Lecture 15
Trajectory Generation

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April 7, 2026

Modern Robotics Ch 9

Administrivia – Final Project

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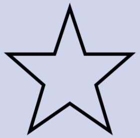
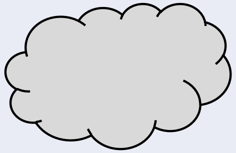
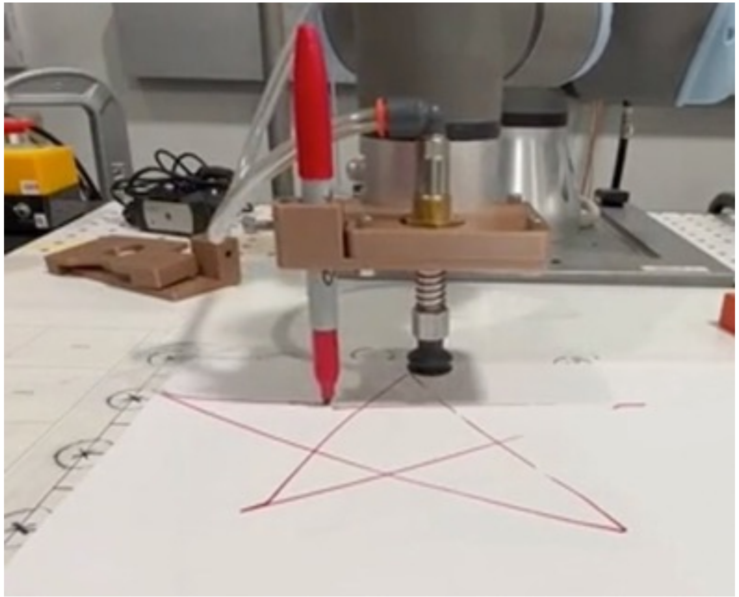
Final Project Goal: Draw some pictures using the UR arms

Project Tasks

Task	Goal	Example
1	Draw a given shape with simple keypoints	
2	Draw a given shape with more complex contours	 
3	Draw a picture of your choice! And compete in an art competition (details will be posted on CW)	 
4	Submit formal report	

Administrivia – Final Project

- Specifics and lab details will be posted on CampusWire
- Freedom in how you go about detecting images and implementing the motions
- Demo drawings to lab TAs before reading day
- Report due end of finals
- Rubrics will be posted in the next few weeks



Administrivia – Exam 2

- mean : 67%
- std: 17%

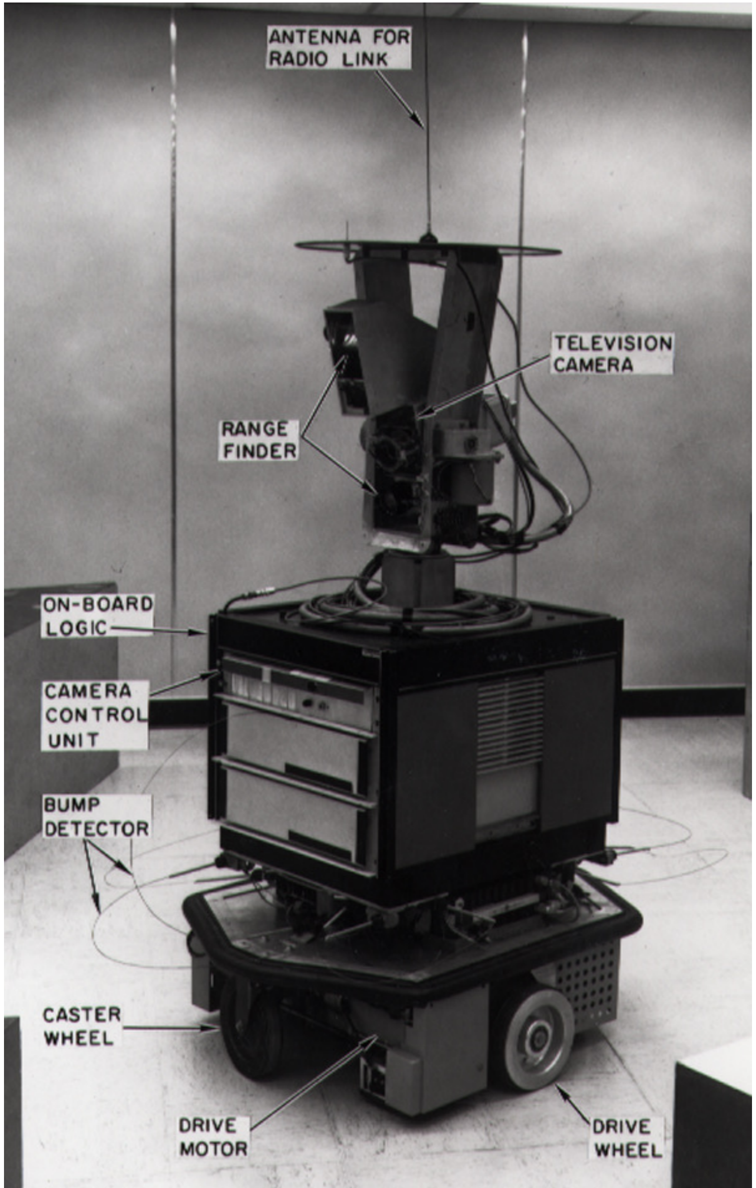
Historical Fun Fact

Meet Shakey the Robot:

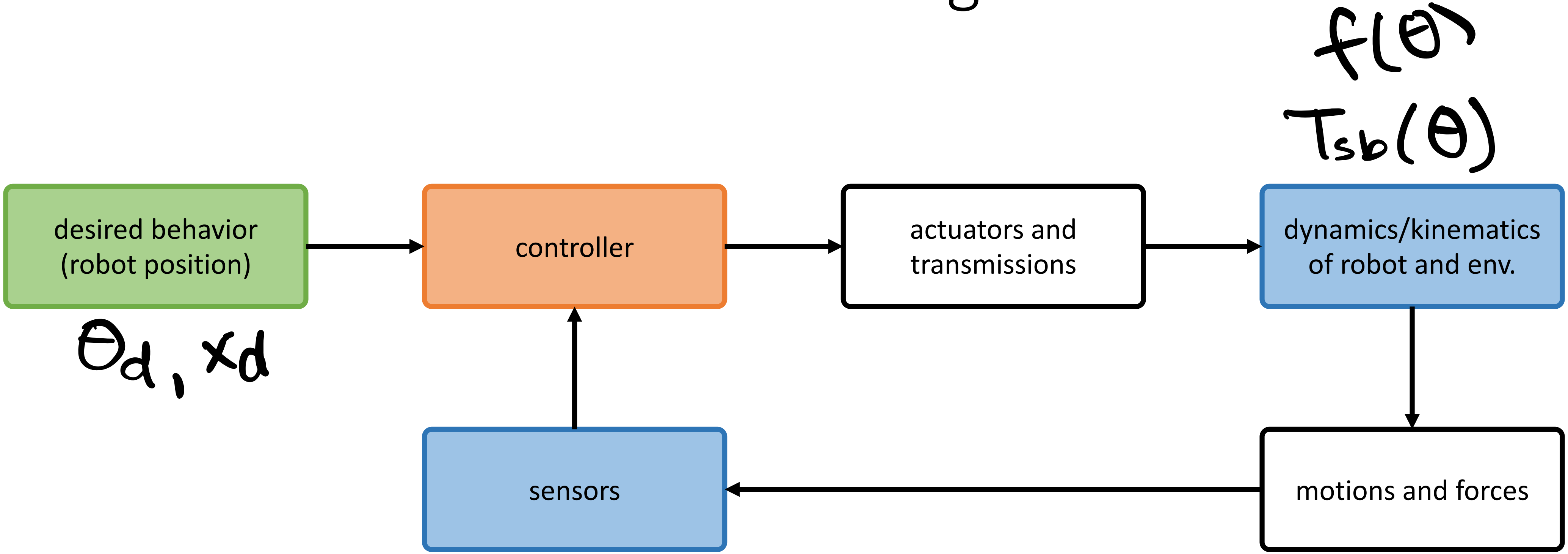
An Experiment in Robot Planning and Learning

Developed by Stanford Research Institute (SRI) (1966 – 1972)

1. An operator types the command "push the block off the platform" at a computer console.
2. Shakey looks around, identifies a platform with a block on it, and locates a ramp in order to reach the platform.
3. Shakey then pushes the ramp over to the platform, rolls up the ramp onto the platform, and pushes the block off the platform.
4. Mission accomplished.



Control Paradigm

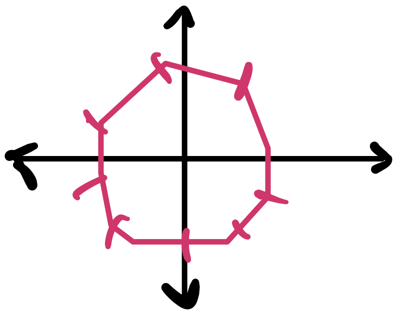


Trajectories and Paths

- The specification of a robot state as a function of time is called a **trajectory**

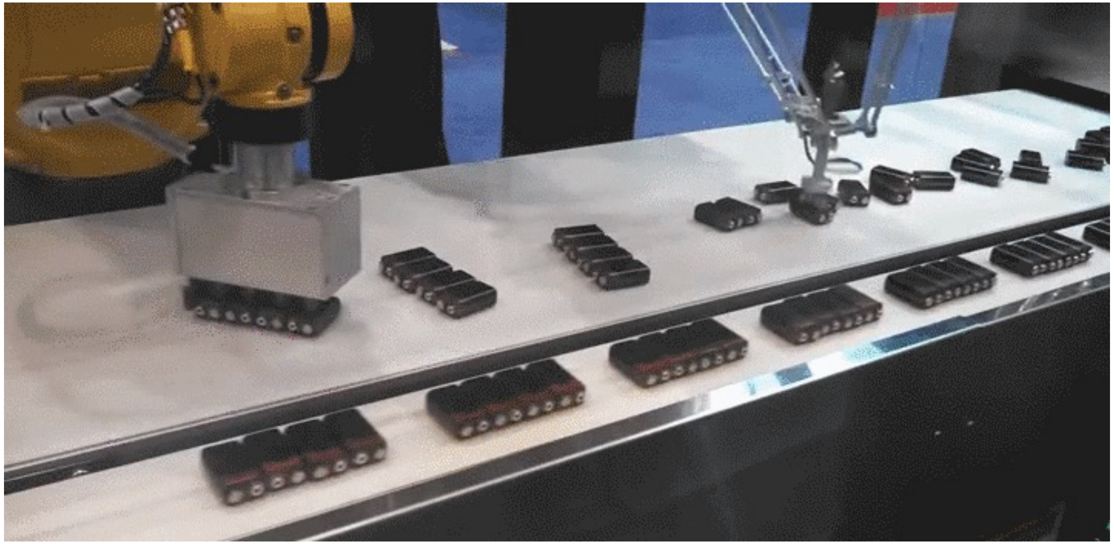
$\theta : [0, T] \longrightarrow \mathbb{R}^n \rightsquigarrow$ $\theta(t)$ gives us joint angles at time t

- Using forward kinematic maps, we can obtain the position of each link given as joint angles
  - The trajectory of the end-effector is then $T_{sb}(\theta(t))$
- A **path** is a set of points



path + specification of timing yields a trajectory

Trajectories for Automation



Normalized Trajectories

- **Path** $\theta(s)$ maps ~~a~~ scalar path parameter $s \in [0,1]$ to a point in the robot's configuration space ← config space

$\theta : [0,1] \to \Theta$

$\theta(0)$ start of path // $\theta(1)$ end of path

- A **time-scaling** $s(t)$ is a monotonically increasing function:

$s : [0,T] \to [0,1]$

→ path + time-scaling define a traj.

$\theta(t) \to \theta(s(t))$

Straight-Line Paths

$s = 0 \rightarrow \theta_0$
$s = 1 \rightarrow \theta_1$

- Given θ_0 and θ_1, find straight-line path: $s \in [0, 1]$

$$\theta(s) = \theta_0 + s(t)(\theta_1 - \theta_0)$$

- Is this in the task or configuration space?

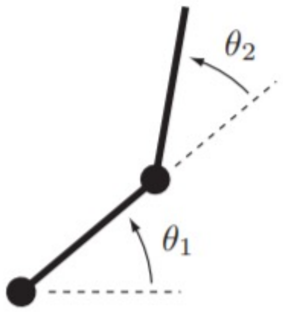
Straight-Line Paths

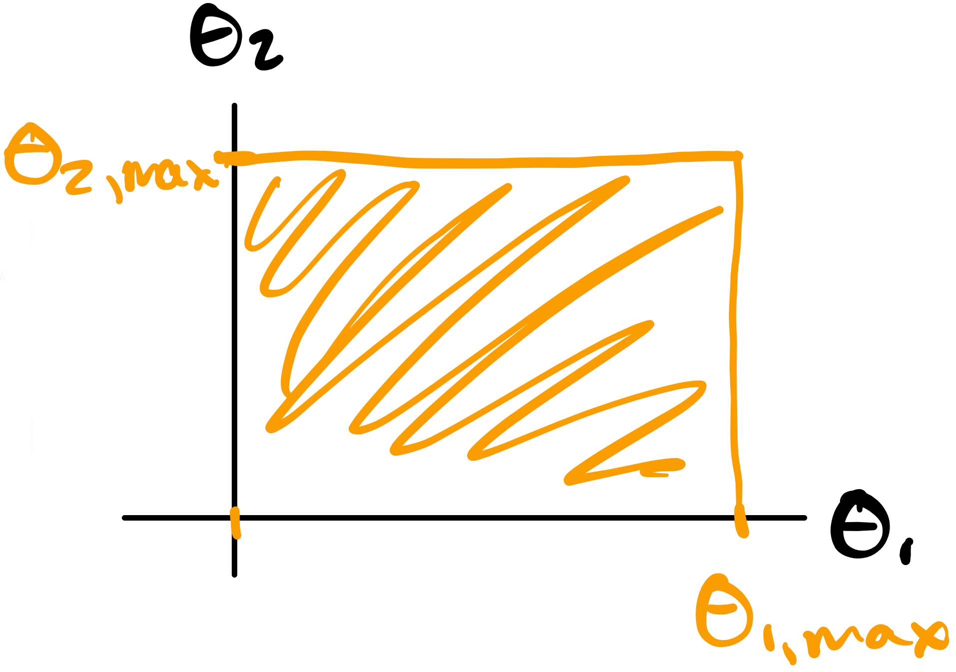
- Given θ_0 and θ_1, find straight-line path:

- Is this in the task or configuration space?
  - Straight lines in joint space do not lead to straight lines in end-effector/task space
- Straight line in task space:

$$X(s) = X_0 + s \cdot (X_1 - X_0)$$

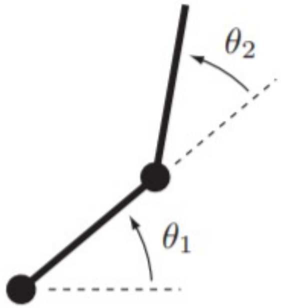
Straight-line Paths

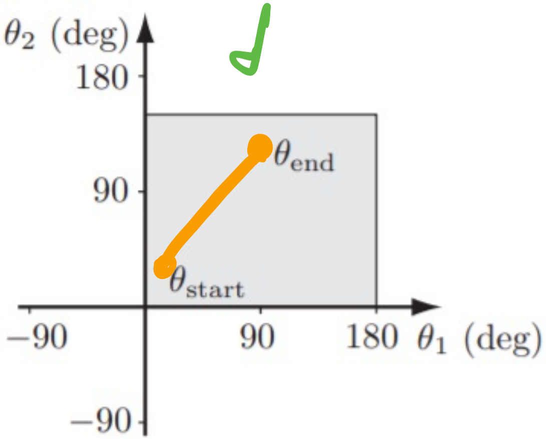


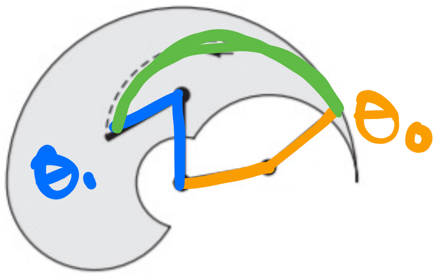


$$\theta_{i,min} \leq \theta \leq \theta_{i,max}$$

Straight-line Paths

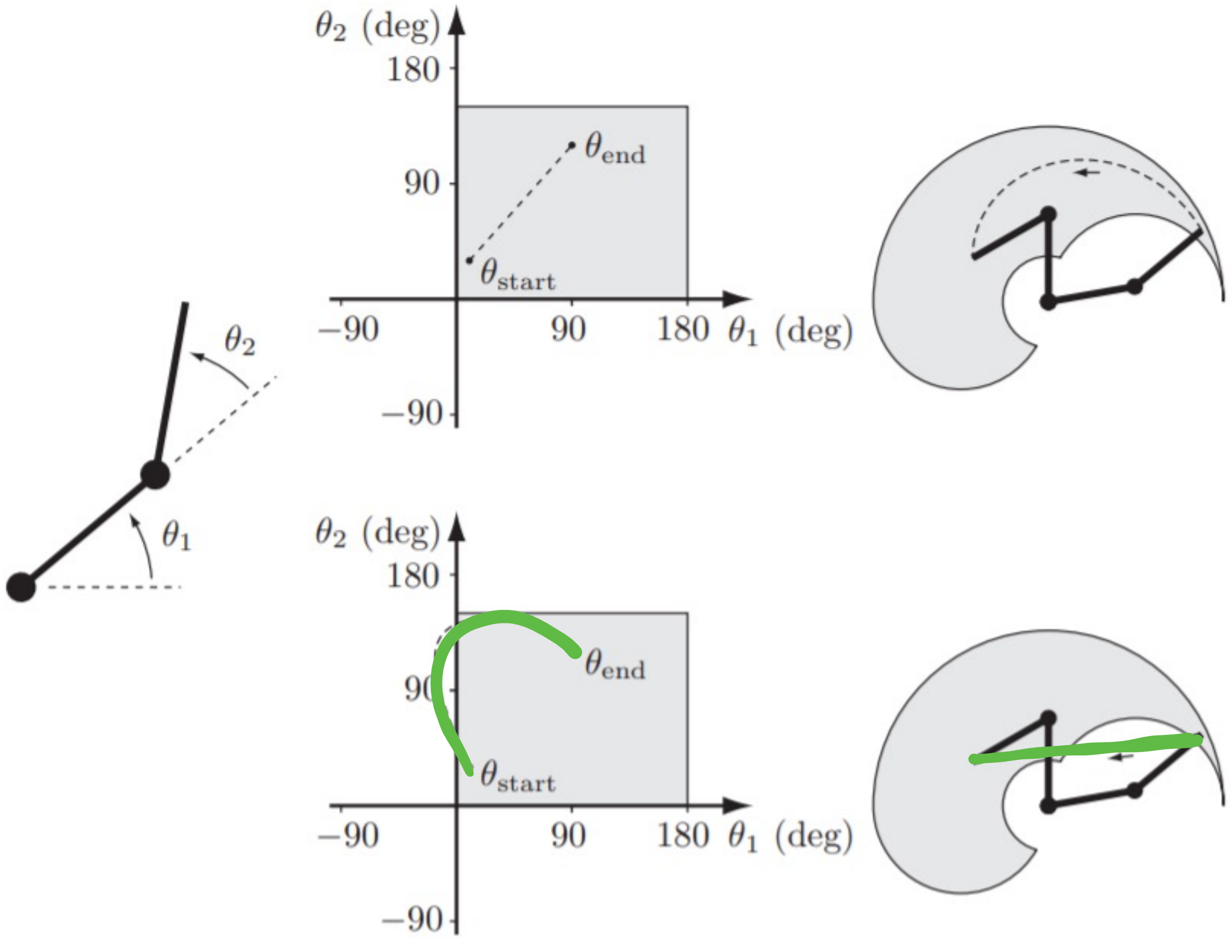






"straight" line in joint space

Straight-line Paths



"straight" line in task (tool) space

Straight lines in SE(3)

In $\mathbb{R}^2$, straight lines are characterized by a constant velocity $\to \dot{y} = v$

$y(t) = y_0 + v(t)$

Recall: $\dot{T} = T[S]$

if S is constant, then: $T(t) = T_0 e^{[S]t}$

Given X_0 + X_1, a "straight" line in SE(3)

is a constant screw motion:

$$X(s) = X_0 e^{\log(\underbrace{X_0^{-1} X_1}) s}$$

$$X_{\substack{start,end \\ 0\ 1}} = \underbrace{X_{start,\{s\}}}_{X_0^{-1}} \cdot \underbrace{X_{\{s\},end}}_{X_1} = X_0^{-1} X_1$$

Straight lines in SE(3)

- We can decouple rotation and translation:

$$X = (R, p) \rightsquigarrow p(s) = p_0 + s(p_1 - p_0)$$

$$R(s) = R_0 e^{\log(R_0^T R_1) s}$$

$\rightarrow$ pass to IK solver

Time-scaling of straight-line paths

- Time scaling ensures that the motion is smooth and constraints are met

$$\theta(s) = \theta_0 + s(t)\cdot(\theta_1 - \theta_0)$$

$$\frac{d\theta(s)}{dt} = \dot{\theta}(s) = \frac{ds}{dt}\cdot(\theta_1 - \theta_0)$$

$$\ddot{\theta}(s) = \frac{d^2s}{dt^2}\cdot(\theta_1 - \theta_0)$$

choose time-scaling $s(t)$ to limit $\dot{\theta}, \ddot{\theta}$

→ often pick a parametric form of $s(t)$, eg polynomial

Polynomial Time-Scaling (1)

let $s(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ (1)

point-to-point motion in time T, impose constraints:

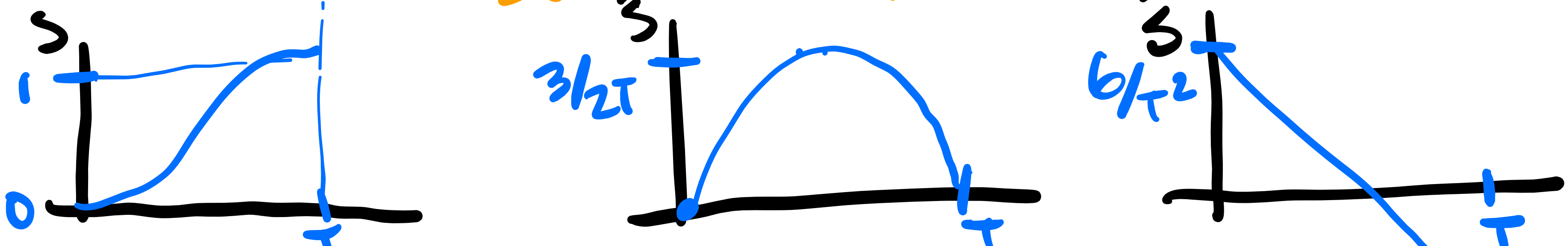
$s(0) = \dot{s}(0) = 0$ and $s(T) = 1$ and $\dot{s}(T) = 0$

(2) $\dot{s}(t) = a_1 + 2a_2 t + 3a_3 t^2$

evaluate (1) + (2) @ $t=0$ + $t=T$, solve for coeff.

$a_0 = a_1 = 0$, $a_2 = 3/T^2$, $a_3 = -2/T^3$

$s(t) = 3t^2/T^2 - 2t^3/T^3$



Polynomial Time-Scaling (2)

① $\theta(t) = \theta_0 + \left(3t^2/T^2 - 2t^3/T^3\right)(\theta_1 - \theta_0)$

② $\dot{\theta}(t) = \left(6t/T^2 - 6t^2/T^3\right)(\theta_1 - \theta_0)$

③ $\ddot{\theta}(t) = \left(6/T^2 - 12t/T^3\right)(\theta_1 - \theta_0)$

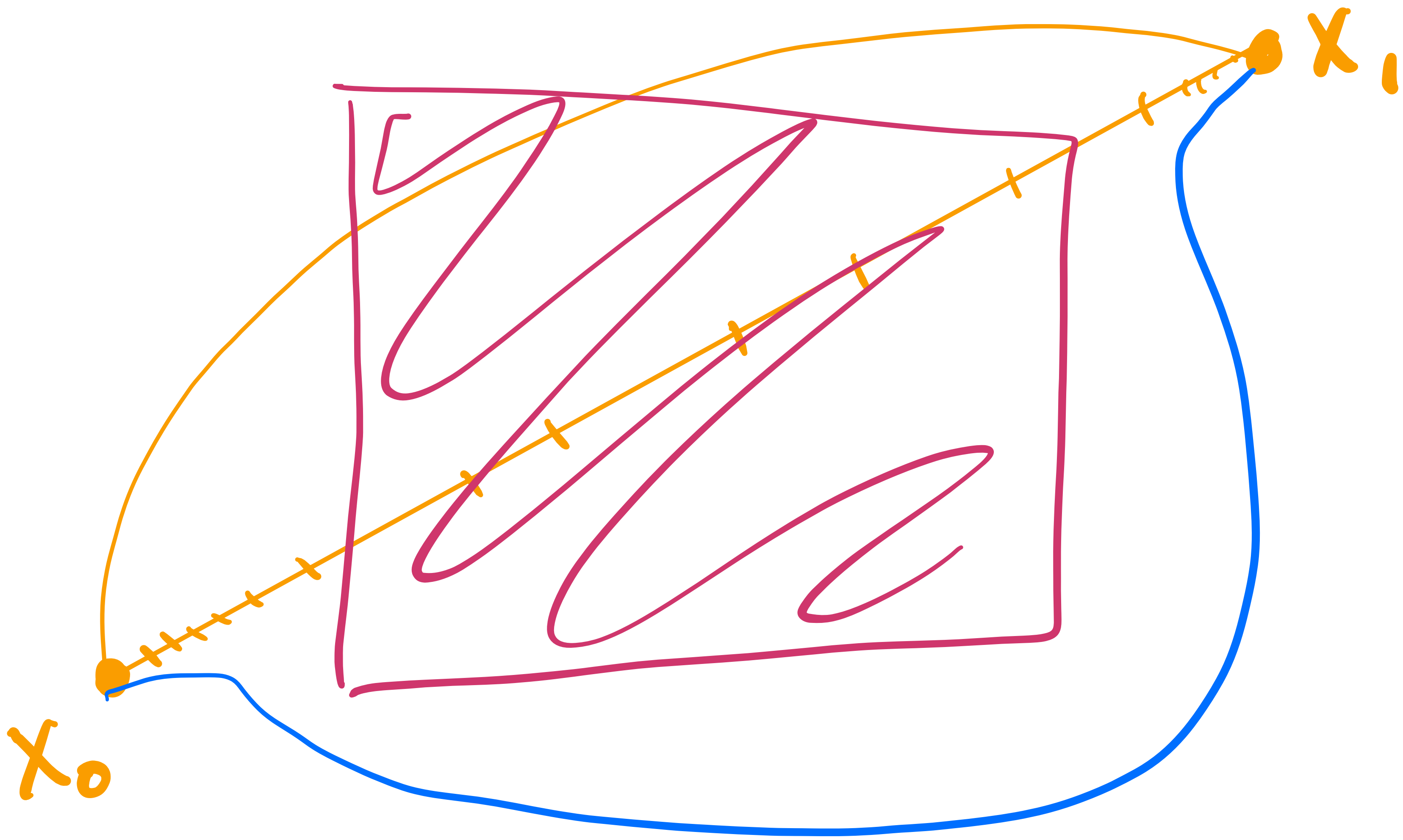
evaluate ② to find max vel @ $t = T/2$

$\rightarrow \dot{\theta}_{max} = \frac{3}{2T}(\theta_1 - \theta_0)$

evaluate ③ to find max acc @ $t = 0$ & $t = T$

$\rightarrow \ddot{\theta}_{max} = \pm \frac{6}{T^2}(\theta_1 - \theta_0)$

④ choose T to meet constraints


X_0
X_1

Summary

- Defined **paths, time-scaling,** and **trajectories**
- Looked at how to find **straight-line paths** in various spaces
- We choose a **parametrization** $\boldsymbol{s(t)}$, and computed the resulting velocity and acceleration profiles of the trajectory
  - Using a third-order polynomial, we tuned their maximal values to meet requirements with one parameter T

Summary

- Defined **paths, time-scaling,** and **trajectories**
- Looked at how to find **straight-line paths** in various spaces
- We choose a **parametrization** $\boldsymbol{s(t)}$, and computed the resulting velocity and acceleration profiles of the trajectory
  - Using a third-order polynomial, we tuned their maximal values to meet requirements with one parameter T
- We can follow the same procedure with different parametrizations for $s(t)$ (e.g., polynomials of order 5, trapezoidal functions, splines, etc.)
  - Having more parameters allows us to meet more constraints. For example, using a fifth order polynomial, we can ensure that $\ddot{\theta}(0) = \ddot{\theta}(T) = 0$, meaning no jerk at beginning and end of the motion
- **Next topics** are on different concepts of / approaches to planning when the path may not be given