Kinematic Calibration and the Product of Exponentials Formula

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Abstract

We present a method for kinematic calibration of open chain mechanisms based on the product of exponentials *(POE) formula. The POE formula represents the forward kinematics of an open chain as a product of matrix exponentials, and is based on a modern geometric interpretation of classical screw theory. Unlike the kinematic representations based on the Denavit-Hartenberg (D-H) parameters, the kinematic parameters in the POE formula vary smoothly with changes in the joint axes; ad hoc methods designed to address the inherent singularities in the D-H parameters are therefore unnecessary. After introducing the POE formula, we derive a least-squares kinematic calibration algorithm for general open chain mechanisms.*

1 Introduction

In practice the actual kinematic parameters of a computer-controlled mechanism rarely agree with those specified by the manufacturer: for example, planar mechanisms are never quite planar, and the axes of a spherical mechanism never quite intersect at a single point. These errors can usually be attributed to imprecisions in the manufacturing process or to the natural wear and tear of daily usage. In any case regular and automatic calibration of the mechanism's parameters is crucial for reliable and accurate performance. For the PUMA 560 robot, for example, endpoint positioning accuracies before kinematic calibration are only on the order of 1cm, whereas after kinematic calibration the accuracy is improved to 0.3mm [12]. This is a significant improvement of almost two orders of magnitude.

Kinematic calibration typically begins with a linearization of the forward kinematic equations. If the nominal forward kinematics is given by the equation $x = f(\theta, p)$, where θ and p are the joint position and kinematic parameter vectors, respectively, and x is the vector of the endpoint position and orientation, the linearized equations are of the form

$$dx = \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial p} dp$$

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Calibration involves making several measurements of the endpoint position and orientation error dx, and determining the optimal $d\theta$ and dp that minimize the error in some least-squares sense, *e.g.*, minimizing $\|dx - \frac{\partial f}{\partial \theta} d\theta - \frac{\partial f}{\partial p} dp\|^2$.

The Denavit-Hartenberg (D-H) parameters are easily the most popular set of parameters used for describing the kinematics of mechanisms. Although they are attractive because of their minimal set property, it is well-known that the D-H parameters are singular when neighboring joint axes are nearly parallel: because the common normal varies wildly with small changes in the axis orientation, these parameters are ill-conditioned. Various modifications have been proposed to handle the parallel axis case. Hayati [3] introduces a modified set of four parameters as well as a five parameter representation that also handles prismatic joints. Stone [11] adds two parameters to the D-H model to allow for arbitrary placement of link frames. Zhuang, Roth, and Hamano [13] also propose a six-parameter kinematic representation, in which four parameters represent the joint axis, and the remaining two parameters allow for arbitrary placement of the link frame. Zhuang *etal* emphasize that their model, unlike the D-H or certain other kinematic models, is continuous and parametrically complete.

While many of the above calibration methods overcome the problem of singularities with the D-H representation, most of these methods are ad hoc, and often unnecessarily complicate the calibration procedure with arbitrarily defined parameters and computational algorithms that depend on whether the joint is prismatic or revolute. Also, the majority of these kinematic models are developed expressly for calibration, and their usefulness for other applications remains unclear. In this article we propose a kinematic calibration method based on representing the forward kinematics as a product of matrix exponentials of the form

$$f(x_1, \ldots, x_n) = e^{A_1 x_1} e^{A_2 x_2} \cdots e^{A_n x_n} M$$

Here $x_1, \ldots, x_n$ represent the joint variables, M is a 4×4 homogeneous transformation representing the tool frame in its home position, and the A_i are 4×4 constant matrices of a special structure that completely specify the kinematics of the mechanism. Unlike the D-H representations, the A_i vary *smoothly* with changes in the joint axes, and form a complete set of parameters for modeling mechanisms with both revolute and prismatic joints. This representation, called the *product-of-exponentials* (POE) formula, is based on the concept of a one-parameter subgroup of a Lie group, and provides a modern geometric interpretation of classical screw theory [1]. The POE formula is a zero reference model that eliminates the need to attach local reference frames to each link, and shares a number of similar features with the kinematic model proposed by Mooring and Tang [5]. By drawing upon well-established principles of classical screw theory and modern differential geometry, the POE formula eliminates many of the ad hoc features found in existing calibration algorithms. More importantly, it offers a general kinematic representation that is useful for applications besides kinematic calibration (Park [8]).

The article is organized as follows. In Section 2 we review the necessary background on SE(3), the Lie group of Euclidean motions, and introduce the POE formula for modeling the kinematics of open chains. In Section 3 we linearize the POE formula with respect to its kinematic parameters, and derive a least-squares algorithm for kinematic calibration. Although we do not assume any prior knowledge of Lie groups and Lie algebras on the part of the reader, we suggest consulting [2] for additional background.

2 Geometric Background

The Lie Group SE(3)

For our purposes it is sufficient to think of SE(3), the Euclidean group of rigid-body motions (or the *Special Euclidean Group*, also known in the robotics literature as the *homogeneous transformations*), as consisting of matrices of the form $\begin{bmatrix} \Theta & b \\ 0 & 1 \end{bmatrix}$, where $\Theta \in SO(3)$ and $b \in \Re^3$. Here SO(3) denotes the group of 3×3 proper rotation matrices. SE(3) has the structure of both a differentiable manifold and an algebraic group, and is an example of a *Lie group*.

Associated with any matrix Lie group is its *Lie algebra*, which for our purposes can be thought of as the set of tangent matrices at the identity element. On SO(3) it can be shown that its Lie algebra, denoted, so(3), consists of 3×3 real skew-symmetric matrices of the following form:

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \triangleq [\omega] \tag{1}$$

where $\omega \in \Re^3$. Note that an element $[\omega] \in so(3)$ can also be represented as a vector $\omega \in \Re^3$. It can also be shown that the Lie algebra se(3) consists of 4×4 matrices of the form $\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$, where $[\omega] \in so(3)$ and $v \in \Re^3$.

An important connection between a Lie group and its Lie algebra is the *exponential mapping*—defined on each Lie algebra is the exponential mapping into the corresponding Lie group. On matrix groups the exponential mapping corresponds to the usual matrix exponential, *i.e.*, if A is an element of the Lie algebra, then $\exp A = I + A + \frac{A^2}{2!} + \ldots$ is an element of the Lie group. On so(3) and se(3) the exponential mapping is *onto*, *i.e.*, for every $X \in SO(3)$ (resp., SE(3)) there exists an $x \in so(3)$ (resp., se(3)) such that $\exp x = X$. The following explicit formula for the exponential mapping on se(3) is derived in Park [10].

Lemma 1 *Let* $[\omega] \in so(3)$ *and* $v \in \Re^3$, *and* $||\omega||^2 = w_1^2 + w_2^2 + w_3^2$. *Then*

$$\exp \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \exp[\omega] & Av \\ 0 & 1 \end{bmatrix} \tag{2}$$

is an element of SE(3), *where*

$$\exp[\omega] = I + \frac{\sin||\omega||}{||\omega||} \cdot [\omega] + \frac{1 - \cos||\omega||}{||\omega||^2} \cdot [\omega]^2 \tag{3}$$

$$A = I + \frac{1 - \cos||\omega||}{||\omega||^2} \cdot [\omega] + \frac{||\omega|| - \sin||\omega||}{||\omega||^3} \cdot [\omega]^2 \tag{4}$$

Note that if A is an element of some Lie algebra, then $e^{At}, t \in \Re$, itself forms a group, in this case a subgroup of the Lie group. Such groups are called ***one-parameter subgroups*** of a Lie group, and play a special role in the kinematic description of mechanisms.

Explicit formulas for the inverse map log : SE(3) $\rightarrow$ se(3) can also be derived (Park [8]). Although SE(3) is not a vector space, se(3) is: the log formula provides a set of ***canonical coordinates*** for representing neighborhoods of SE(3) as open sets in a vector space.

Lemma 2 *Let* $\Theta \in SO(3)$ *such that* $\mathrm{Tr}(\Theta) \neq -1$, *and let* ϕ *satisfy* $1+2\cos\phi = \mathrm{Tr}(\Theta)$, $|\phi| < \pi$. *Finally, suppose* $b \in \Re^3$. *Then*

$$\log \begin{bmatrix} \Theta & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} [\omega] & A^{-1}b \\ 0 & 0 \end{bmatrix} \tag{5}$$

where $[\omega] = \frac{\phi}{2\sin\phi}(\Theta - \Theta^T)$, *and*

$$A^{-1} = I - \frac{1}{2}\cdot[\omega] + \frac{2\sin||\omega|| - ||\omega||(1+\cos||\omega||)}{2||\omega||^2 \sin||\omega||}\cdot[\omega]^2 \tag{6}$$

Also, $||\log\Theta||^2 = \phi^2$.

The matrix exponential and logarithm show explicitly the connection between SE(3) and the classical screw theory of rigid-body motions: the screw $(\omega, v) \in \Re^6$ defines a rigid-body motion given by the matrix exponential of Lemma 1.

An element of a Lie group can also be identified with a linear mapping between its Lie algebra as follows. Let $\mathbf{G}$ be a matrix Lie group with Lie algebra $\mathbf{g}$. For every $X \in \mathbf{G}$ the mapping $\mathrm{Ad}_X : \mathbf{g} \rightarrow \mathbf{g}$ defined by $\mathrm{Ad}_X(x) = XxX^{-1}$ is linear. Ad_X is known as the ***adjoint map*** (or ***adjoint representation***) of X. If $X = \begin{bmatrix} \Theta & b \\ 0 & 1 \end{bmatrix}$ is an element of SE(3), then its adjoint map acting on an element $x = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$ of se(3) is given by

$$\mathrm{Ad}_X(x) = \begin{bmatrix} \Theta & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Theta^T & -\Theta^T b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \Theta[\omega]\Theta^T & [b]\Theta\omega + \Theta v \\ 0 & 0 \end{bmatrix} \tag{7}$$

It is not difficult to verify that $\Theta[\omega]\Theta^T = [\Theta\omega]$. Alternatively if x is regarded as a six-dimensional column vector the Ad mapping admits the 6×6 matrix representation

$$\mathrm{Ad}_X(x) = \begin{bmatrix} \Theta & 0 \\ [b]\Theta & \Theta \end{bmatrix}\begin{bmatrix} \omega \\ v \end{bmatrix} \tag{8}$$

It is also easily verified that $\mathrm{Ad}_X^{-1} = \mathrm{Ad}_{X^{-1}}$ and $\mathrm{Ad}_X \mathrm{Ad}_Y = \mathrm{Ad}_{XY}$ for any $X, Y \in$ SE(3). These identities will be particularly useful in linearizing the POE formula.

The Product-of-Exponentials Formula

The product-of-exponentials (POE) formula is a useful theoretical and computational tool for modeling the kinematics of open chains (see Brockett [1], Paden and Sastry [7], Park [8]) in which the the connections between kinematics and Lie theory are shown explicitly. If a right-handed reference frame is fixed at the tip of each link of the chain, then the Euclidean transformation which describes the position and orientation of the i^{th} frame in terms of the $(i-1)^{st}$ frame is $f_{i-1,i} = e^{P_i x_i} M_i$, where $M_i \in SE(3)$, $P_i \in se(3)$, and $x_i \in \Re$ is the joint variable, $i = 1, 2, \ldots, n$. The frame fixed at the tip is related to that at the base by the product

$$f(x_1, \ldots, x_n) = e^{P_1 x_1} M_1 e^{P_2 x_2} M_2 \cdots e^{P_n x_n} M_n \tag{9}$$

This equation can be simplified using the fact that $M^{-1} e^P M = e^{M^{-1} P M}$, so that by repeatedly applying the identity $M e^P = e^{M P M^{-1}} M$, f can be written

$$f(x_1, x_2, \ldots, x_n) = e^{A_1 x_1} e^{A_2 x_2} \ldots e^{A_n x_n} M \tag{10}$$

where $A_1 = P_1$, $A_2 = M_1 P_2 M_1^{-1}, A_3 = (M_1 M_2) P_3 (M_1 M_2)^{-1}$, *etc.* Similarly, one can apply the identity $e^A M = M e^{M^{-1} A M}$ to rewrite the above as

$$f(x_1, x_2, \ldots, x_n) = M e^{B_1 x_1} e^{B_2 x_2} \ldots e^{B_n x_n} \tag{11}$$

where $B_i = M^{-1} A_i M$, $i = 1, \ldots, n$. The matrix exponentials in this formula can be easily computed from the earlier closed-form formulas.

The POE formula can alternatively be derived independently of the D-H parameters, by appealing to the interpretation of the A_i (B_i) as the screw parameters for joint i's motion (see Paden and Sastry [7]). Specifically, denote the rotational and translational components of A_i (B_i) by ω_i and v_i, respectively. Then ω_i is a unit vector in the direction of joint axis i, expressed in inertial (tip) frame coordinates; v_i is a vector, also described in inertial (tip) frame coordinates, such that the pitch of the screw motion generated by joint i is $\omega_i^T v_i$. Note that for revolute joints the pitch is zero, and $\omega_i \times v_i$ is required to be a point lying on the joint axis; for prismatic joints $\omega_i = 0$ and v_i is in the direction of movement.

Example 1 Let the forward kinematics for the 3R open chain of Figure 1 be of the form $f(x_1, x_2, x_3) = e^{A_1 x_1} e^{A_2 x_2} e^{A_3 x_3} M$. Setting the joint axes to zero, the tip frame $M = (\Theta, b)$ relative to the base frame is given by $\Theta = I$, $b = (L_1 + L_2, 0, 0)$. Joint 1 has a screw axis in the direction $\omega_1 = (0, 0, 1)$; since $\omega_1 \times v_1$ must be a point lying on the joint axis, such that v_1 is normal to ω_1, it follows that $v_1 = (0, 0, 0)$. Similarly, for joints 2 and 3 we have $\omega_2 = (0, -1, 0)$, $v_2 = (0, 0, 0)$, and $\omega_3 = (0, -1, 0)$, $v_3 = (0, 0, -L_1)$.

As the above example illustrates, the POE formula for an open kinematic chain can be obtained directly without resorting to the D-H parameters. The POE formula also has a number of modeling advantages over the D-H representations which make it attractive for calibration. First, it is a type of *zero reference position description* of the kinematics: once a tip and inertial frame have been chosen, and a zero position selected

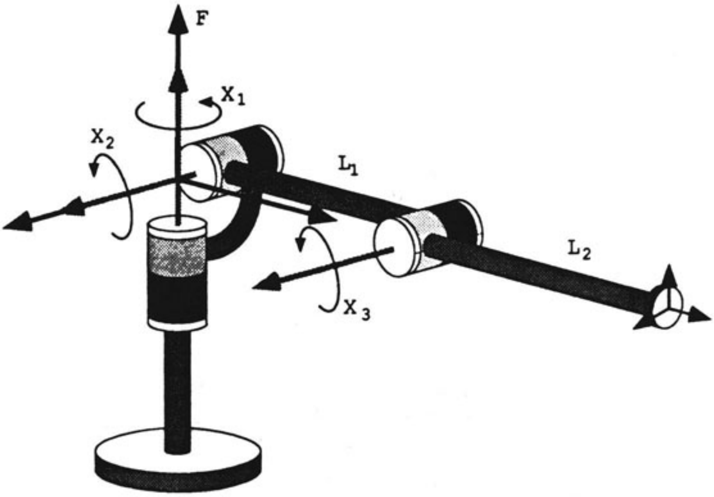

Figure 1: A 3R elbow manipulator.

for each of the joints, there exists a unique set of constant matrices $A_1, A_2, \ldots, A_n \in$ se(3) and $M \in$ SE(3) that describes the forward kinematics of the mechanism. Unlike the D-H representations it is not necessary to attach reference frames to each link. Second, the POE formula treats both revolute and prismatic joints in a uniform way; recall that using the D-H parameters the joint variable can be either θ_i or d_i depending on whether the joint is revolute or prismatic. Third, as discussed earlier, the D-H parameters are extremely sensitive to small kinematic variations when neighboring joint axes are nearly parallel. This sensitivity is what makes kinematic calibration routines based on DH parameters unnecessarily complicated. On the other hand the POE formula is a continuous parametric model, in the sense that the A_i's vary smoothly with variations in the joint axes.

One of the most attractive features of the POE formula is the compact expression for the Jacobian. If f describes the position and orientation of the tip frame relative to the inertial frame, then recall that $\dot{f}f^{-1}$ is an element of se(3) corresponding to the generalized velocity of the tip frame relative to the inertial frame. Using the fact that $(e^{Ax})^{-1} = e^{-Ax}$, a direct calculation shows that

$$\dot{f}f^{-1} = A_1\dot{x}_1 + \mathrm{Ad}_{e^{A_1x_1}}(A_2\dot{x}_2) + + \ldots \mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}}(A_n\dot{x}_n) \tag{12}$$

If $\dot{f}f^{-1}$ is rearranged as a six-dimensional vector, then the right-hand side can be expressed as the product of a $6 \times n$ Jacobian matrix $J(x)$ with an n-dimensional joint velocity vector $(\dot{x}_1, \ldots, \dot{x}_n)$, so that each term in the right-hand side can be identified with a column of the Jacobian matrix.

3 Kinematic Calibration Using the POE Formula

3.1 Linearizing the POE Formula

The following result is the key to our POE-based calibration algorithm:

Lemma 3 *Let* $X(t) = e^{A(t)}$ *be a curve in* SE(3). *Then*

$$\dot{X}X^{-1} = \int_0^1 e^{A(t)s}\dot{A}(t)e^{-A(t)s}\, ds \tag{13}$$

In fact, this integral can be evaluated to the following explicit formula. Denoting $A(t)$ by $\begin{bmatrix} [\omega(t)] & v(t) \\ 0 & 0 \end{bmatrix}$ and $e^{A(t)}$ by $\begin{bmatrix} \Theta(t) & b(t) \\ 0 & 1 \end{bmatrix}$, a direct calculation shows that $\dot{\Theta}\Theta^{-1} = [R\dot{\omega}]$, where

$$R = I + \frac{1 - \cos\|\omega\|}{\|\omega\|^2} \cdot [\omega] + \frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^3} \cdot [\omega]^2$$

Also, $b = Rv$, from which a closed-form expression for $\dot{b} - \dot{\Theta}\Theta^{-1}b$ can be obtained.

We apply this result to linearize the POE formula with respect to its kinematic parameters. As before let the forward kinematics be given by Equation (10), where each $A_i = \begin{bmatrix} [\omega_i] & v_i \\ 0 & 0 \end{bmatrix}$. Note that M can also be expressed using the log formula as $M = e^{\Gamma}$ for some constant $\Gamma \in \mathrm{se}(3)$. Then

$$\begin{aligned} df \cdot f^{-1} &= d(e^{A_1x_1})e^{-A_1x_1} + e^{A_1x_1}d(e^{A_2x_2})e^{-A_2x_2}e^{-A_1x_1} \\ &+ \ldots + e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}d(e^{A_nx_n})e^{-A_nx_n}\cdots e^{-A_1x_1} \\ &+ e^{A_1x_1}\cdots e^{A_nx_n}(dM)M^{-1}e^{-A_nx_n}\cdots e^{-A_1x_1} \end{aligned} \tag{14}$$

From Lemma 3, each $d(e^{A_ix_i})e^{-A_ix_i}$ in (14) is expanded as

$$d(e^{A_ix_i})e^{-A_ix_i} = x_i \int_0^1 e^{A_ix_is}dA_ie^{-A_ix_is}ds + A_idx_i \tag{15}$$

Similarly, $(dM)M^{-1}$ in (14) becomes

$$(dM)M^{-1} = d(e^{\Gamma})e^{-\Gamma} = \int_0^1 e^{\Gamma s}d\Gamma e^{-\Gamma s}ds \tag{16}$$

In terms of the adjoint map (14) can now be expressed as

$$\begin{aligned} df \cdot f^{-1} &= A_1\, dx_1 + \mathrm{Ad}_{e^{A_1x_1}}(A_2)\, dx_2 + \ldots + \mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}}(A_n)\, dx_n \\ &+ x_1 \int_0^1 \mathrm{Ad}_{e^{A_1x_1s}}(dA_1)\, ds + x_2\mathrm{Ad}_{e^{A_1x_1}}\left(\int_0^1 \mathrm{Ad}_{e^{A_2x_2s}}(dA_2)\, ds\right) \\ &+ \ldots + x_n\mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_{n-1}x_{n-1}}}\left(\int_0^1 \mathrm{Ad}_{e^{A_nx_ns}}(dA_n)\, ds\right) \\ &+ \mathrm{Ad}_{e^{A_1x_1}\cdots e^{A_nx_n}}\left(\int_0^1 \mathrm{Ad}_{e^{\Gamma s}}(d\Gamma)\, ds\right) \end{aligned} \tag{17}$$

The linearized equation above can also be expressed in the usual matrix form $\mathbf{A}\mathbf{p} = \mathbf{y}$, where $\mathbf{p} \in \Re^{7n+6}$ is the vector of kinematic parameters

$$\mathbf{p} = [dx_1 \; dx_2 \cdots dx_n \; d\omega_1^T \; dv_1^T \cdots d\omega_n^T \; dv_n^T \; d\omega_M^T \; dv_M^T \,] \tag{18}$$

and $\mathbf{y}$ is the six-dimensional representation of $df \cdot f^{-1}$. Let

$$e^{A_i x_i} = \begin{bmatrix} \Theta_i & b_i \\ 0 & 1 \end{bmatrix}, \quad e^{A_i x_i s} = \begin{bmatrix} R_i(s) & d_i(s) \\ 0 & 1 \end{bmatrix} \tag{19}$$

$$M = e^{\Gamma} = \begin{bmatrix} \Theta_M & b_M \\ 0 & 1 \end{bmatrix}, \quad e^{\Gamma s} = \begin{bmatrix} R_M(s) & d_M(s) \\ 0 & 1 \end{bmatrix} \tag{20}$$

with $e^{A_0 x_0}$ defined to be the identity matrix, and

$$r_k = \left(\prod_{i=0}^{k-1} \begin{bmatrix} \Theta_i & 0 \\ [b_i]\Theta_i & \Theta_i \end{bmatrix}\right) \begin{bmatrix} \omega_k \\ v_k \end{bmatrix} \tag{21}$$

$$Q_k = \left(\prod_{i=0}^{k-1} \begin{bmatrix} \Theta_i & 0 \\ [b_i]\Theta_i & \Theta_i \end{bmatrix}\right) x_k \int_0^1 \begin{bmatrix} R_k(s) & 0 \\ [d_k(s)]R_k(s) & R_k(s) \end{bmatrix} ds \tag{22}$$

$$Q_M = \left(\prod_{i=0}^{n} \begin{bmatrix} \Theta_i & 0 \\ [b_i]\Theta_i & \Theta_i \end{bmatrix}\right) \int_0^1 \begin{bmatrix} R_M(s) & 0 \\ [d_M(s)]R_M(s) & R_M(s) \end{bmatrix} ds \tag{23}$$

Then (17) can be expressed as

$$\begin{aligned} \mathbf{y} &= \Big[r_1 \mid r_2 \mid \cdots \mid r_n \mid Q_1 \mid Q_2 \mid \cdots \mid Q_n \mid Q_M \Big] \mathbf{p} \\ &\triangleq \mathbf{A}\mathbf{p} \end{aligned} \tag{24}$$

Next we consider the left-hand side $df \cdot f^{-1}$. Let T_n and T_a be the nominal and actual end-effector frames, respectively,where T_a is obtained from measurement data and T_n is computed using the nominal parameter values. Then $df \cdot f^{-1} = (T_a - T_n)T_n^{-1} = T_a T_n^{-1} - I$. If $T_a T_n^{-1} = I + \Omega + \Omega^2/2! + \ldots$, where $\Omega = \log(T_a T_n^{-1})$, and T_a and T_n are sufficiently close, then to first order

$$df \cdot f^{-1} = \Omega = \log(T_a T_n^{-1})$$

3.2 A Least-Squares Algorithm for Kinematic Calibration

Calibration proceeds by positioning the mechanism in many points of the workspace and combining the error vectors $\mathbf{y}_i$ and the Jacobians $\mathbf{A}_i$ into a single equation:

$$[\; \mathbf{y}_1 \;\; \cdots \;\; \mathbf{y}_m \;]^T = [\; \mathbf{A}_1 \;\; \cdots \;\; \mathbf{A}_m \;]^T \mathbf{p} \tag{25}$$

or more compactly, $\mathcal{Y} = \mathcal{A}\mathbf{p}$. The least-squares solution for $\mathbf{p}$ is, as well known, $\mathbf{p} = (\mathcal{A}^T\mathcal{A})^{-1}\mathcal{A}^T\mathcal{Y}$. If $\mathbf{P}$ denotes the vector of original kinematic parameters, the updated kinematic parameter values $\mathbf{P}'$ are obtained by $\mathbf{P}' = \mathbf{P} + \mathbf{p}$. Since this is a nonlinear estimation problem, this procedure is iterated until the variations $\mathbf{p}$ approach zero and the parameters $\mathbf{P}$ have converged to some stable values. At each iteration the $\mathbf{A}_i$'s are evaluated with the current parameters.

Note that each $A_i = \begin{bmatrix} [\omega_i] & v_i \\ 0 & 0 \end{bmatrix}$ in the POE formula (10) models a general helical joint, in which the joint axis is directed along ω_i, and the pitch is $\omega_i^T v_i$. Hence, purely revolute joints must satisfy the condition $\omega_i^T v_i = 0$, while for prismatic joints $\omega_i = 0$. If revolute joints are to be modeled as being perfectly revolute (*i.e.*, without any kinematic imperfections that may result in a slightly helical motion) then this additional constraint can be imposed. Hence, to calibrate an n-revolute joint manipulator we minimize

$$J(\mathbf{p}) = ||\mathcal{A}\mathbf{p} - \mathcal{Y}||^2$$

subject to the quadratic equality constraint $\mathbf{p}^T \mathbf{Q} \mathbf{p} = 0$, where $\mathbf{Q}$ is a $6n \times 6n$ constant block-diagonal matrix with each diagonal block of the form $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Here $I \in \Re^{3\times 3}$ is the identity matrix. For prismatic joints only v_i need be identified.

4 Conclusions

In this article we have formulated a kinematic calibration algorithm based on the product of exponentials formula for open kinematic chains. The POE formula enjoys a number of modeling advantages over other comonly used representations, and calibration can be performed in a straightforward and uniform way, without the ad hoc methods of other calibration algorithms. In particular, no special procedures are required for adjacent joints whose axes are nearly parallel. (Further details as well as simulation examples can be found in Okamura [6]). The POE formula can be viewed as a modern geometric formulation of classical screw theory, and as such is a general tool that is useful for a range of applications besides calibration. For example, using the POE formula the dynamic equations can be formulated elegantly and in a form that admits easy factorization and differentiation (Park *et al*, [9]).

The calibration algorithm discussed in this article determines the least-squares solution, but in principle other estimation algorithms can be used, *e.g.*, Levenberg-Marquardt methods, total least-squares, *etc.* Also, only geometric errors have been considered; for more accurate modeling one may wish to include non-geometric effects such as backlash, gear transmission error, and joint compliance. Many of these issues are discussed in Hollerbach [4] and the references cited, and extending the POE calibration method to include these effects merits further study.

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