Kinematics Calibration

Kinematic Calibration and the Product of Exponentials Formula

Advances in Robot Kinematics and Computational Geometry, 1994

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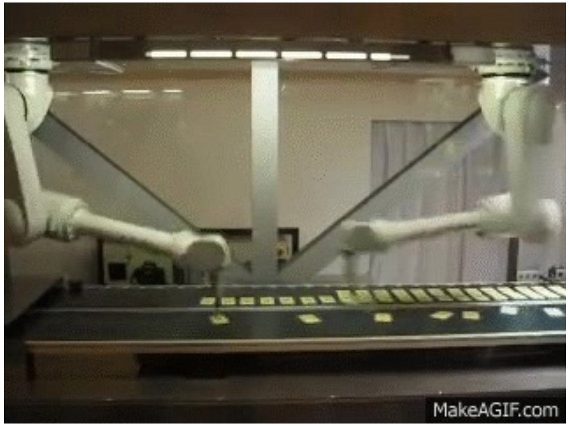
Fun Facts about Industrial Robots

- As of 2026, there are over 4.5 million industrial robots in operation
  - In 2025, nearly 550,000 new robots were installed in industrial operations
- Majority of industrial robots are used in automotive and electronics assembly
  - Metals (welding) and machinery industries are the fastest growing areas
- Industry robots typically operate for 10 years (80,000-10,000 hours)
  - In contrast, domestic robots (vacuums, mowers) typically operate for less than two hours per day and last about 5 years

Robot Lifespans and Calibration



- D-H parameters are extremely sensitive to small kinematic variations
  - Especially when join axes are nearly parallel
- PoE is a *continuous parametric model,* the exponential coordinates vary smoothly with variations in the joint axes
  - The Jacobian also has a nice compact expression!

PoE-Based Calibration Steps

1. **Modeling:** Find screw axes as representation for parameterization
   - ω_i and v_i provide all information about links and joints
2. **Linearization:** Use the Jacobian to linearize the PoE formula
3. **Data Collection:** Gather ground truth tool pose across various postures
4. **Optimization:** Update screw axes to minimize different between measured data and computed FK pose

Effective calibration can improve position control errors by up to 90% and increase precision by orders of magnitude!

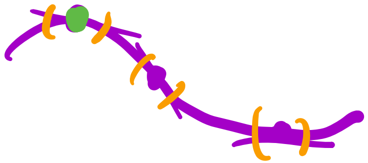
Linearization for Calibration

In general: $x = f(\theta, p)$, θ, p are joint positions + kin parameters

x end effector pose

linearize eq:

$$dx = \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial p} dp \rightarrow x' = x + dx$$



calibration to minimize errors

- measure errors: $\ell \rightarrow dx$
- find $d\theta, dp$ to match/min err

$$\min \left\| dx - \frac{\partial f}{\partial \theta} d\theta - \frac{\partial f}{\partial p} dp \right\|^2$$

Linearizing the PoE Formula

- Using the fact that for curve in SE(3), $X(t) = e^{A(t)}$:

$$\dot{X}X^{-1} = \int_0^1 e^{A(t)s}\dot{A}(t)e^{-A(t)s}ds$$

- We may linearize the PoE formula as follows, were $A = \begin{bmatrix} [\omega_i] & v_i \\ 0 & 0 \end{bmatrix}$:

$$
\begin{aligned}
J = df \cdot f^{-1} \quad &= \quad A_1\,dx_1 + \mathrm{Ad}_{e^{A_1x_1}}(A_2)\,dx_2 + \ldots + \mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}}(A_n)\,dx_n \\
\dot{T}\cdot T^{-1} \quad &+ \quad x_1\int_0^1 \mathrm{Ad}_{e^{A_1x_1s}}(dA_1)\,ds + x_2\mathrm{Ad}_{e^{A_1x_1}}\left(\int_0^1 \mathrm{Ad}_{e^{A_2x_2s}}(dA_2)\,ds\right) \\
&+ \quad \ldots + x_n\mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}}\left(\int_0^1 \mathrm{Ad}_{e^{A_nx_ns}}(dA_n)\,ds\right) \\
&+ \quad \mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_nx_n}}\left(\int_0^1 \mathrm{Ad}_{e^{\Gamma s}}(d\Gamma)\,ds\right)
\end{aligned}
$$

x_i is joint value
Γ is M parameters

Why linearize?

rearrange: $\underset{\substack{\downarrow \\ df \cdot f^{-1}}}{Y} = \underset{\substack{\uparrow \\ \text{variation} \\ \text{of } J}}{A} P$ ← all parameters $d\theta_i, dw_i, dv_i$

error vectors — Jacobians

$$[Y_1 \ \cdots \ Y_m]^T = [A_1 \ \cdots \ A_m]^T P$$ ← parameters

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T_n is nominal frame

T_a is actual frame

$$df \cdot f^{-1} = (T_a - T_n) T_n^{-1} = \underbrace{T_a T_n^{-1}}_{\sim I + \Omega + \cancel{\Omega^2/2!} + \cdots} - I$$

$$\Omega \to \log(T_a T_n^{-1})$$

Least-Squares Solutions

$\underline{Ax = b}$ → does a solution exist?

→ yes? → perfect world

→ no? → what is best soln?

best soln: x^*

any vec

$$\text{dist}(b, Ax^*) \le \text{dist}(b, Ax)$$

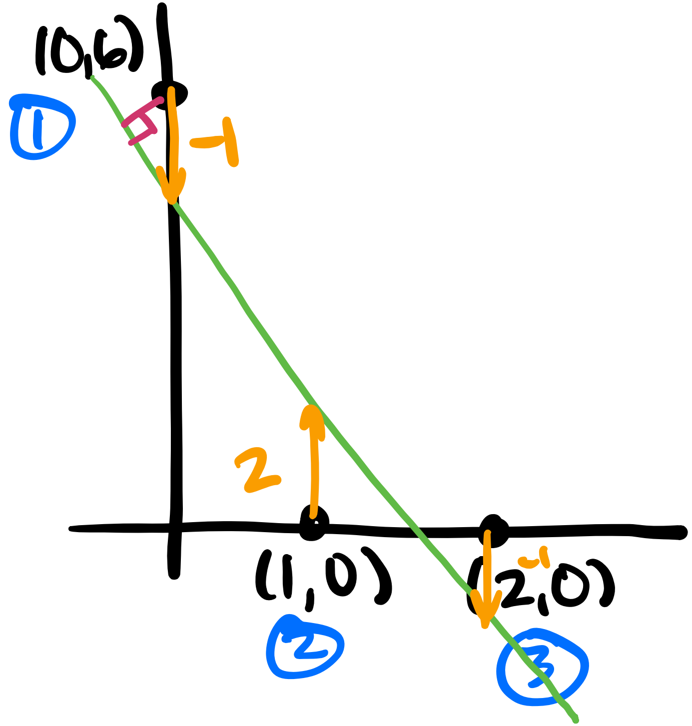
$$\|b - Ax^*\| \le \|b - Ax\|$$

for a general $A \in \mathbb{R}^{m \times n}$

soln match $A^T A x = A^T b$

$$x^* = (A^T A)^{-1} A^T b$$

Least Squares Example



$y = mx + b \rightarrow$ parameters: m, b

1 $6 = m\cdot 0 + b$

2 $0 = m\cdot 1 + b$

3 $0 = m\cdot 2 + b$

$$y = \begin{bmatrix}6\\0\\0\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\1 & 1\\2 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}m\\b\end{bmatrix}}_{p}$$

$$p^* = \begin{bmatrix}-3\\5\end{bmatrix}$$

$$\underline{y - Ap^*} = \begin{bmatrix}-1\\2\\-1\end{bmatrix}$$

Least Squares for Kinematic Calibration

- Errors and Jacobians in a single equation:

$$[y_i \cdots y_m]^\top = [\mathbf{A_1} \cdots \mathbf{A_m}]^\top \mathbf{p} = \mathbf{Y} = \boldsymbol{\mathcal{A}}\mathbf{p}$$

- Least squares solution:

$$\mathbf{p} = (\mathcal{A}^\top \mathcal{A})^{-1} \mathcal{A}^\top \mathbf{Y}$$

- Parameter update:

$$\mathbf{P}' = \mathbf{P} + \mathbf{p}$$

- Since this is a nonlinear problem, we iteratively solve until variations in **p** approach zero (parameters converge)
    - At each iteration, the new parameters for $[S_i]$ are used

A few things to note

- In the general formulation, we are free to optimize both ω_i, v_i simultaneously
  - This captures a helical joint! And useful since no joint is perfect and kinematic imperfections may result in a slightly helical motion
  - Can add constraints if you have perfect joints
- Any optimization or estimation algorithms can be used – not just least squares!
- This method only accounts for geometric errors
  - Other errors include backlash, gear transmission error, joint compliance may impact your model

Summary

- **Calibration** is super important for precise tasks!
  - Parameters vary from manufacturing defects or wear and tear over time
- The **PoE formulation is great for calibration** due to its simple and continuous representation
- Presented a method for **linearizing** and optimizing PoE parameters using **least squares** method
- Note that no method is perfect, but this geometric calibration can improve accuracy from the order of cm to sub-mm