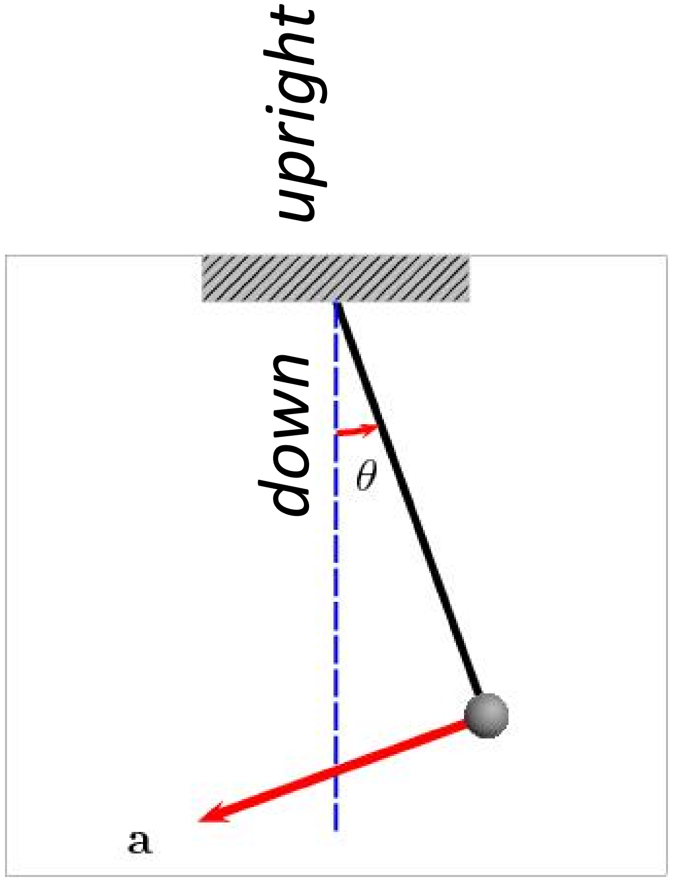
$x_2 = \dot{x}_1$

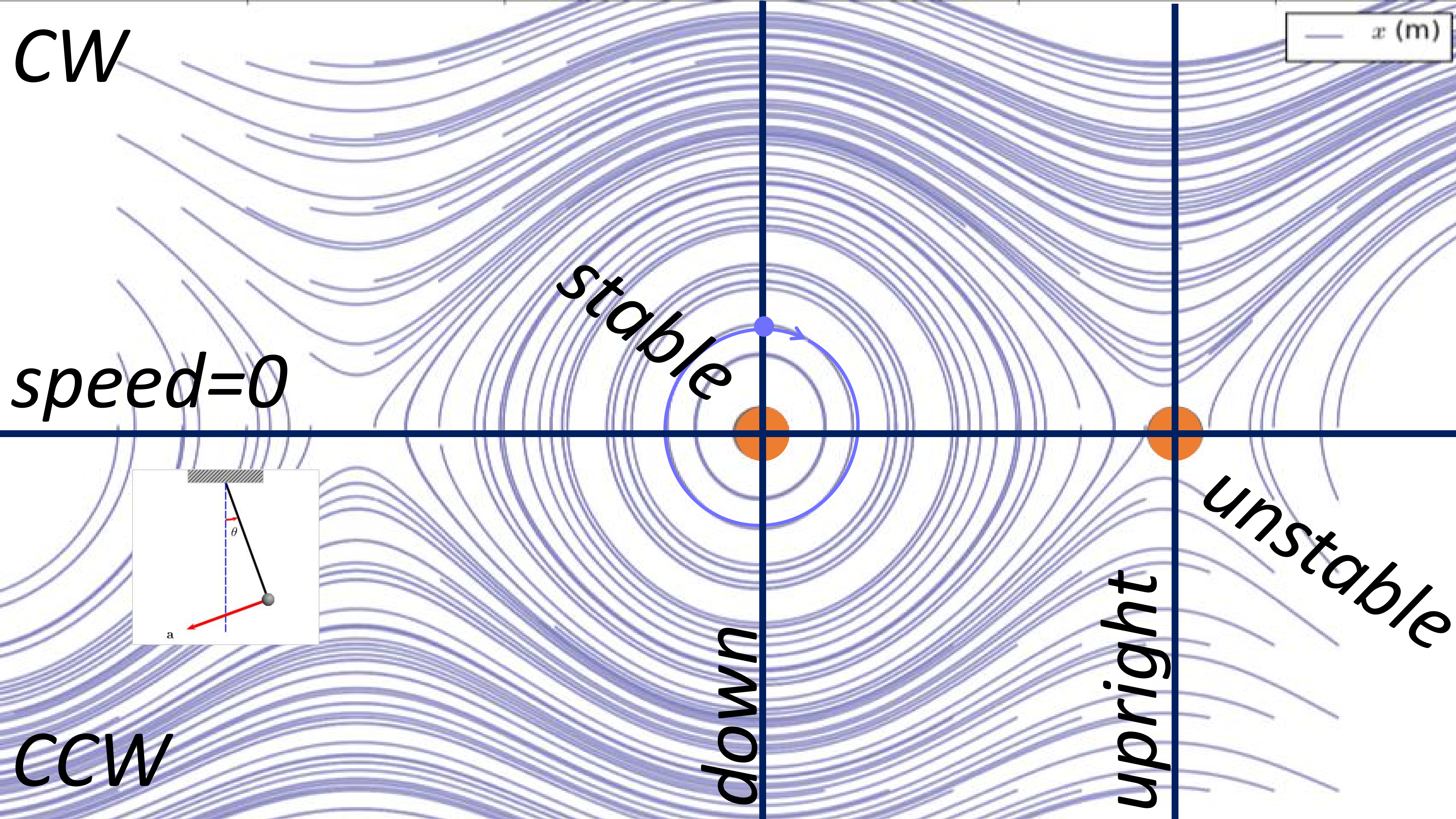
$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$

$\begin{bmatrix}\dot{x_2}\\ \dot{x_1}\end{bmatrix} = \begin{bmatrix}-\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2\\ x_2\end{bmatrix}$

k: friction coefficient

Two equilibrium points: $(0,0),\ (\pi, 0)$





CW
speed=0
CCW
stable
unstable
down
upright
x (m)
θ
a