

Lecture 10: Velocity Kinematics II and Manipulability

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Modern Robotics Ch 5.1-5.3

Leonardo Da Vinci's Robot

- Leonardo's robot, or mechanical knight, was a humanoid automaton designed and possibly constructed by Leonardo da Vinci around 1495
- The robot knight could stand, sit, raise its visor and independently maneuver its arms, and had an anatomically correct jaw, all operated by a series of pulleys and cables
 - The robot has been built faithfully based on Leonardo's design, and was found to be fully functional
- **Question:** is this a robot?



Computing the Jacobian

- Recall forward kinematics:

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} \dots e^{[S_n]\theta_n} M$$

- Take the derivative:

$$\begin{aligned}\dot{T}(\theta) &= \frac{d}{dt} (e^{[S_1]\theta_1}) e^{[S_2]\theta_2} \dots e^{[S_n]\theta_n} M + \dots + e^{[S_1]\theta_1} e^{[S_2]\theta_2} \dots \frac{d}{dt} (e^{[S_n]\theta_n}) M \\ &= [S_1] \dot{\theta}_1 e^{[S_1]\theta_1} \dots e^{[S_n]\theta_n} M + \dots + e^{[S_1]\theta_1} \dots [S_n] \dot{\theta}_n e^{[S_n]\theta_n} M\end{aligned}$$

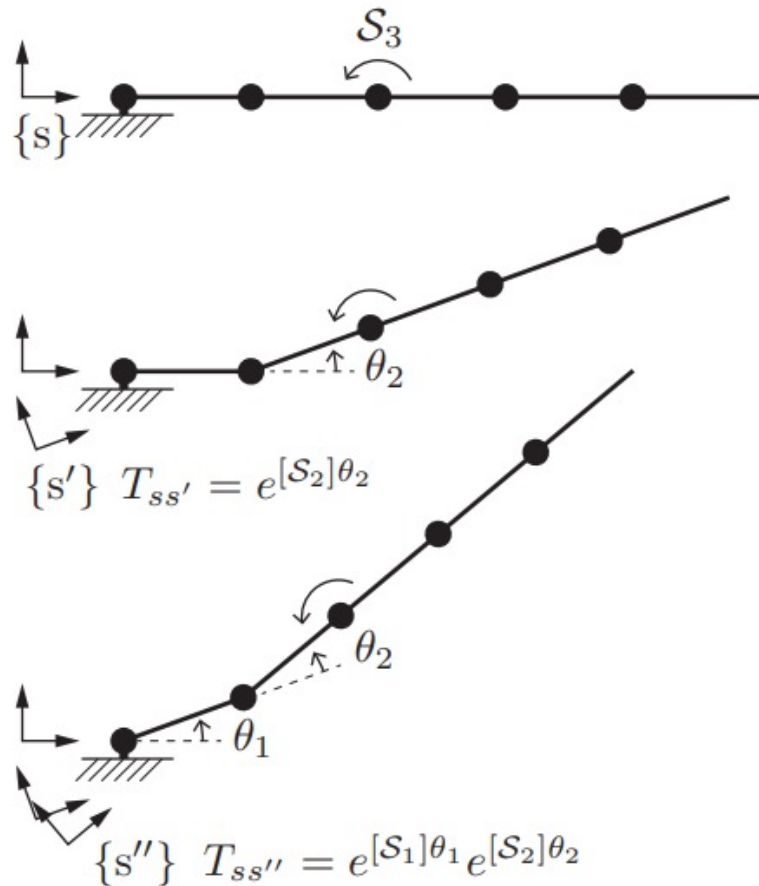
- Take the inverse:

$$T^{-1}(\theta) = M^{-1} e^{-[S_n]\theta_n} \dots e^{-[S_1]\theta_1}$$

- Recall: $[\mathcal{V}_s] = \dot{T} T^{-1}$

$$\dot{T} T^{-1} = [S_1] \dot{\theta}_1 + e^{[S_1]\theta_1} [S_2] e^{-[S_1]\theta_1} \dot{\theta}_2 + e^{[S_1]\theta_1} e^{[S_2]\theta_2} [S_3] e^{-[S_2]\theta_2} e^{-[S_1]\theta_1} \dot{\theta}_3 + \dots$$

Visualizing the Jacobian



- By inspection, let's find J_{s3}
- Ignore joints $\theta_3, \theta_4, \theta_5$, as they do not displace axis 3 relative to $\{s\}$
- If $\theta_1 = 0$ and θ_2 is arbitrary, then $T_{ss'} = e^{[s_2]\theta_2}$
- If θ_1 is arbitrary, then $T_{ss''} = e^{[s_1]\theta_1} e^{[s_2]\theta_2}$
- $J_{s3} = [Ad_{T_{ss''}}]s_3 = [Ad_{e^{[s_1]\theta_1} e^{[s_2]\theta_2}}]s_3$

Methods for Computing the Jacobian

By Inspection

$J_{si} = (\omega_{si}, v_{si})$ where $Ad_{T_{i-1}}$ is implicit

By Definition

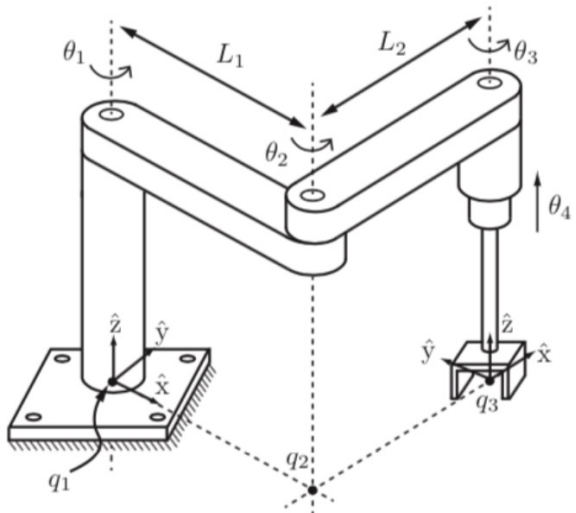
The **space Jacobian** $J_s(\theta) \in \mathbb{R}^{6 \times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$ via

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}. \quad (5.10)$$

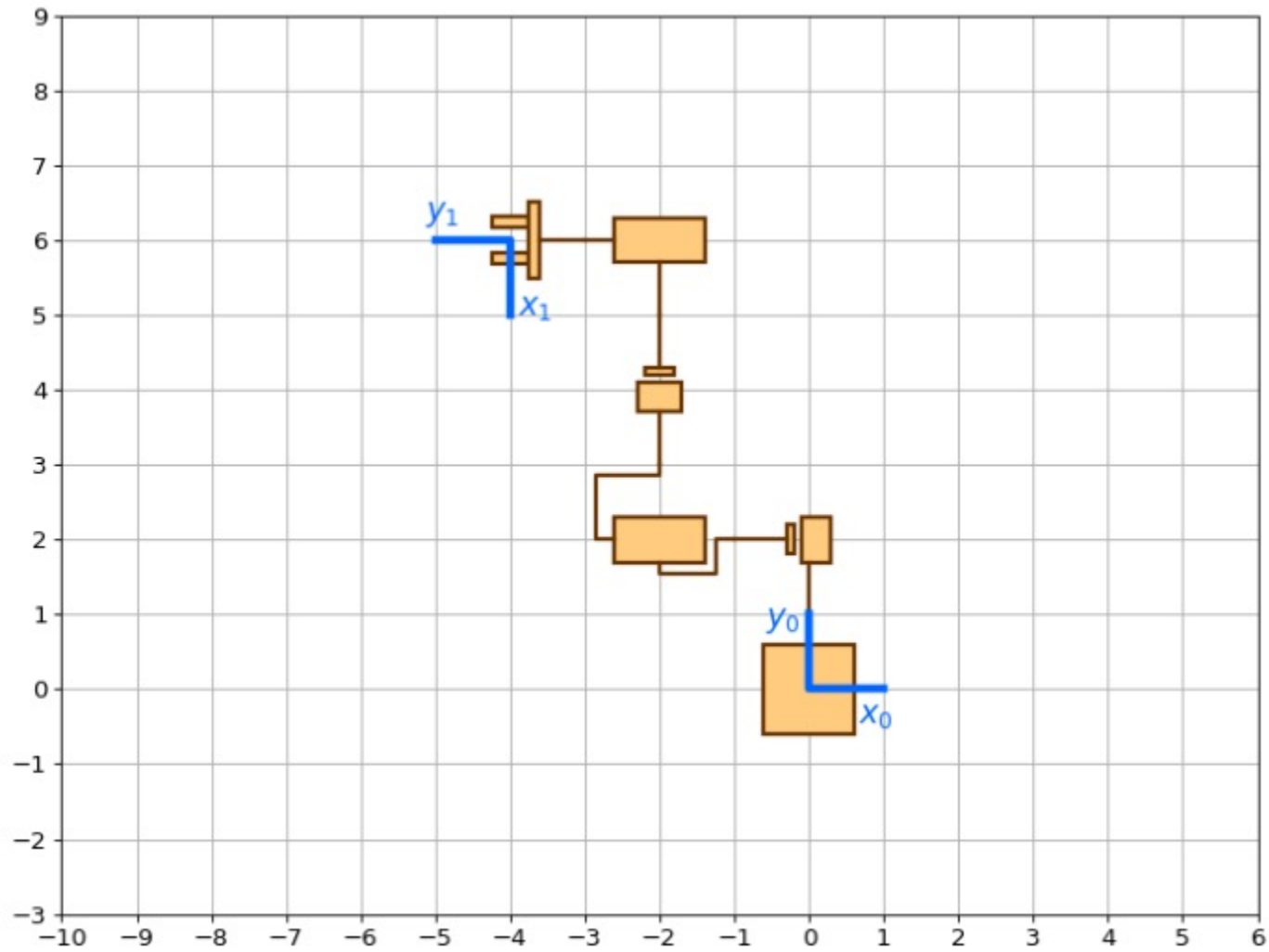
The i th column of $J_s(\theta)$ is

$$J_{si}(\theta) = \text{Ad}_{e^{[S_1]\theta_1} \dots e^{[S_{i-1}]\theta_{i-1}}} (S_i), \quad (5.11)$$

for $i = 2, \dots, n$, with the first column $J_{s1} = S_1$.



Example by Definition



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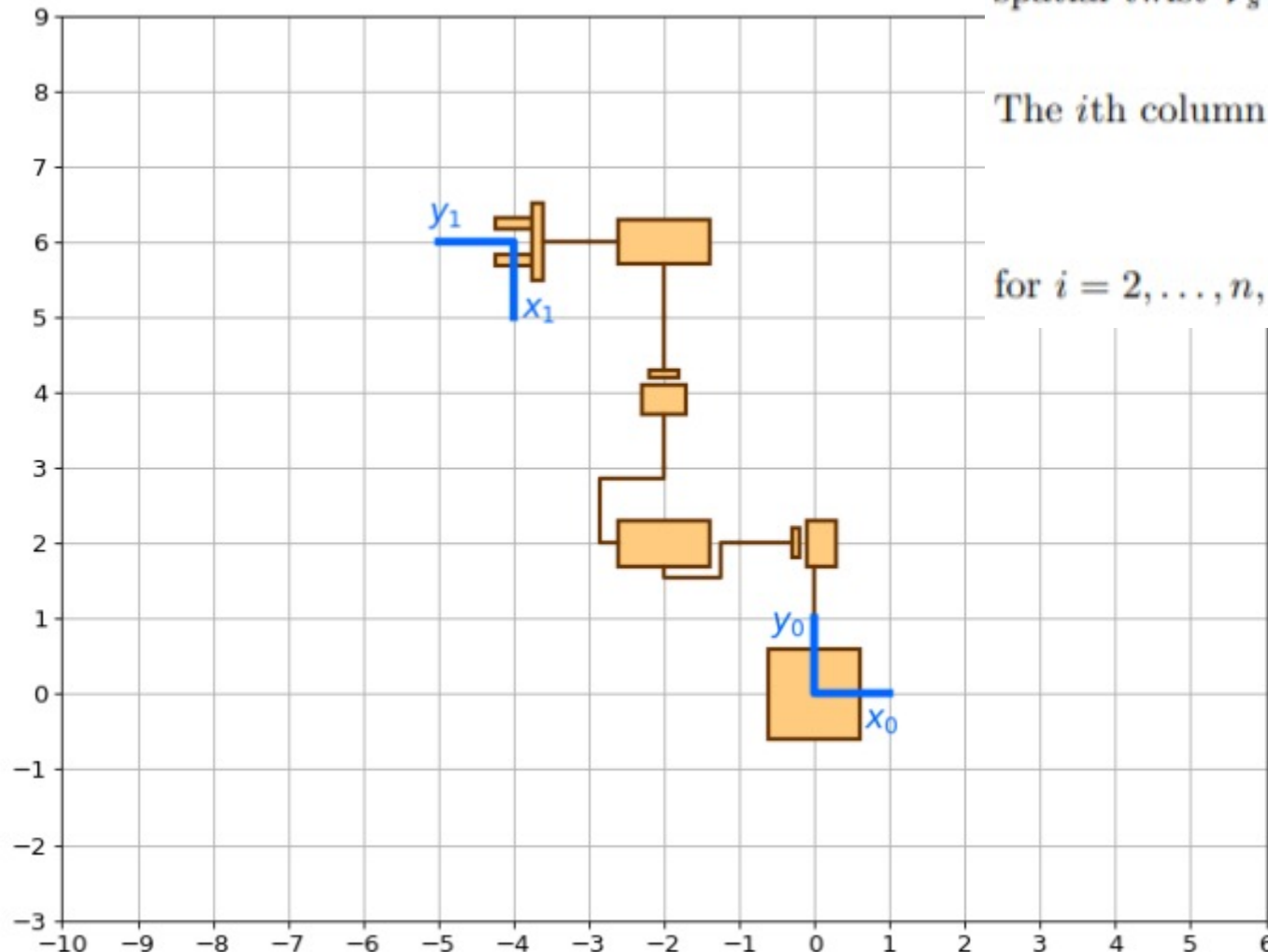
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So why do we care about
velocity kinematics?

Recall Our 2R Example

- Forward Kinematics:

$$x = L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2)$$

$$y = L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

- First Derivative:

$$\dot{x} = -L_1 \dot{\theta}_1 \sin \theta_1 - L_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2)$$

$$\dot{y} = L_1 \dot{\theta}_1 \cos \theta_1 + L_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2)$$

- Jacobian:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ &= J_1(\theta) \dot{\theta} + J_2(\theta) \dot{\theta} = J(\theta) \dot{\theta} \end{aligned}$$

Kinematic Singularities

- What does the velocity kinematics intuitively give us?

$$\mathcal{V} = [J_1(\theta) \quad J_2(\theta) \quad \cdots \quad J_n(\theta)] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

Kinematic Singularities

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- What is the maximum rank? What happens $\text{rank}(J) < n$?

Kinematic Singularities- 2R example

- Consider the robot with link lengths of 1 at the typical zero position.
- What is the Jacobian?

$$J \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -\sin 0 & -\sin 0 & -\sin 0 \\ \cos 0 & +\cos 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

Kinematic Singularities - 2R example

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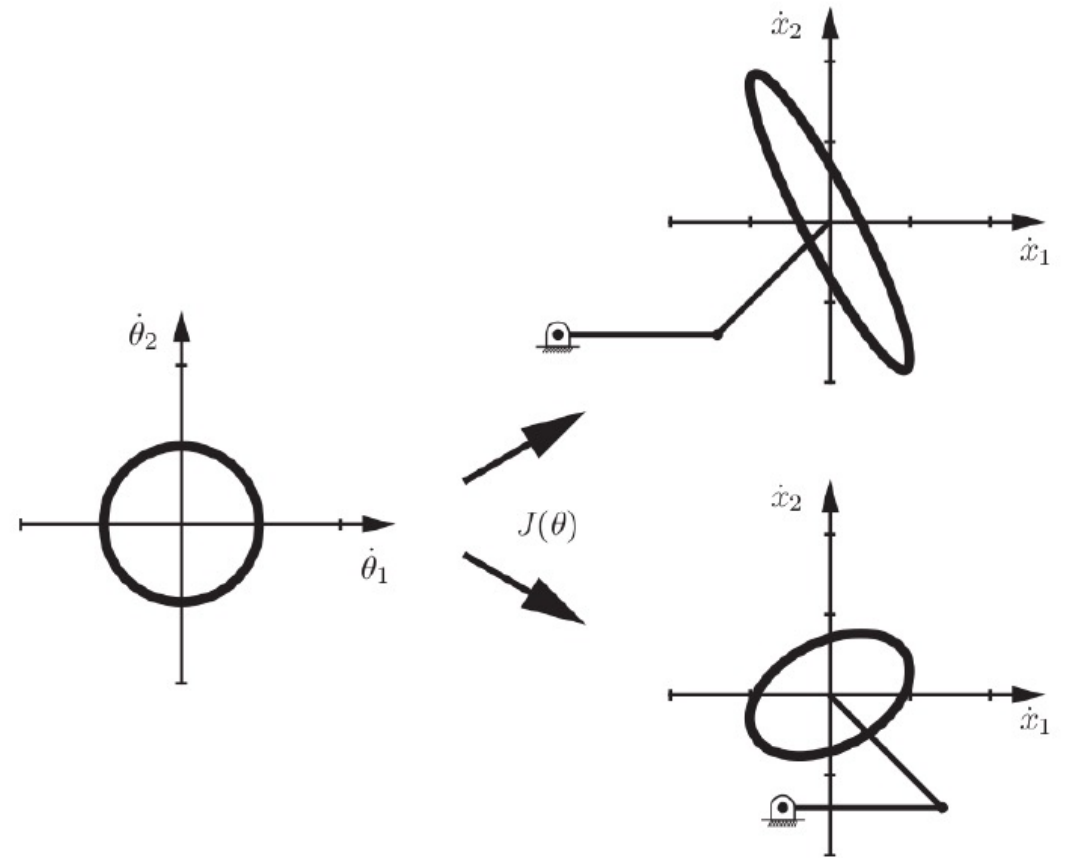
- Let's look at another configuration where $\theta = \begin{bmatrix} 0 & \frac{\pi}{4} \end{bmatrix}^T$
- What is the Jacobian?

$$J \left(\begin{bmatrix} 0 \\ \frac{\pi}{4} \end{bmatrix} \right) = \begin{bmatrix} -\sin 0 - \sin \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \cos 0 + \cos \frac{\pi}{4} & \cos \frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} -.71 & -.71 \\ 1.71 & .71 \end{bmatrix}$$

Jacobian Mapping

Jacobian and Manipulability

- Suppose joint limits are $\dot{\theta}_1^2 + \dot{\theta}_2^2 \leq 1$
- The ellipsoid obtained by mapping through the Jacobian is called **the manipulability ellipsoid**



This circle represents an “iso-effort” contour in the joint velocity space, where total actuator effort is considered to be the sum of squares of the joint velocities.

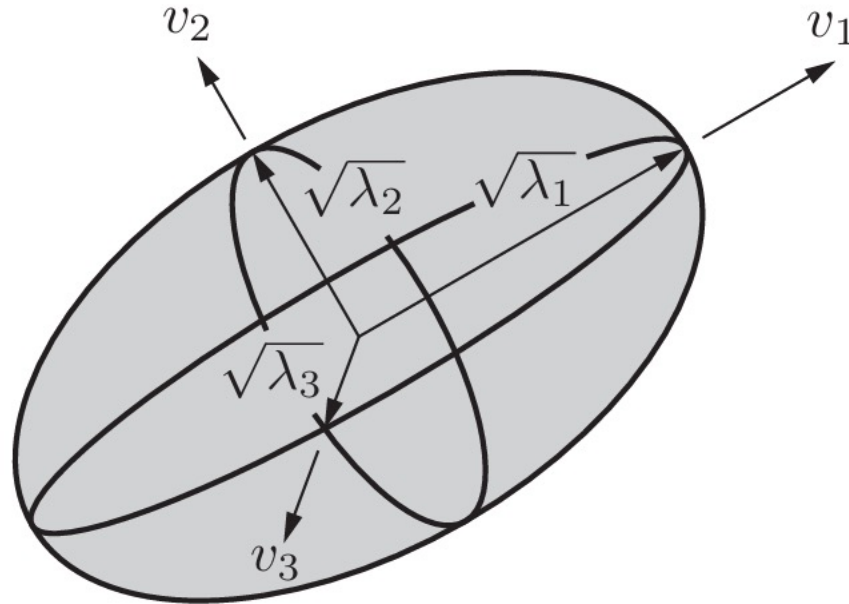
Manipulability

Recall Linear Algebra and Ellipsoids

- An ellipsoid can be written:

$$\xi^T Q \xi = 1$$

- The eigenvectors v_i and values λ_i give you the principle semi-axes v_i and lengths $\sqrt{\lambda_i}$



Measure of “closeness” to singularity?

Jacobian and Statics

- Assume a force f_{tip} is applied on the end-effector. What torque (τ) must be applied at the joint to keep the fixed position?

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- By conservation of power:

$$f_{tip}^\top v_{tip} = \tau^\top \dot{\theta}, \quad \forall \dot{\theta}$$

- Since $v_{tip} = J(\theta)\dot{\theta}$, we have

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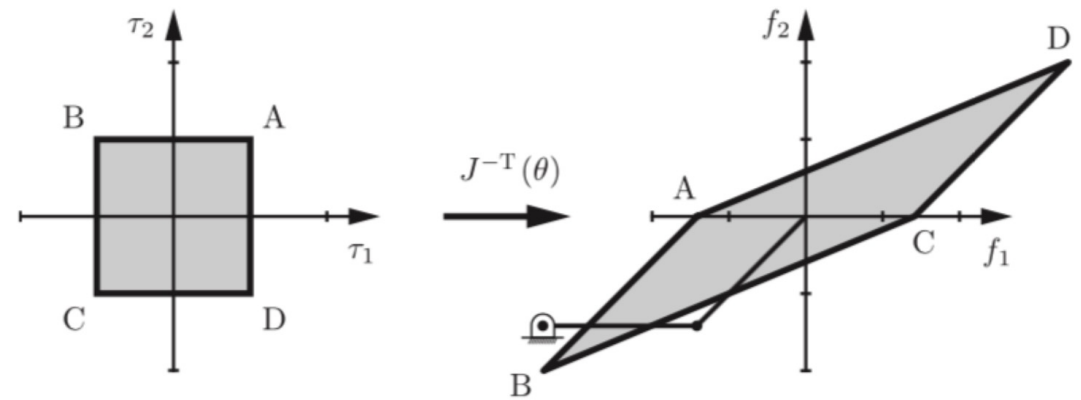
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- If $J^\top(\theta)$ is invertible, then given the limits of the torques at the joints, we can compute all the forces that can be counteracted at the end-effector



Summary

- Gave example on how to **compute velocity kinematics by definition**, instead of by inspection
- Introduced the concept of **kinematic singularities**, configurations in which there is a direction we cannot move
- The **manipulability ellipsoid** gives us a sense of nearness to singularity
- The **Jacobian is also useful in understanding statics** and forces required for simple tasks
 - What if the inverse doesn't exist? Will post notes on this.