Lecture 10: Velocity Kinematics II and Manipulability

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Modern Robotics Ch 5.1-5.3

Leonardo Da Vinci’s Robot

- Leonardo's robot, or mechanical knight, was a humanoid automaton designed and possibly constructed by Leonardo da Vinci around 1495
- The robot knight could stand, sit, raise its visor and independently maneuver its arms, and had an anatomically correct jaw, all operated by a series of pulleys and cables
  - The robot has been built faithfully based on Leonardo's design, and was found to be fully functional
- **Question:** is this a robot?

Computing the Jacobian

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

- Take the derivative:

$$\dot{T}(\theta) = \frac{d}{dt}\left(e^{[\mathcal{S}_1]\theta_1}\right) e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots \frac{d}{dt}\left(e^{[\mathcal{S}_n]\theta_n}\right) M$$
$$= [\mathcal{S}_1]\,\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} \dots [\mathcal{S}_n]\,\dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M$$

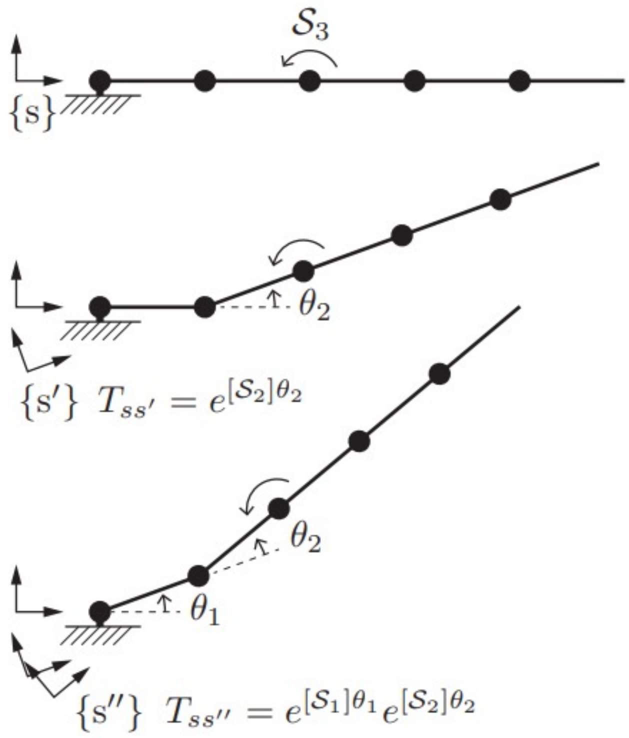
- Take the inverse:

$$T^{-1}(\theta) = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \dots e^{-[\mathcal{S}_1]\theta_1}$$

- Recall: $[\mathcal{V}_s] = \dot{T}\, T^{-1}$

$$\dot{T}T^{-1} = [\mathcal{S}_1]\dot{\theta}_1 + \underbrace{e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}}_{Ad_{e^{[S_1]\theta_1}}(S_2)}\dot{\theta}_2 + \underbrace{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}}\dot{\theta}_3 + \cdots$$

Visualizing the Jacobian



- By inspection, let's find J_{s3}
- Ignore joints $\theta_3, \theta_4, \theta_5$, as they do not displace axis 3 relative to {s}
- If $\theta_1 = 0$ and θ_2 is arbitrary, then $T_{ss'} = e^{[\mathcal{S}_2]\theta_2}$
- If θ_1 is arbitrary, then $T_{ss''} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}$
- $J_{s3} = [Ad_{T_{ss''}}]\mathcal{S}_3 = [Ad_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}]\mathcal{S}_3$

Methods for Computing the Jacobian

By Inspection/ Geometry + Trig

$J_{si} = (\omega_{si}, v_{si})$ where $Ad_{T_{i-1}}$ is implicit

By Definition

The **space Jacobian** $J_s(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$ via

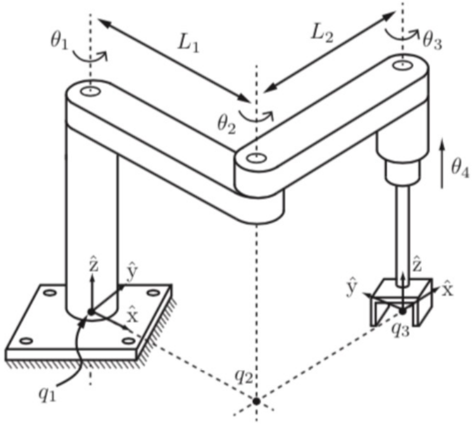
$$\mathcal{V}_s = J_s(\theta)\dot{\theta}. \tag{5.10}$$

The ith column of $J_s(\theta)$ is

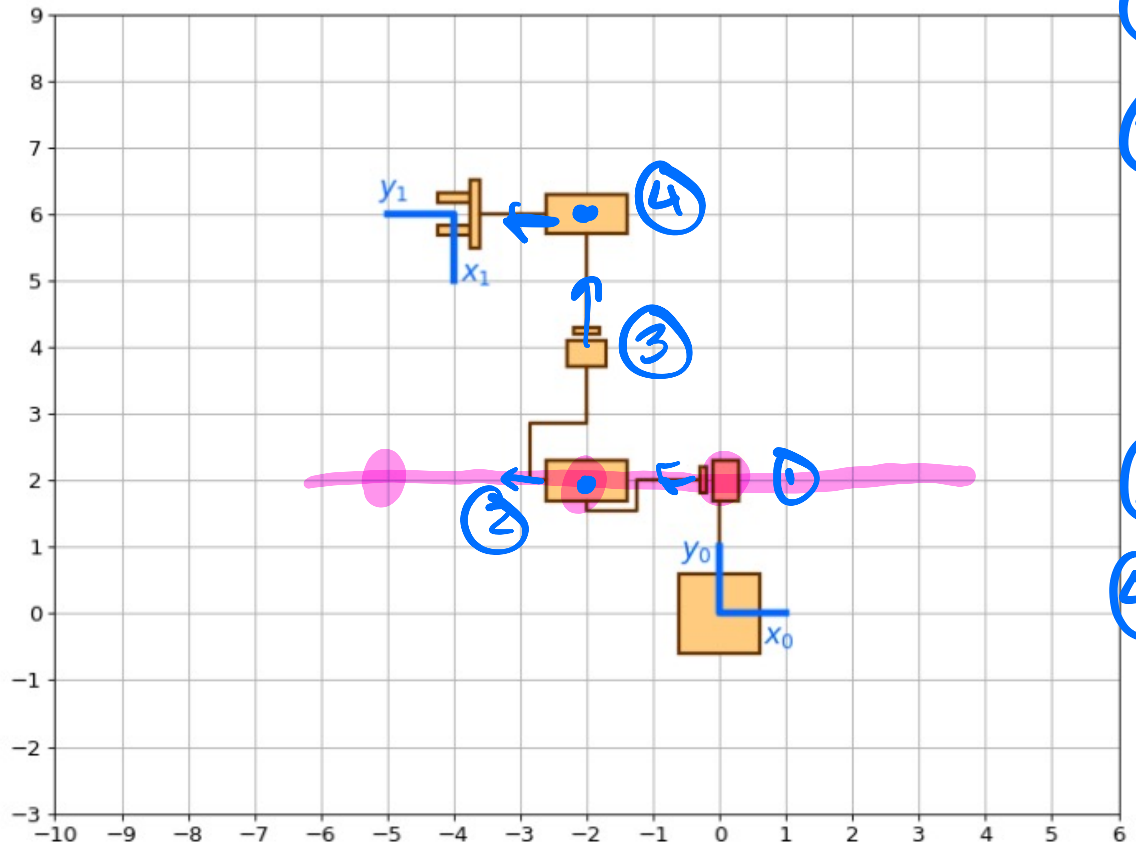
$$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i), \tag{5.11}$$

for $i = 2, \ldots, n$, with the first column $J_{s1} = \mathcal{S}_1$.

$$J_{si} = Ad_{e^{[S_1]\theta_1} \cdots e^{[S_{i-1}]\theta_{i-1}}}(S_i)$$



Example by Definition step 1 find S_i



① $\omega_1 = 0, \quad v_1 = [-1 \ 0 \ 0]^T$

② $\omega_2 = [-1 \ 0 \ 0]^T$

$q_2 = [-2 \ 2 \ 0]^T$

$v_2 = [0 \ 0 \ 2]^T$

③ $\omega_3 = 0, \quad v_3 = [0 \ 1 \ 0]^T$

④ $\omega_4 = [-1 \ 0 \ 0]^T$

$q_4 = [-2 \ 6 \ 0]^T$

Example by Definition

step 2 | compute transformation and adjoint

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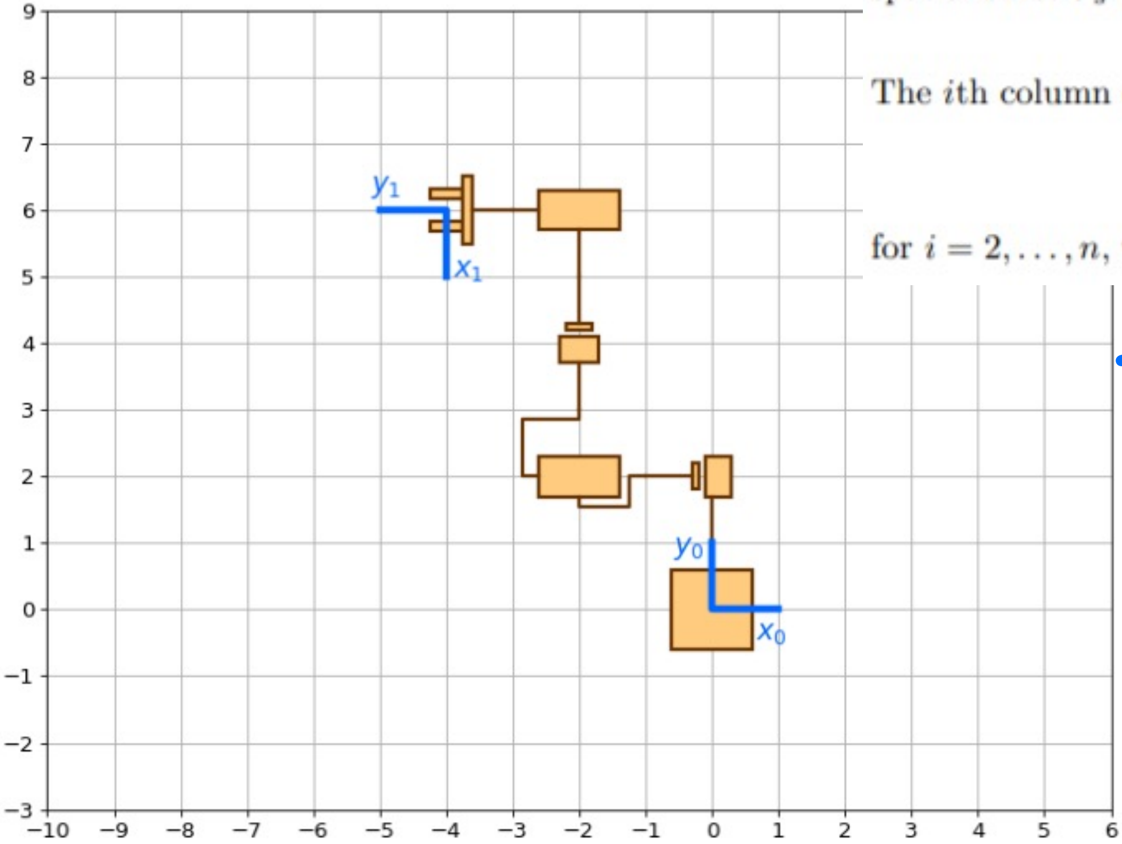


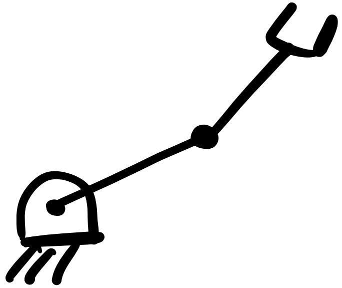
$$J_{s_4} = [\mathrm{Ad}_{\underbrace{\overbrace{e^{[S_1]\theta_1}}^{T_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3}}_{}}] S_4$$

$$T = T_1 T_2 T_3 = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

So why do we care about velocity kinematics?

Recall Our 2R Example



- Forward Kinematics:

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$\mathrm{y} = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

- First Derivative:

$$\dot{x} = -L_1\dot{\theta}_1 \sin\theta_1 - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin(\theta_1 + \theta_2)$$
$$\dot{\mathrm{y}} = L_1\dot{\theta}_1 \cos\theta_1 + L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos(\theta_1 + \theta_2)$$

- Jacobian:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$
$$= J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2 = J(\theta)\dot{\theta}$$

Kinematic Singularities

- What does the velocity kinematics intuitively give us?

$$\mathcal{V} = \begin{bmatrix} J_1(\theta) & J_2(\theta) & \cdots & J_n(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

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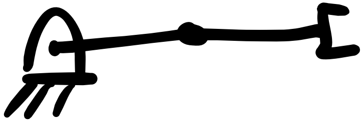
- What is the maximum rank? What happens rank(J)<n?

$\theta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Kinematic Singularities- 2R example



- Consider the robot with link lengths of 1 at the typical zero position.
- What is the Jacobian?

$$J\left(\underset{}{\overset{\theta}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}}\right) = \begin{bmatrix} -\sin 0 - \sin 0 & -\sin 0 \\ \cos 0 + \cos 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

Kinematic Singularities - 2R example

- Consider the robot with link lengths of 1 at the typical zero position.
- What is the Jacobian?

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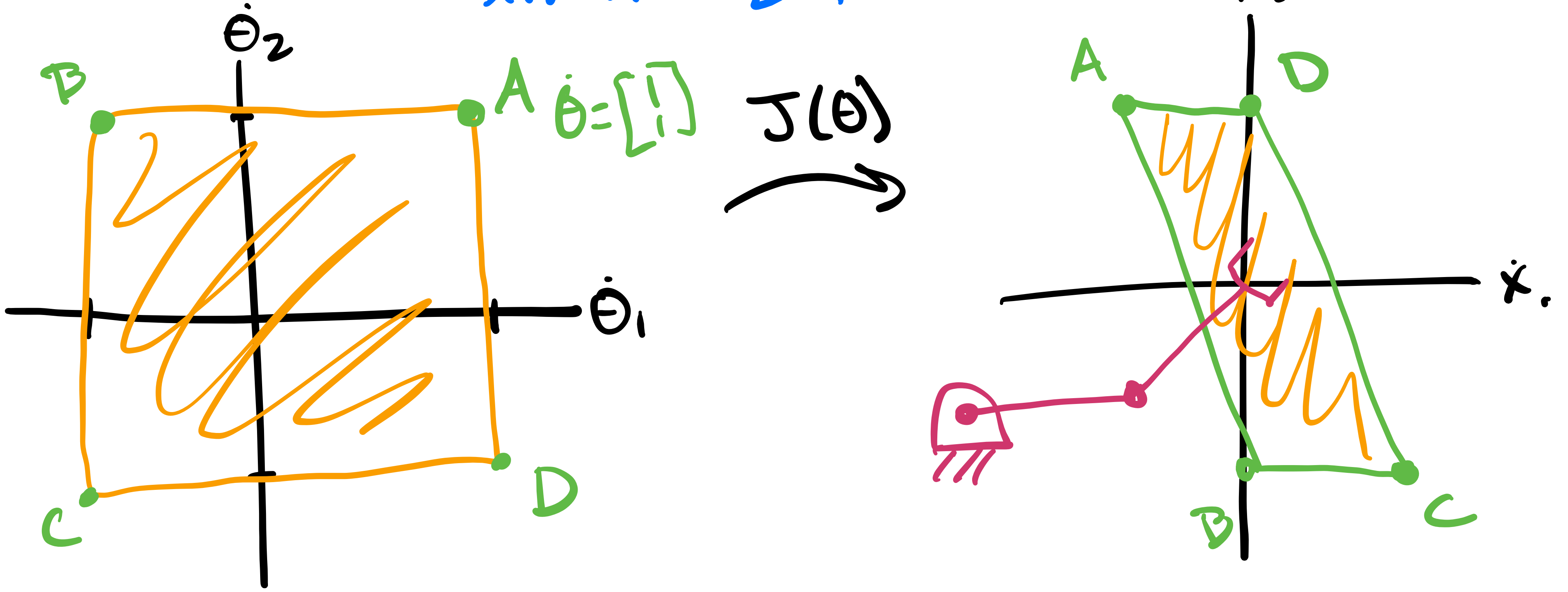
- Let's look at another configuration where $\theta=\begin{bmatrix}0 & \frac{\pi}{4}\end{bmatrix}^{\mathsf{T}}$
- What is the Jacobian?

$$J\left(\begin{bmatrix}0\\ \frac{\pi}{4}\end{bmatrix}\right)=\begin{bmatrix}-\sin 0-\sin\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \cos 0+\cos\frac{\pi}{4} & \cos\frac{\pi}{4}\end{bmatrix}=\begin{bmatrix}-.71 & -.71\\ 1.71 & .71\end{bmatrix}$$

Jacobian Mapping

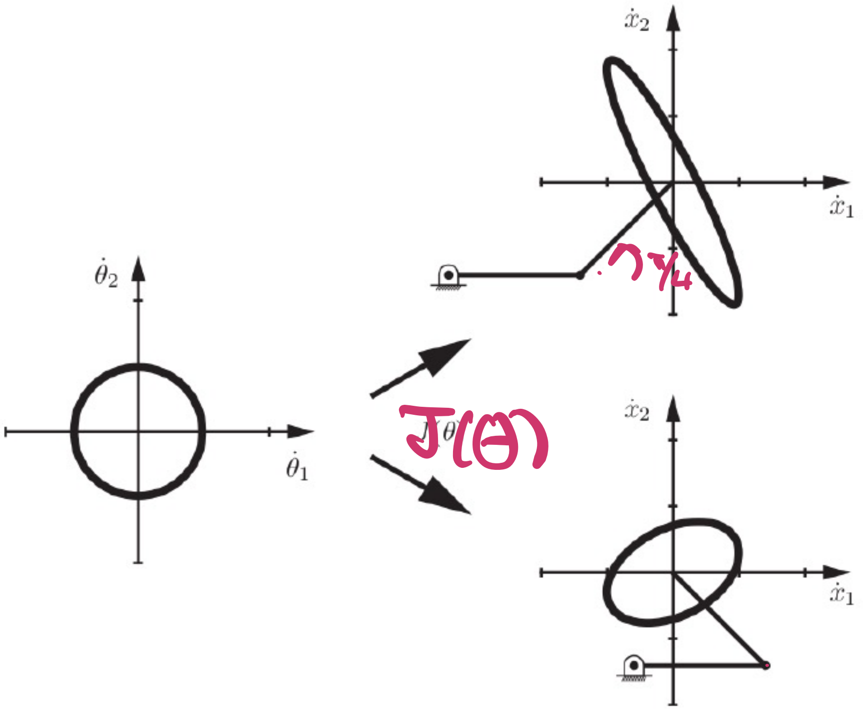
$J\left(\begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}\right) = \begin{bmatrix} -.71 & -.71 \\ 1.71 & .71 \end{bmatrix}$

assume some ~~ea~~actuator limits: $[-1, 1]$



Jacobian and Manipulability

- Suppose joint limits are $\dot{\theta}_1^2 + \dot{\theta}_2^2 \leq 1$
- The ellipsoid obtained by mapping through the Jacobian is called **the manipulability ellipsoid**



This circle represents an “iso-effort” contour in the joint velocity space, where total actuator effort is considered to be the sum of squares of the joint velocities.

Manipulability

end effector coor: $x \in \mathbb{R}^m$

joint velocities on iso-effort sphere: $\|\dot{\theta}\| = 1$

$\dot{\theta}^T \dot{\theta} = 1$

recall:

$\dot{x} = J(\theta)\dot{\theta}$

$J^{-1}(\theta)\dot{x} = \dot{\theta}$

$$1 = \dot{\theta}^T \dot{\theta} = (J^{-1}\dot{x})^T (J^{-1}\dot{x})$$

$$= \dot{x}^T J^{-T} J^{-1} \dot{x}$$

$$= \dot{x}^T \underbrace{(J^T J)^{-1}}_{A^{-1}} \dot{x}$$

this is eq for ellipse

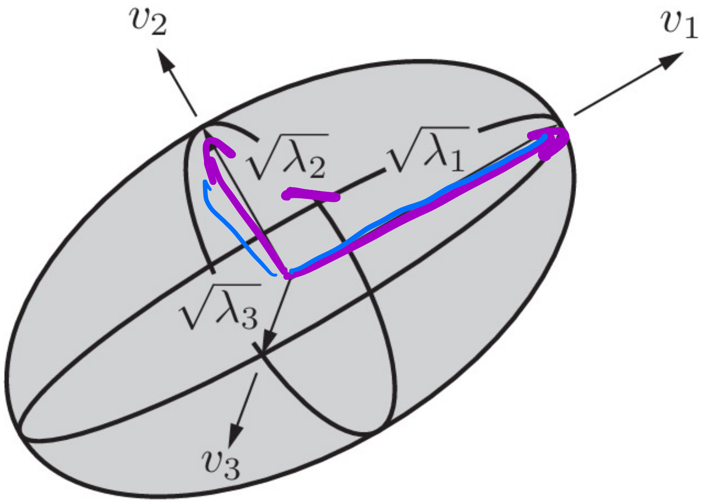
Recall Linear Algebra and Ellipsoids

$x^T A^{-1} x = 1$

- An ellipsoid can be written:

$$\xi^\top Q \, \xi = 1$$

- The eigenvectors v_i and values λ_i give you the principle semi-axes v_i and lengths $\sqrt{\lambda_i}$



Measure of "closeness" to singularity?

$A = JJ^T$

$$\mu_1(A) = \frac{\sqrt{\lambda_{max}(A)}}{\sqrt{\lambda_{min}(A)}} = \sqrt{\frac{\lambda_{max}(A)}{\lambda_{min}(A)}} \geq 1$$

→ closer to 1 is "better"

→ nearly spherical or isotropic

$$\mu_2(A) = \frac{\lambda_{max}}{\lambda_{min}} \rightarrow \text{condition number}$$

Jacobian and Statics

- Assume a force f_{tip} is applied on the end-effector. What torque (τ) must be applied at the joint to keep the fixed position?

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- By conservation of power:
$$f_{tip}^{\top} v_{tip} = \tau^{\top}\dot{\theta}, \qquad \forall\dot{\theta}$$
- Since $v_{tip} = J(\theta)\dot{\theta}$, we have
$$f_{tip}^{\top} J(\theta)\dot{\theta} = \tau^{\top}\dot{\theta}, \qquad \forall\dot{\theta}$$
- Which gives:
$$\tau = J^{\top}(\theta) f_{tip}$$

$J^{-T}(\theta)\tau = f_{tip}$

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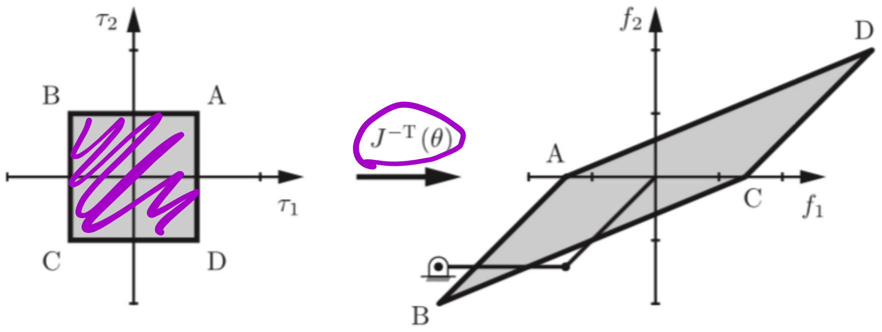
- If $J^{\top}(\theta)$ is invertible, then given the limits of the torques at the joints, we can compute all the forces that can be counteracted at the end-effector



Summary

- Gave example on how to **compute velocity kinematics by definition,** instead of by inspection
- Introduced the concept of **kinematic singularities,** configurations in which there is a direction we cannot move
- The **manipulability ellipsoid** gives us a sense of nearness to singularity
- The **Jacobian is also useful in understanding statics** and forces required for simple tasks
  - What if the inverse doesn't exist? Will post notes on this.