Grubler's Formula

$$Dof = m(N - J - 1) + \sum_i f_i$$

N: number of links

J: number of joints

m: Dof of rigid body, if planner m=3, if spatial m=6

f_i: number of freedom from joint i

2D Rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3D Rotation matrix

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3×3 skew-symmetric matrix representation

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Angular velocity in the fixed frame and body frame

$$[\omega_s] = \dot{R}R^T$$
$$[\omega_b] = R^T\dot{R}$$

Rotation matrix in exponential coordinate representation

$$R(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + sin\theta[\hat{\omega}] + (1 - cos\theta)[\hat{\omega}]^2$$

Matrix Logarithm Method

$$tr(R) = r_{11} + r_{22} + r_{33} = 1 + 2cos\theta$$
$$[\hat{\omega}] = \frac{1}{2sin\theta}(R - R^T)$$

The inverse of the transformation matrix

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

Twist

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \quad [\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

Twist in fixed and body frame

$$[\mathcal{V}_b] = T^{-1}\dot{T} \quad [\mathcal{V}_s] = \dot{T}T^{-1}$$
$$\mathcal{V}_s = [Ad_T]\mathcal{V}_b \quad [Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

Screw

$$\text{screw axis: } \mathcal{S} = \{q, \hat{s}, h\}, \quad \text{velocity: } \dot{\theta}$$

Screw to twist

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ h\hat{s}\dot{\theta} - \hat{s}\dot{\theta} \times q \end{bmatrix}$$

Twist to screw

$$\hat{s} = \frac{\omega}{||\omega||} = \hat{\omega}, \quad \dot{\theta} = ||\omega||, \quad h = \frac{\omega^T v}{||\omega||^2}$$
$$\text{if } \omega \neq 0, \quad q = \frac{\hat{s} \times v}{\dot{\theta}}$$

Screw axis

$$||\omega|| = 1 : \ v = -\omega \times q + h\omega$$
$$||v|| = 1 \ , \ \omega = 0 : \ \text{translate along v}$$

Exponential Coordinates for Rigid Body Motions(S $\rightarrow$ T)

$$T = e^{[S]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}$$
$$G(\theta) = I\theta + (1 - cos\theta)[w] + (\theta - sin\theta)[w]^2$$

Logarithm of rigid body motions(T $\rightarrow$ S)

$$tr(R) = r_{11} + r_{22} + r_{33} = 1 + 2cos\theta$$
$$[\hat{\omega}] = \frac{1}{2sin\theta}(R - R^T)$$
$$v = G^{-1}(\theta)p \quad G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[w] + (\frac{1}{\theta} - \frac{1}{2}cot(\frac{\theta}{2}))[w]^2$$

Forward Kinematics

$$\text{Screw in fixed frame: } T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} \cdots e^{[S_n]\theta_n} M$$
$$\text{Screw in body frame: } T(\theta) = M e^{[B_1]\theta_1} e^{[B_2]\theta_2} \cdots e^{[B_n]\theta_n}$$

Velocity Kinematics

Space Jacobian

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} = [J_{s,1} \; J_{s,2} \; \cdots \; J_{s,n}] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$
$$J_{s,1} = S_1 \quad J_{s,2} = Ad_{e^{[S_1]\theta_1}}(S_2) \quad \cdots \quad J_{s,n} = Ad_{e^{[S_1]\theta_1} \ldots e^{[S_{n-1}]\theta_{n-1}}}(S_n)$$

Body Jacobian

$$\mathcal{V}_b = J_b(\theta)\dot{\theta} = [J_{b,1} \; J_{b,2} \; \cdots \; J_{b,n}] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$
$$J_{b,n} = \mathcal{B}_n \quad J_{b,n-1} = Ad_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1}) \quad \cdots \quad J_{b,1} = Ad_{e^{-[\mathcal{B}_n]\theta_n} \ldots e^{-[\mathcal{B}_2]\theta_2}}(\mathcal{B}_1)$$

Relationship between Space and Body Jacobian

$$[\mathcal{V}_s] = \dot{T}_{sb} T_{sb}^{-1} \qquad [\mathcal{V}_b] = T_{sb}^{-1} \dot{T}_{sb}$$
$$\mathcal{V}_s = Ad_{T_{sb}}(\mathcal{V}_b) \quad \mathcal{V}_b = Ad_{T_{bs}}(\mathcal{V}_s)$$
$$J_s(\theta) = Ad_{T_{sb}}(J_b(\theta)) = [Ad_{T_{sb}}] J_b(\theta)$$
$$J_b(\theta) = Ad_{T_{bs}}(J_s(\theta)) = [Ad_{T_{bs}}] J_s(\theta)$$

Cross Product

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = i \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - j \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + k \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

Path & Trajectory

Straight-Line Path

$$\theta(s) = \theta_0 + s(\theta_1 - \theta_0) \quad s \in [0, 1]$$
$$X(s) = X_0 + s(X_1 - X_0) \quad s \in [0, 1]$$

Decouple Rotation and Translation

$$X = (R, p)$$
$$p(s) = p_0 + s(p_1 - p_0)$$
$$R(s) = R_0 e^{log(R_0^T R_1)s}$$

Polynomial Time-Scaling

$$s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \quad t \in [0, T]$$

$$s(0) = 0, \quad \dot{s}(0) = 0, \quad s(T) = 1, \quad \dot{s}(T) = 0$$

$$s(t) = \frac{3}{T^2} t^2 - \frac{2}{T^3} t^3, \quad t \in [0, T]$$

$$\theta(t) = \theta_0 + \left(\frac{3}{T^2} t^2 - \frac{2}{T^3} t^3 \right) (\theta_1 - \theta_0), \quad t \in [0, T]$$

Motion Planning

Distance Measures

$$d(q, B) = \min_{i,j} \left(\| r_i(q) - b_j \| - R_i - B_j \right)$$

Given a robot at q represented by k spheres of radius R_i centered at $r_i(q)$, and an obstacle B represented by l spheres of radius B_j centered at b_j.

Collision Detection

$$d(q, B) > 0 \quad \rightarrow \quad \text{no contact}$$

$$d(q, B) = 0 \quad \rightarrow \quad \text{contact}$$

$$d(q, B) < 0 \quad \rightarrow \quad \text{penetration}$$

Manipulation

Wrenches

$$m_a = p_a \times f_a$$

$$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix}$$

Impenetrability Constraint

$$\dot{d}(q) = \frac{\partial d}{\partial q} \dot{q} = \hat{n}^\top \left(\dot{p}_A - \dot{p}_B \right)$$

$$\dot{p}_A = v_A + \omega_A \times p_A = v_A + [\omega_A] p_A$$

$$\mathcal{F} = \begin{bmatrix} p_a \times \hat{n} \\ \hat{n} \end{bmatrix}$$

$$\mathcal{F}^\top \left(\mathcal{V}_A - \mathcal{V}_B \right) \geq 0$$

Active Constraint

$$\mathcal{F}^{\top}\left(\mathcal{V}_A - \mathcal{V}_B\right) = 0$$

Rolling Contacts

$$\dot{p}_A = \dot{p}_B$$

Planar Center of Rotation

$$\mathcal{V} = \begin{bmatrix} \omega_z \\ v_x \\ v_y \end{bmatrix}$$

$$c = \left(-\frac{v_y}{\omega_z}, \frac{v_x}{\omega_z}, \omega_z\right)$$