Lecture 12: Modeling Physics

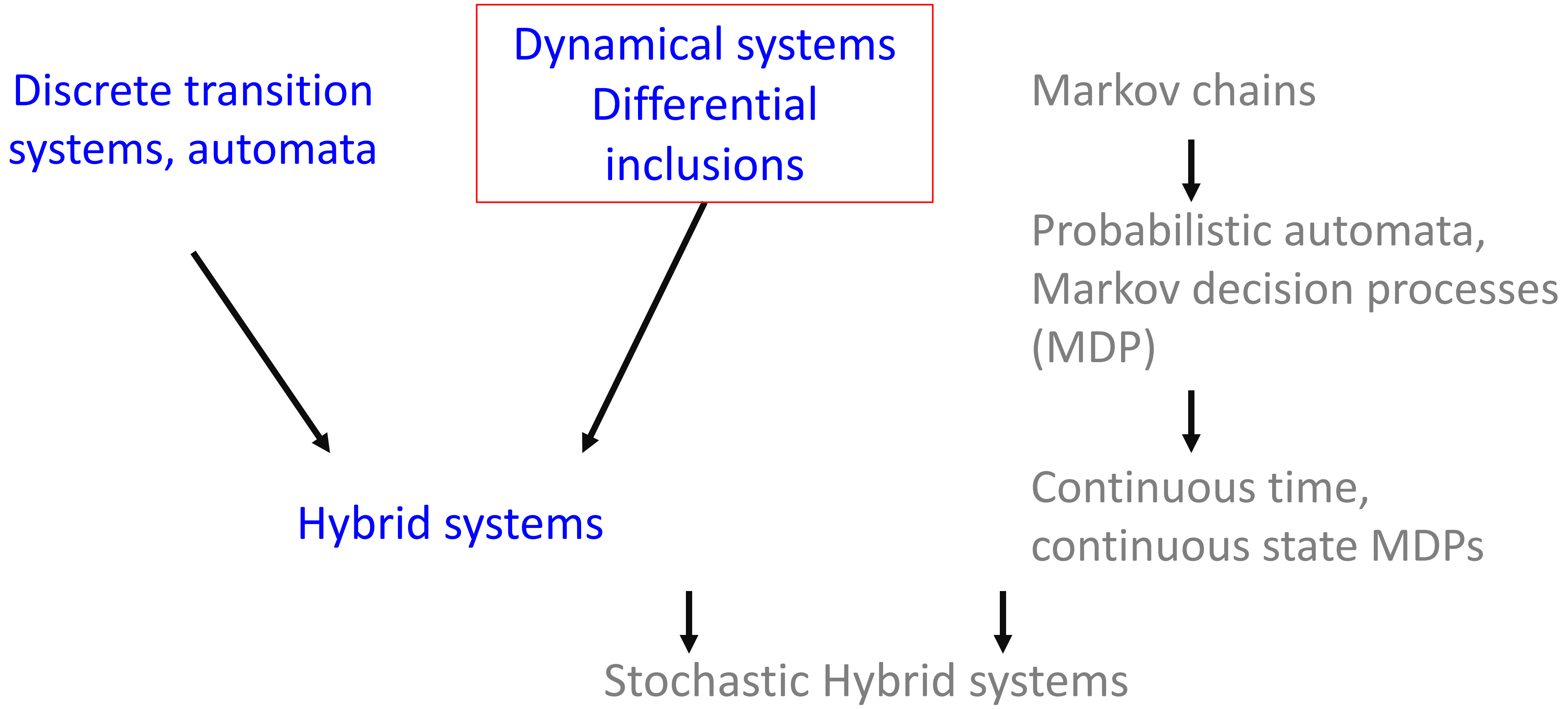
Huan Zhang
huan@huan-zhang.com

Slides adapted from Prof. Sayan Mitra's slides in Fall 2021

Plan

- Dynamical system models
  - notions of solutions
  - Linear dynamical systems
  - Connection to automata
  - Verification requirement: Stability
  - Lyapunov method to verify stability

Map of CPS models



All this was in the two plague years 1665 and 1666, for in those days I was in my prime of age for invention, and minded mathematics and philosophy more than at any time since.

---Isaac Newton

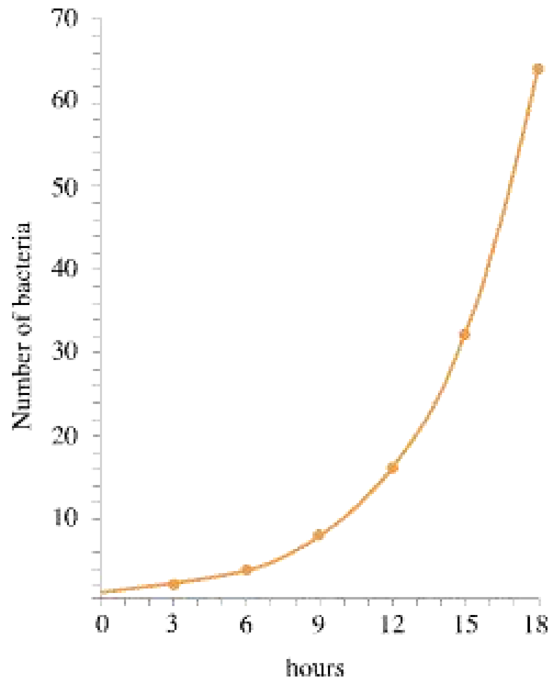
From: Wilczek, Frank. A Beautiful Question: Finding Nature's Deep Design (p. 87).

Introduction to dynamical systems

Behaviors of physical processes are described in terms of instantaneous laws

Example: growth of bacteria

$$\frac{dx(t)}{dt} = x$$



Vehicles, weather, circuits, biomedical processes, ...

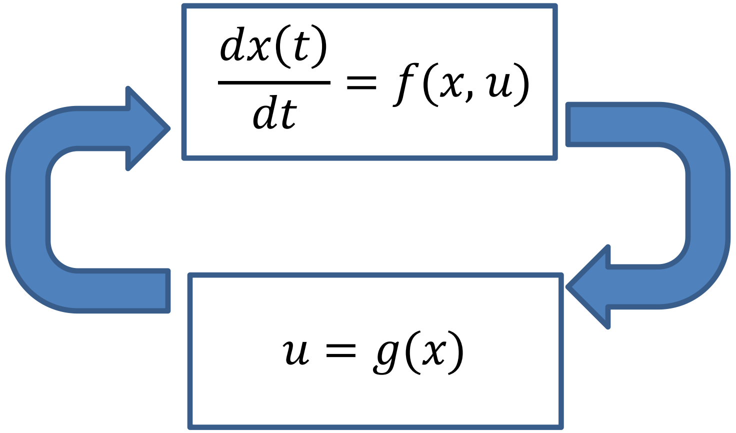
Introduction to dynamical systems

Behaviors of physical processes are described in terms of instantaneous laws

Common notation: $\frac{dx(t)}{dt} = f(x(t), u(t), t) \; - \; Eq. (1)$

where time $t \in \mathbb{R}$; **state** $x(t) \in \mathbb{R}^n$; **input** $u(t) \in \mathbb{R}^m$; $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$

Example. $\frac{dx(t)}{dt} = v(t) \; ; \frac{dv(t)}{dt} = -g$



Example: Pendulum

Pendulum equation

$x_1 = \theta \;\; x_2 = \dot{\theta}$

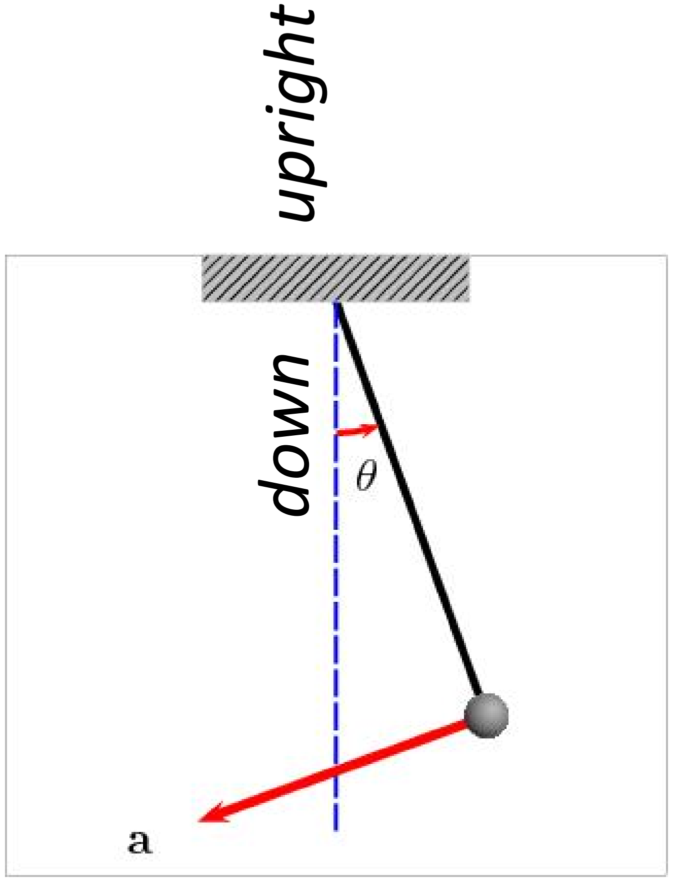
$x_2 = \dot{x}_1$

$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$

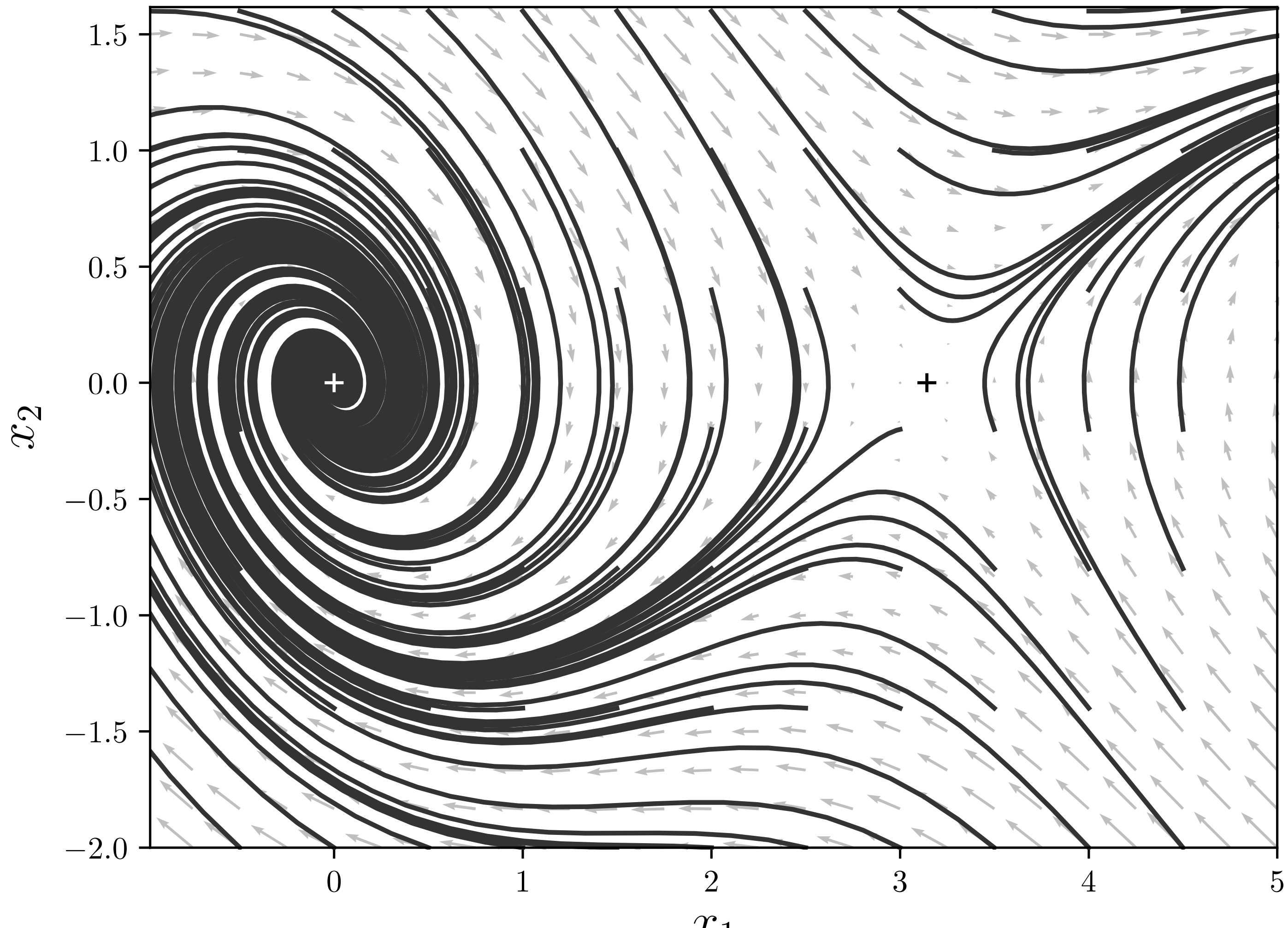
$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \\ x_2 \end{bmatrix}$

k: friction coefficient

Two equilibrium points: $(0,0)$, $(\pi, 0)$

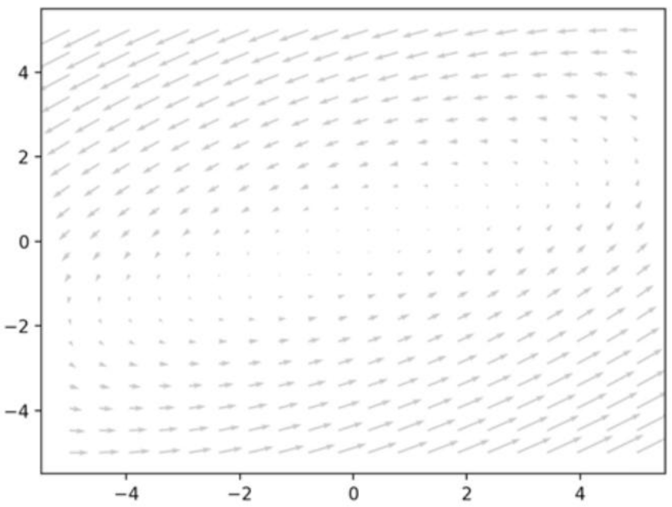


Phase portrait of pendulum with friction

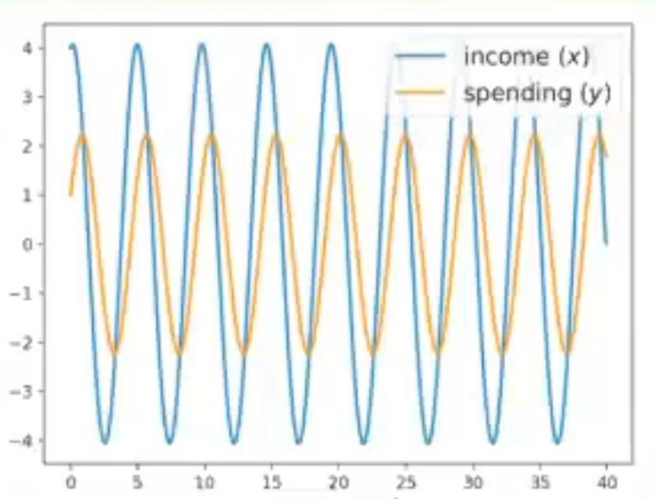


Example: Simple model of an economy

- x: national income
- y: rate of consumer spending
- g: rate government expenditure (control)



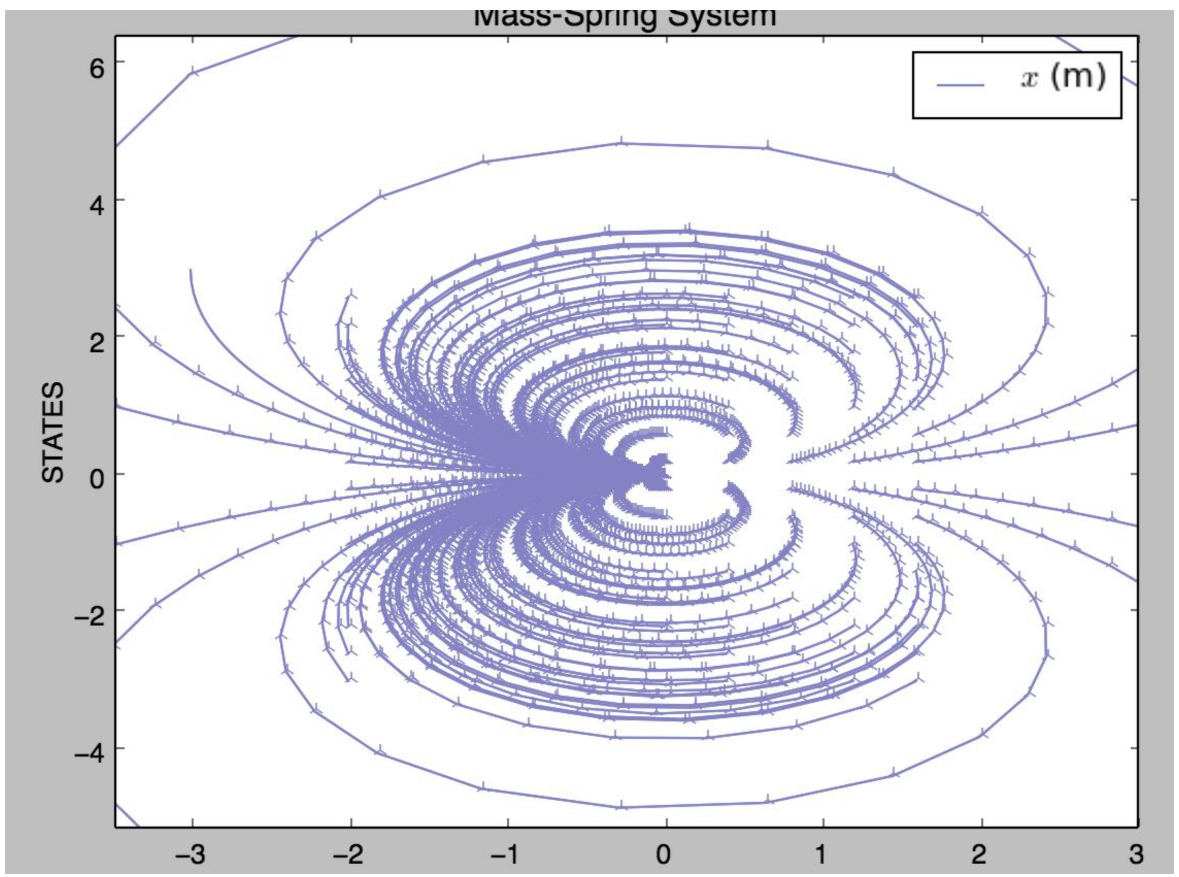
- $\dot{x} = x - \alpha y$
- $\dot{y} = \beta(x - y - g)$



Butterfly

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$$

To plot ODE like this you can use odeint from scipy



Van der pol oscillator

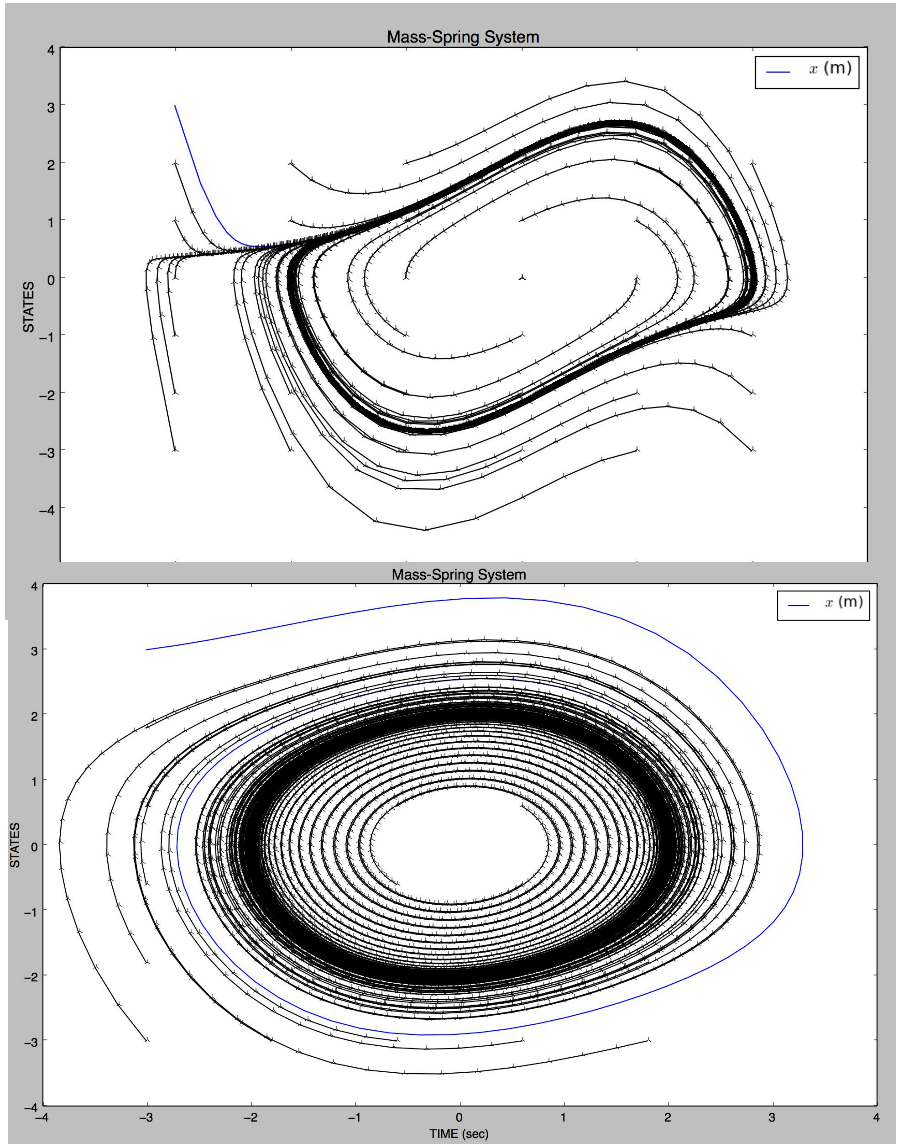
Van der pol oscillator

$$\frac{dx^2}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

$x_1 = x; x_2 = \dot{x}_1;$

coupling coefficient $\mu = 2 \;\; 0.1$

$$\begin{bmatrix} \dot{x_2} \\ \dot{x_1} \end{bmatrix} = \begin{bmatrix} \mu(1 - x_1^2)x_2 - x_1 \\ x_2 \end{bmatrix}$$



Introduction to dynamical systems

Behaviors of physical processes are described in terms of instantaneous laws

Common notation: $\frac{dx(t)}{dt} = f(x(t), u(t), t) \; - \; Eq.(1)$

where time $t \in \mathbb{R}$; **state** $x(t) \in \mathbb{R}^n$; $\mathbf{input}$ $u(t) \in \mathbb{R}^m$; $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$

Example. $\frac{dx(t)}{dt} = v(t)$; $\frac{dv(t)}{dt} =- g$

Initial value problem: Given system (1) and initial state $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and input $\mathrm{u}: \mathbb{R} \rightarrow \mathbb{R}^m$, find a state trajectory or *solution* of (1).

Notions of solution

What is a solution? Many different notions.

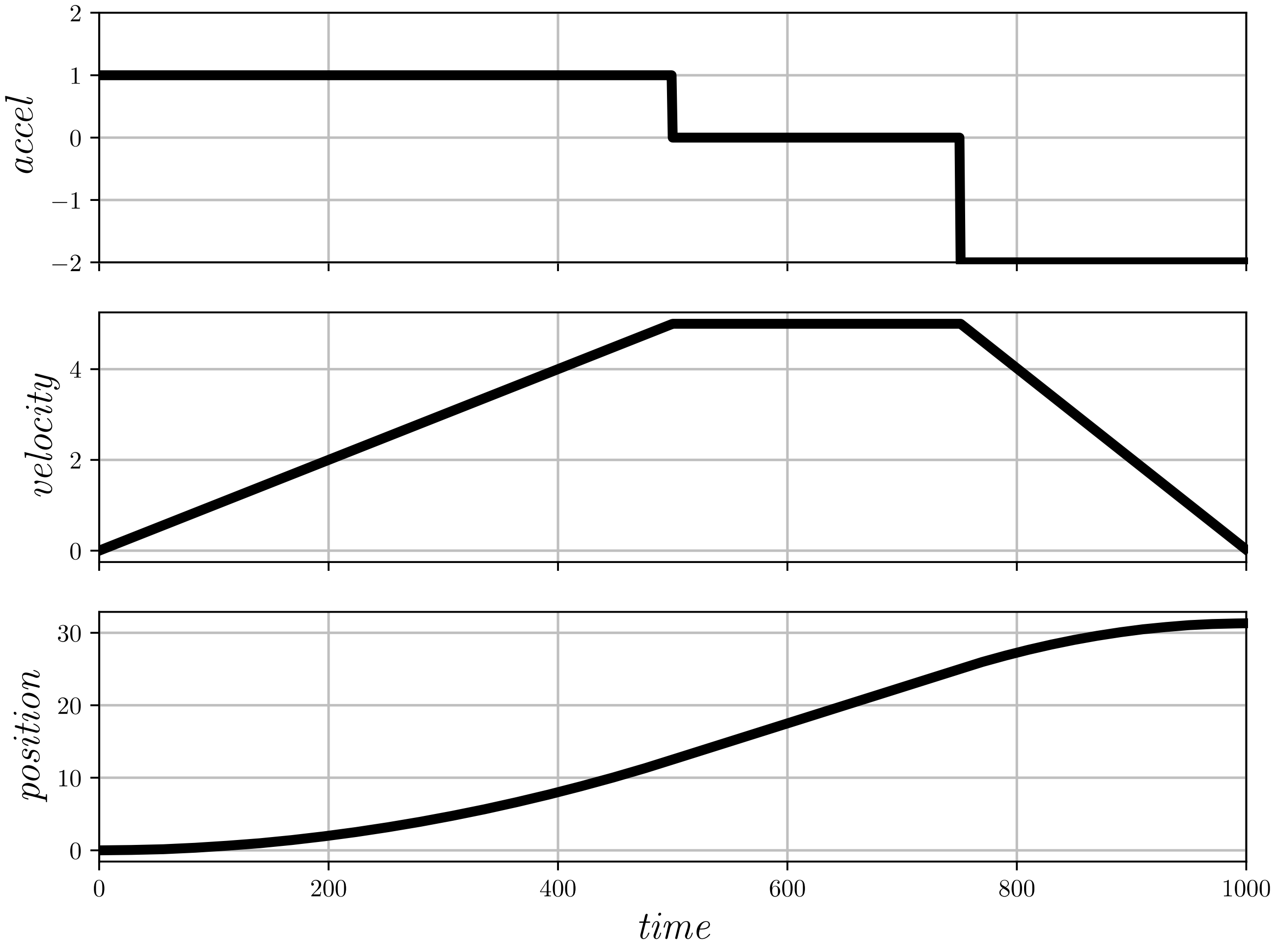
Definition 1. (First attempt) Given x_0 and u, ξ: $\mathbb{R} \to \mathbb{R}^n$ is a solution or trajectory iff

(1)$\xi(t_0) = x_0$ and

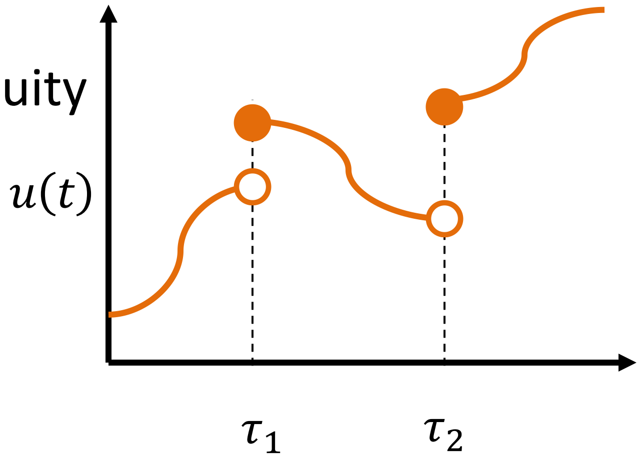
(2)$\frac{d}{dt}\xi(t) = f(\xi(t), u(t), t))$, $\forall t \in \mathbb{R}$.

Mathematically makes sense, but too restrictive. Assumes that ξ is not only continuous, but also differentiable. This disallows $\mathrm{u}(t)$ to be discontinuous, which is often required for optimal control.

Getting from point a to point b



Modified notion



Definition. $u(\cdot)$ is a piece-wise continuous with set of discontinuity points $D \subseteq \mathbb{R}^m$ if

(1) $\forall \tau \in D, \lim_{t \to \tau^+} u(t) < \infty; \lim_{t \to \tau^-} u(t) < \infty$

(2) Continuous from right $\lim_{t \to \tau^+} u(t) = u(t)$

(3) $\forall\, t_0 < t_1$, $[t_0, t_1] \cap D$ is finite

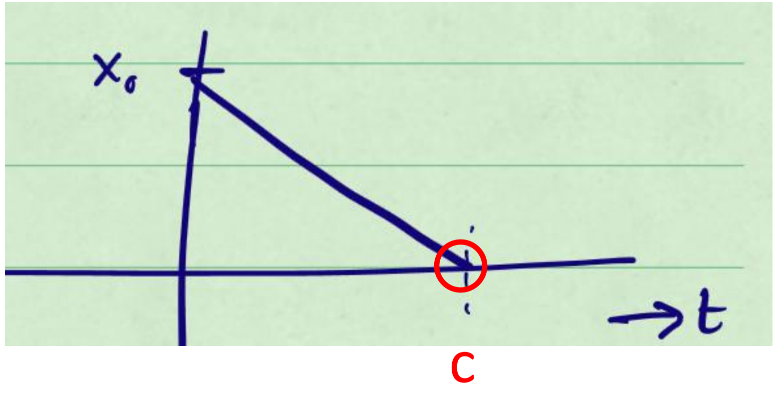
$PC([t_0, t_1], \mathbb{R}^m)$ is the set of all piece-wise continuous functions over the domain $[t_0, t_1]$

Definition 2. **Given** $\boldsymbol{x_0}$ **and** $\boldsymbol{u}$, $\xi\colon \mathbb{R} \to \mathbb{R}^n$ is a **solution** or trajectory iff (1) $\xi(t_0) = x_0$ and (2) $\frac{d}{dt}\xi(t) = f(\xi(t), u(t), t)$, $\forall t \in \mathbb{R} \setminus \mathrm{D}$.

When can we guarantee the existence of solutions?

Example. $\dot{x}(t) =- sgn\big(x(t)\big); x_0 = c; t_0 = 0; c > 0$
Solution: $\xi(t) = c - t$ for $t \leq c$; check $\dot{\xi} =- 1 =- sgn\big(\xi(t)\big)$
Problem: $-sgn\big(x(t)\big)$ is discontinuous, cannot find ξ such that $\dot{\xi}$ exists and suddenly changes

$$sgn(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$



When can we guarantee the existence of solutions?

Example. $\dot{x}(t) = x^2; x_0 = c; t_0 = 0; c > 0$

Solution: $\xi(t) = \frac{c}{1-tc}$ works for $t \neq 1/c$; $\dot{\xi} = \frac{-c\,(-c)}{(1-tc)^2} = (\xi(t))^2$

Problem: As $t \to \frac{1}{c}$ then $\xi(t) \to \infty$; $f(x)$ grows too fast

Lipschitz continuity

A function $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous if there exist $L > 0$ such that for any pair $x, x' \in \mathbb{R}^n$, $\left|\left|f(x) - f(x')\right|\right| \leq L\left|\left|x - x'\right|\right|$

Examples: $6x + 4$; $|x|$; all differentiable functions with bounded derivatives

Non-examples: $\sqrt{x}$; x^2 (locally Lipschitz)

Existence and uniqueness of solutions

Theorem. If $f(x(t), u(t), t)$ is Lipschitz continuous in the first argument, and $u(t)$ is PC then (1) has unique solutions.

In general, for nonlinear dynamical systems we do not have closed form solutions for $\xi(t)$, but there are numerical solvers

Linear system and solutions

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

For a given initial state $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}$ and $u(.) \in PC(\mathbb{R}, \mathbb{R}^n)$ the *solution* is a function $\xi(.\,, t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$

We studied several properties of ξ: continuity with respect to first and third argument, linearity, decomposition

Linear time-varying systems

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ --- (2)

$u(t)$ continuous everywhere except D_x

Theorem. Let $\xi(t, t_0, x_0, u)$ be the solution for (2) with points of discontinuity , D_x

1. $\forall t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(\cdot\,, t_0, x_0, u): \mathbb{R} \to \mathbb{R}^n$ is continuous and differentiable $\forall\, t \in \mathbb{R} \setminus D_x$
2. $\forall t, t_0 \in \mathbb{R},\ u \in PC(\mathbb{R}, \mathbb{R}^m), \xi(t, t_0, \cdot\,, u): \mathbb{R}^n \to \mathbb{R}^n$ is continuous
3. $\forall t, t_0 \in \mathbb{R}, x_{01}, x_{02} \in \mathbb{R}^n, u_{1,} u_2 \in PC(\mathbb{R}, \mathbb{R}^m), a_1, a_2 \in \mathbb{R},\ \xi(t, t_0, a_1 x_{01} + a_2 x_{02}, a_1 u_1 + a_2 u_2) = a_1 \xi(t, t_0,\ x_{01},\ u_1) + a_2 \xi(t, t_0,\ x_{02},\ u_2)$
4. $\forall t, t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^n, u \in PC(\mathbb{R}, \mathbb{R}^m),\ \xi(t, t_0,\ x_0,\ u) = \xi(t, t_0,\ x_0,\ \mathbf{0}) + \xi(t, t_0,\ 0,\ u)$