Lecture 12: inverse kinematics I

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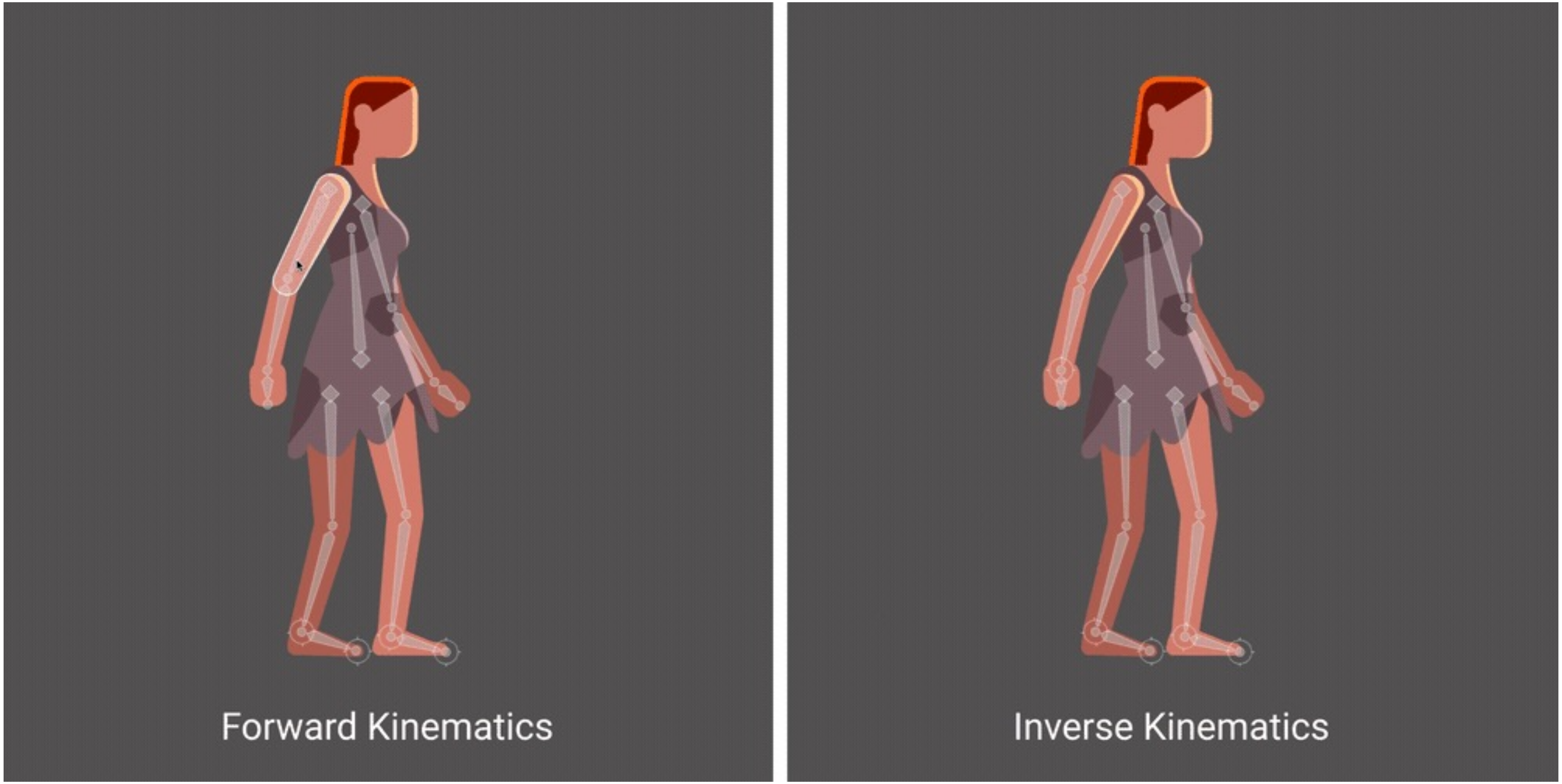
Inverse Kinematics in Animation



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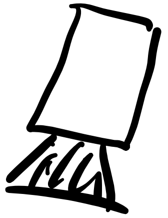
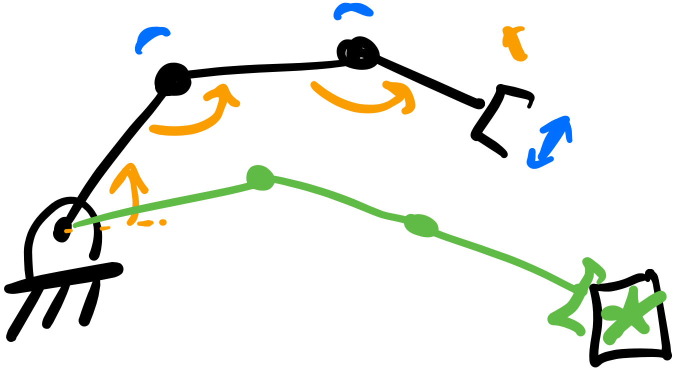
What is going on in this class?

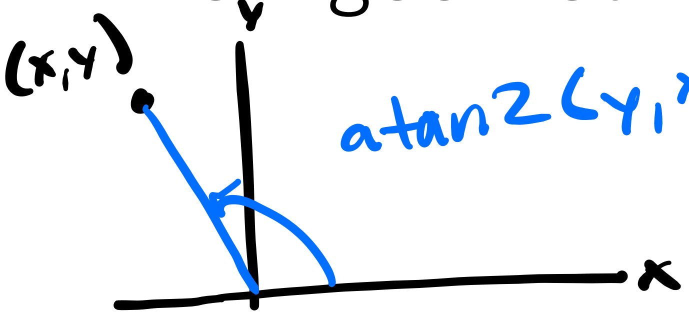


- Rigid Body Motion
- Forward Kinematics
  - Calculate the position of the end-effector of an open chain given joint angles
    $x(t) = \underbrace{f(\theta(t))} = T(\theta)$
- Velocity Kinematics
  - Calculate the velocity (twist!) of the end-effector
    $\frac{d}{dt}x(t) = \frac{\partial f}{\partial \theta}(\theta) \cdot \dot{\theta}$



- Inverse Kinematics
  - Computes the possible joint angles from the pose of the end-effector
  - Given $T(\theta)$, find solutions θ that satisfy $\underbrace{T(\theta)} = \underline{X}$

A few geometric tricks



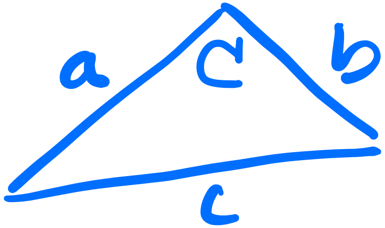
$\mathrm{atan2}(y,x) \in (-\pi, \pi)$

Two argument arctan function atan2

$$\mathrm{atan2}(y,x) = \begin{cases} \mathrm{atan}\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \mathrm{atan}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \mathrm{atan}\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ \frac{-\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$



Euler Angles

Given some arbitrary rotation, how to find the associated angles?

ZYX Euler angles represent a rotation as follows:

$R(\alpha, \beta, \gamma) = \mathrm{Rot}(\hat{z}, \alpha)\mathrm{Rot}(\hat{y}, \beta)\mathrm{Rot}(\hat{x}, \gamma)$

$\alpha, \gamma \in (-\pi, \pi]$ and $\beta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right)$

where

$$\mathrm{Rot}(\hat{z}, \alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles (see Appendix B)

$$R(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta s_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

If r_{ij} denotes the ij^{th} element in R, then

1. If $r_{31} \neq \pm 1$, set $\beta = \text{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right), \alpha = \text{atan2}(r_{21}, r_{11}), \gamma = \text{atan2}(r_{32}, r_{33})$
2. If $r_{31} = -1$, then $\beta = \frac{\pi}{2}$ and there are infinite solutions for α and γ. One solution is: $\alpha = 0,\ \gamma = \text{atan2}(r_{12}, r_{22})$
3. If $r_{31} = 1$, then $\beta = -\frac{\pi}{2}$ and there are infinite solutions for α and γ. One solution is: $\alpha = 0,\ \gamma = -\text{atan2}(r_{12}, r_{22})$

Inverse Kinematics

- **Forward Kinematics:** computes the end-effector position from joint angles:

  joint angles $\longrightarrow$ SE(3)

  $\theta \longmapsto T(\theta)$

- **Inverse Kinematics:** computes the possible joint angles from the pose of the end-effector

  SE(3) $\longrightarrow$ joint space

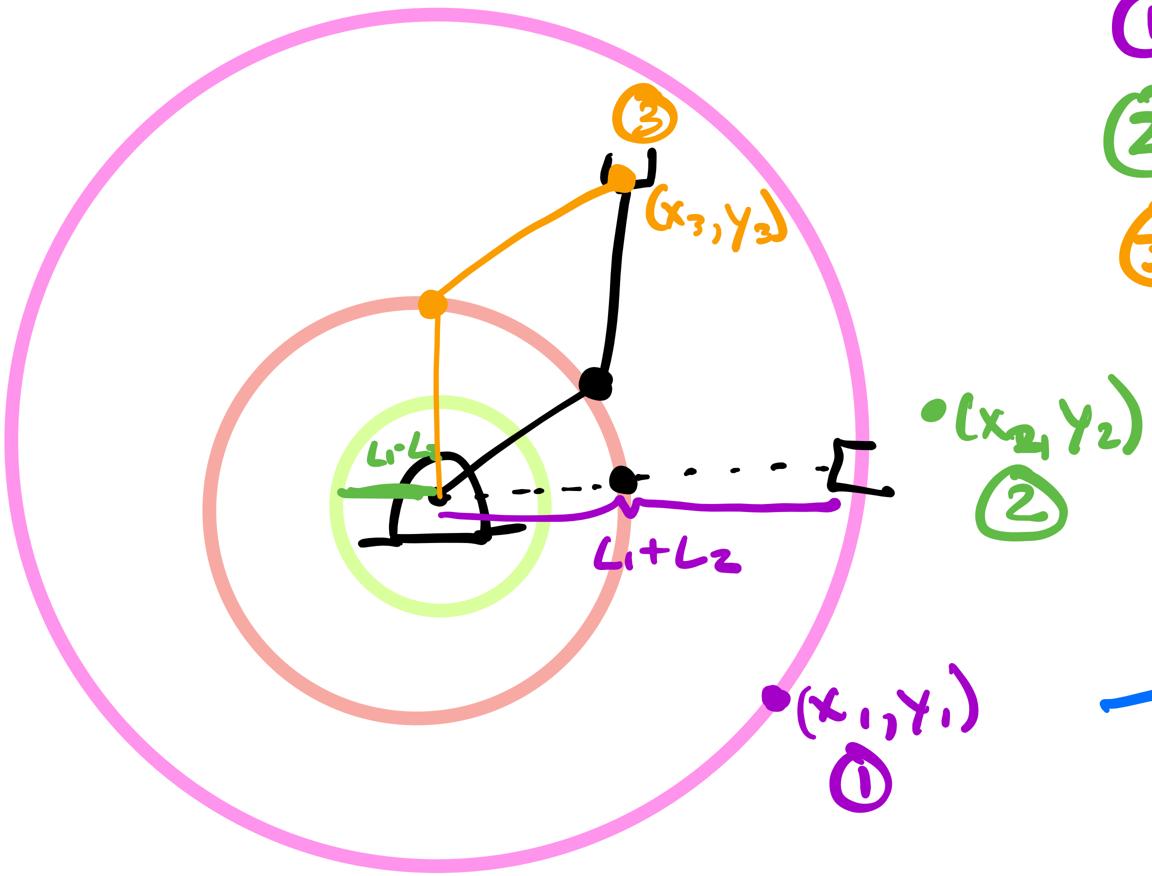
  $X \longmapsto \theta$

- Given $T(\theta)$, find solutions θ that satisfy $T(\theta) = X$

  soln methods $\longrightarrow$ analytic

  $\longrightarrow$ numerical

Inverse Kinematics (IK) Solutions



① one soln

② no soln

③ two solns

→ lefty, righty

→ elbow up/down

→ IK is often multi valued

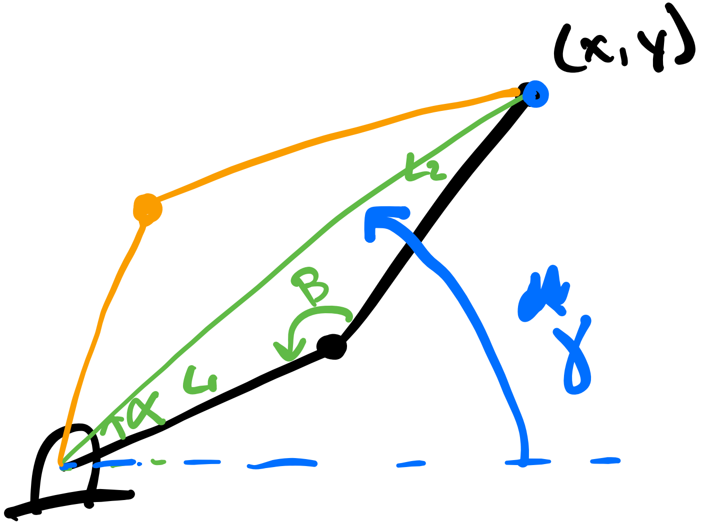
Analytic Inverse Kinematics

use law of cosines:

$$L_1^2 + L_2^2 - 2L_1L_2 \cos\beta = x^2 + y^2$$

$$\rightarrow \beta = \cos^{-1}\left(\frac{L_1^2 + L_2^2 - x^2 - y^2}{2L_1L_2}\right)$$

$$\rightarrow \alpha = \cos^{-1}\left(\frac{x^2 + y^2 + L_1^2 - L_2^2}{2L_2\sqrt{x^2+y^2}}\right)$$



find γ w/ atan2(y, x)

righty: $\theta_1 = \gamma - \alpha$, $\theta_2 = \pi - \beta$

lefty: $\theta_1 = \gamma + \alpha$, $\theta_2 = \beta - \pi$

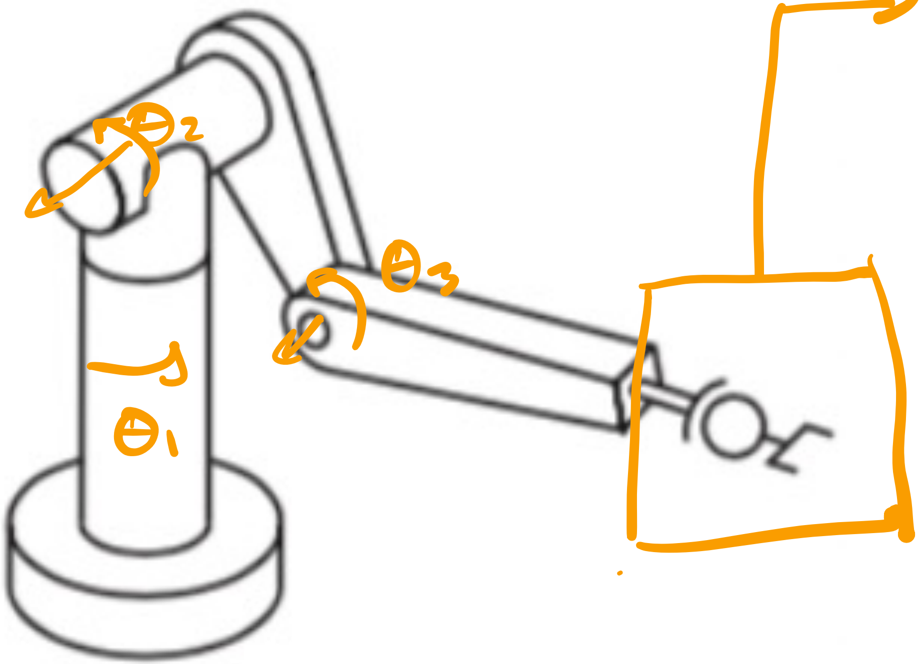
Analytic inverse kinematics

- In the 2R example, there were 2 DOFs for the end-effector and 2 joint angles. This implied that there was a **finite number of solutions.**
- If there are more joint angles than DOFs of the end-effector, there may be **an infinite number of solutions.**
- In practice, we will assume that #DOFs of end-effector = #joint angles.
- For now, assume in general:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_6]\theta_6} M$$

and that we are given an end-effector pose $X \in SE(3)$.

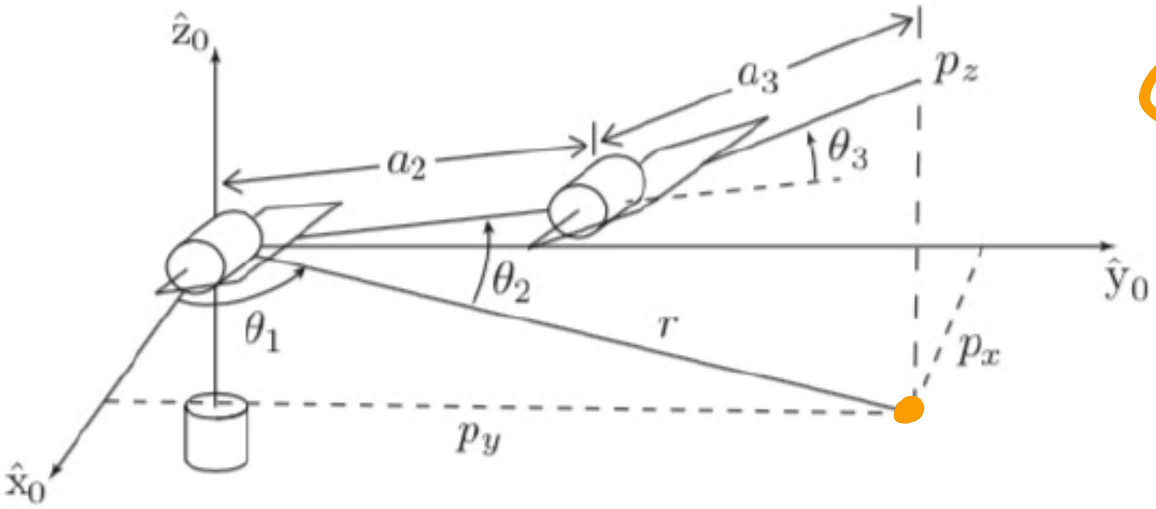
6R PUMA arm



→ wrist: joint 4,5,6

→ solve for pos
→ solve for ori

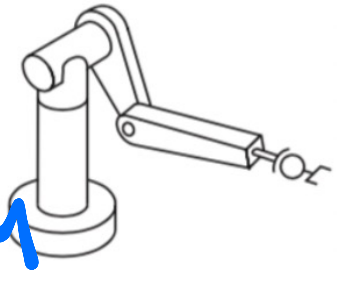
6R PUMA arm



① $\theta_1 = \text{atan2}(p_y, p_x)$

② given θ_1, we have 2R problem!

6R PUMA arm – orientation



recall: $\underbrace{X}_{\text{des. pose}} = \underbrace{e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3}}\ \underbrace{\cdots e^{[S_6]\theta_6}}_{?}\ \underbrace{M}_{\checkmark}$

need: $\underbrace{e^{[S_4]\theta_4} e^{[S_5]\theta_5} e^{[S_6]\theta_6}} = \underbrace{e^{-[S_3]\theta_3} e^{-[S_2]\theta_2} e^{-[S_1]\theta_1} X M^{-1}}_{\text{compute \& pull out}}$

a wrist is designed:

$\omega_4 = [0\ 0\ 1]^T$, $\omega_5 = [0\ 1\ 0]^T$, $\omega_6 = [1\ 0\ 0]^T$

use ZYX Euler Angles:

$Rot(\hat{z},\theta_4)\,Rot(\hat{y},\theta_5)\,Rot(\hat{x},\theta_6) = R$

R

Numerical Inverse Kinematics

- **Forward Kinematics:** $\theta \rightarrow \mathrm{T}(\theta)$
- **Inverse Kinematics:** $X \rightarrow \theta$
- When the analytic solution is hard or impossible to come by, we **numerically** solve:

$T(\theta) = x \longrightarrow T(\theta) - x = 0$

general: $\underbrace{f(\theta) - x}_{g(\theta)} = 0 \rightarrow$ find roots of g

Summary

- Wrapped up **velocity kinematics** by showing example of computing the Jacobian by definition using adjoint maps
- Introduced analytic **inverse kinematics**, which helps us find the joint angles that will produce a desired end-effector pose
  - We can do this with the help of **atan2** and **ZYX Euler Angles**
  - However, this is only works for relatively simple robot chains