

Lecture 9

Velocity Kinematics I

Prof. DC

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Modern Robotics Ch. 5.1-5.3



Who is Carl Gustav Jacob Jacobi?

- A German mathematician who contributed to elliptic functions, dynamics, differential equations, determinants, and number theory
- In Germany during the Revolution of 1848, Jacobi was politically involved with the Liberal club, which after the revolution ended led to his royal grant being temporarily cut off
- His PhD student, Otto Hesse, is known for the Hessian Matrix (second-order partial derivatives of a scalar-valued function, or scalar field)

Velocity Kinematics

Forward Kinematics

Calculate the position of the end-effector of an open chain given joint angles

Velocity Kinematics

Calculate the velocity (twist!) of the end-effector frame

Refresher on Twists

- What is a **twist**?

- Spatial velocity: $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}$, $[\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1}$

- What is the **adjoint map**?

- Given $T = (R, p)$, $[Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6 \times 6}$
 - $\mathcal{V}' = [Ad_T]\mathcal{V} = Ad_T(\mathcal{V})$
 - $[\mathcal{V}'] = T[\mathcal{V}]T^{-1}$

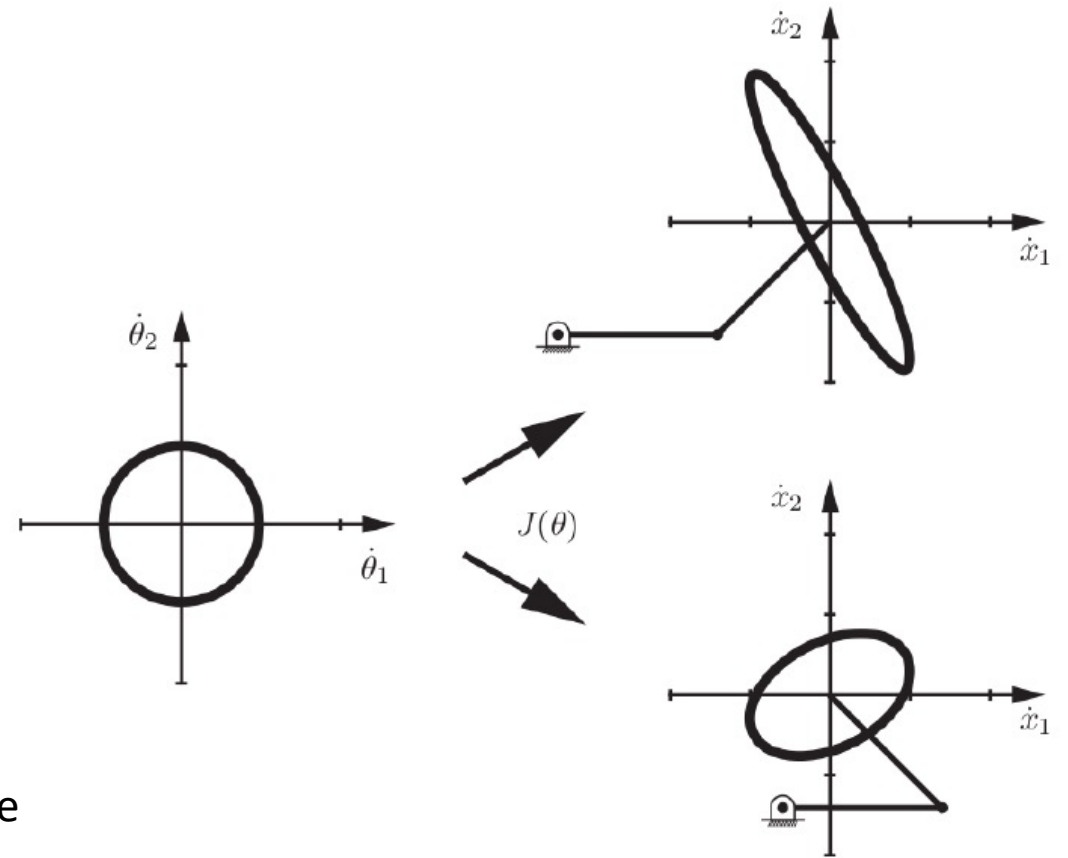
Example

Extra

Jacobian Mapping

Jacobian and Manipulability

- Suppose joint limits are $\dot{\theta}_1^2 + \dot{\theta}_2^2 \leq 1$
- The ellipsoid obtained by mapping through the Jacobian is called **the manipulability ellipsoid**



Configurations for which the end-effector cannot move in one or more directions instantaneously is a kinematic singularity

$J(\theta)$ is not full rank, <6 linearly independent columns

Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[s_1]\theta_1} e^{[s_2]\theta_2} \dots e^{[s_n]\theta_n} M$$

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- Take the derivative:

$$\begin{aligned}\dot{T}(\theta) &= \frac{d}{dt} \left(e^{[S_1]\theta_1} \right) e^{[S_2]\theta_2} \dots e^{[S_n]\theta_n} M + \dots + e^{[S_1]\theta_1} e^{[S_2]\theta_2} \dots \frac{d}{dt} \left(e^{[S_n]\theta_n} \right) M \\ &= [S_1] \dot{\theta}_1 e^{[S_1]\theta_1} \dots e^{[S_n]\theta_n} M + \dots + e^{[S_1]\theta_1} \dots [S_n] \dot{\theta}_n e^{[S_n]\theta_n} M\end{aligned}$$

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- Take the inverse:

$$T^{-1}(\theta) = M^{-1} e^{-[S_n]\theta_n} \dots e^{-[S_1]\theta_1}$$

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- Take the inverse:

$$T^{-1}(\theta) = M^{-1} e^{-[S_n]\theta_n} \dots e^{-[S_1]\theta_1}$$

- Recall: $[\mathcal{V}_s] = \dot{T} T^{-1}$

$$\dot{T} T^{-1} = [S_1] \dot{\theta}_1 + e^{[S_1]\theta_1} [S_2] e^{-[S_1]\theta_1} \dot{\theta}_2 + e^{[S_1]\theta_1} e^{[S_2]\theta_2} [S_3] e^{-[S_2]\theta_2} e^{-[S_1]\theta_1} \dot{\theta}_3 + \dots$$

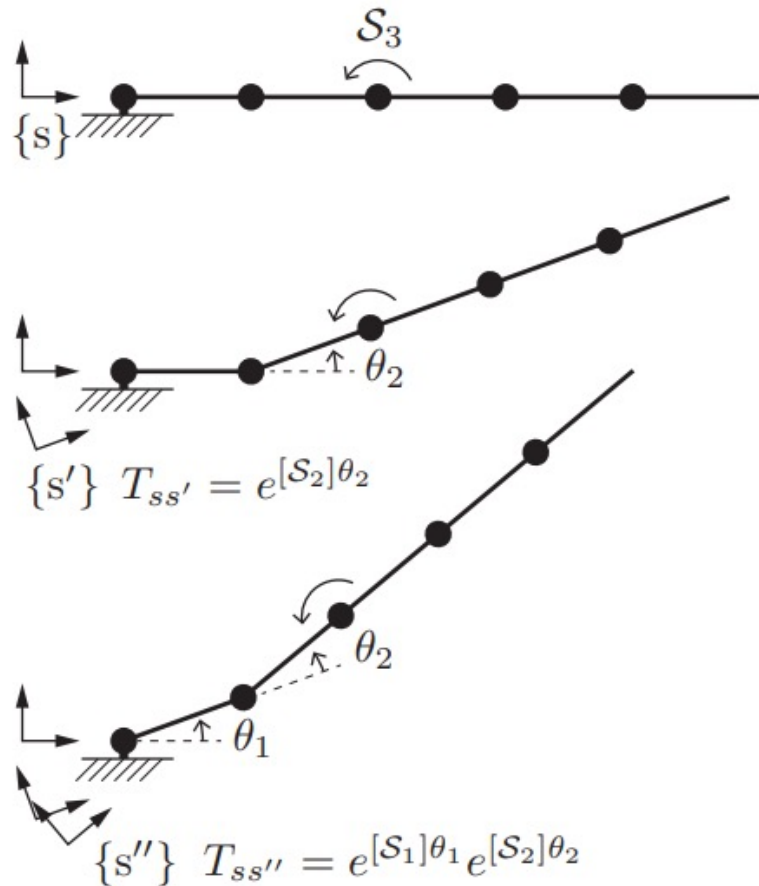
Computing the Jacobian (2)

- Expressed via Adjoint:

$$\mathcal{V}_s = \mathcal{S}_1 \dot{\theta}_1 + Ad_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2) \dot{\theta}_2 + Ad_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3) \dot{\theta}_3 + \dots$$

- Gives our Space Jacobian: $\mathcal{V}_s = J_s(\theta) \dot{\theta}$
- **Intuition:** If $T_i M$ is the configuration when you set joints $\theta_1, \dots, \theta_i$ and leave remaining joints at 0, then J_{si} is the screw vector at joint i , for an arbitrary θ

Visualizing the Jacobian



- By inspection, let's find J_{s3}
- Ignore joints $\theta_3, \theta_4, \theta_5$, as they do not displace axis 3 relative to $\{s\}$
- If $\theta_1 = 0$ and θ_2 is arbitrary, then $T_{ss'} = e^{[s_2]\theta_2}$
- If θ_1 is arbitrary, then $T_{ss''} = e^{[s_1]\theta_1} e^{[s_2]\theta_2}$
- $J_{s3} = [Ad_{T_{ss''}}]s_3 = [Ad_{e^{[s_1]\theta_1} e^{[s_2]\theta_2}}]s_3$

Methods for Computing the Jacobian

By Inspection

$J_{si} = (\omega_{si}, v_{si})$ where $Ad_{T_{i-1}}$ is implicit

By Definition

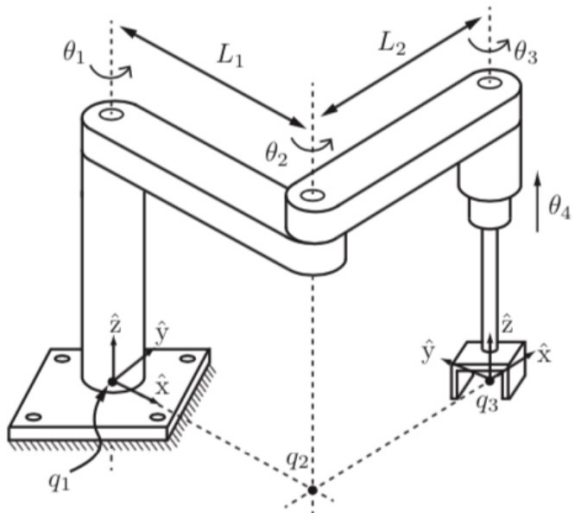
The **space Jacobian** $J_s(\theta) \in \mathbb{R}^{6 \times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$ via

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}. \quad (5.10)$$

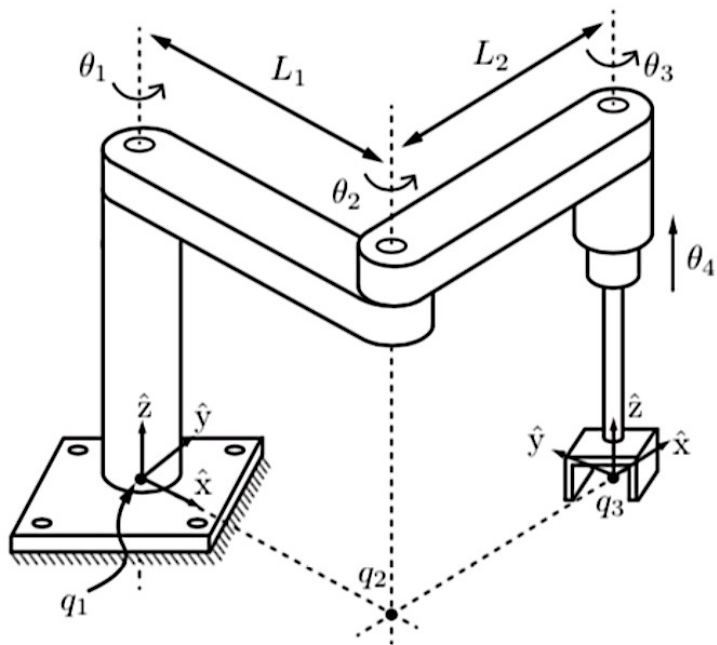
The i th column of $J_s(\theta)$ is

$$J_{si}(\theta) = \text{Ad}_{e^{[S_1]\theta_1} \dots e^{[S_{i-1}]\theta_{i-1}}} (S_i), \quad (5.11)$$

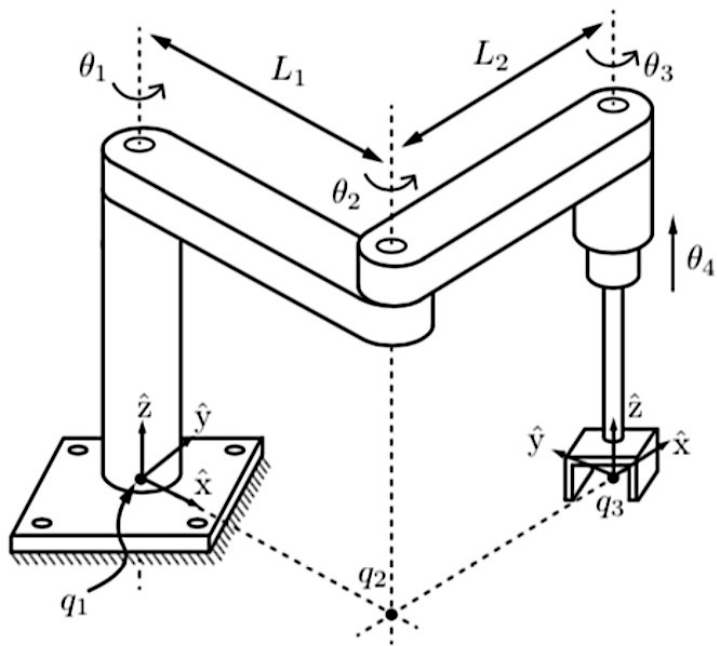
for $i = 2, \dots, n$, with the first column $J_{s1} = S_1$.



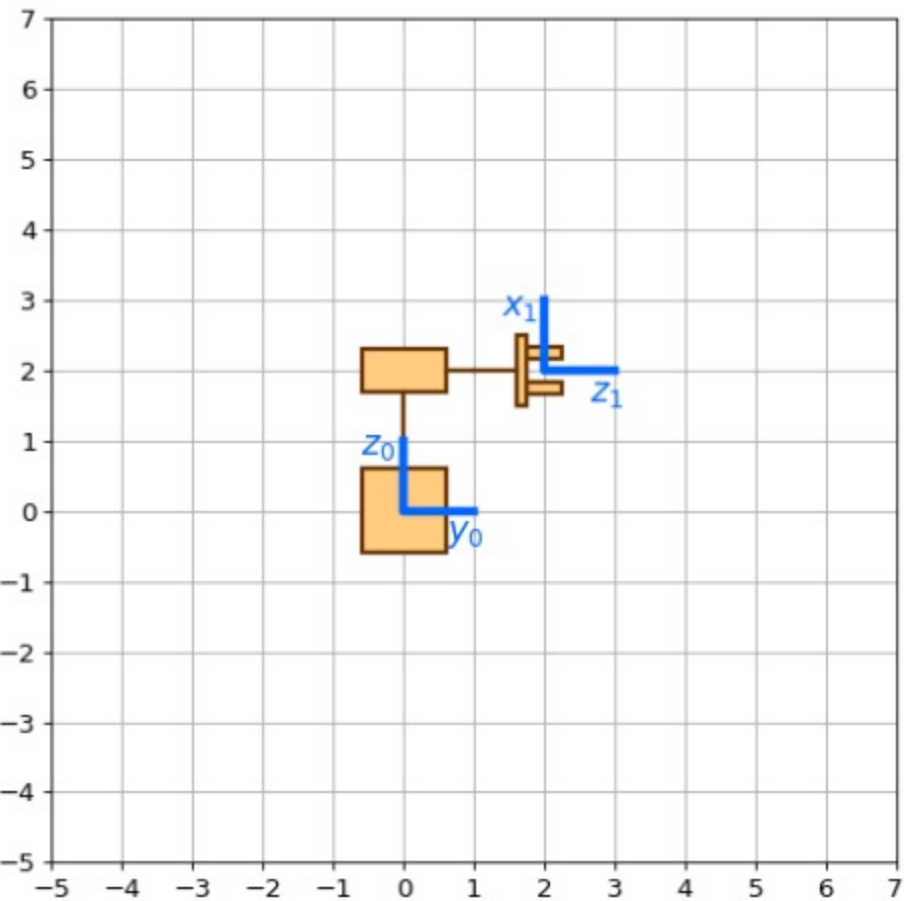
Example 1-1



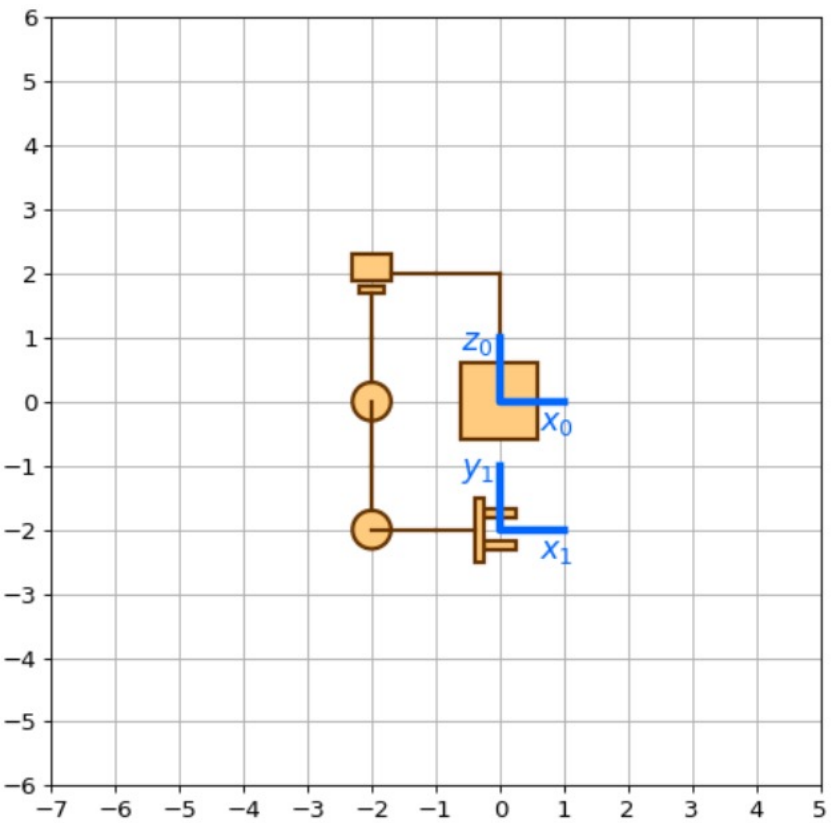
Example 1-2



Example 2



Example 3



Body Jacobians

Our space Jacobian is $[\mathcal{V}_s] = \dot{T} T^{-1}$, and our body Jacobian is $[\mathcal{V}_b] = T^{-1} \dot{T}$

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} \dots e^{[S_n]\theta_n} M = M e^{[B_1]\theta_1} e^{[B_2]\theta_2} \dots e^{[B_n]\theta_n}$$

$$\dot{T}(\theta) = M[B_1] \dot{\theta}_1 e^{[B_1]\theta_1} \dots e^{[B_n]\theta_n} + \dots + M e^{[B_1]\theta_1} \dots [B_n] \dot{\theta}_n e^{[B_n]\theta_n}$$

$$T^{-1}(\theta) = e^{-[B_n]\theta_n} \dots e^{-[B_1]\theta_1} M^{-1}$$

$$\begin{aligned} T^{-1} \dot{T} &= [B_n] \dot{\theta}_n + e^{-[B_n]\theta_n} [B_{n-1}] e^{[B_n]\theta_n} \dot{\theta}_{n-1} \\ &+ e^{-[B_n]\theta_n} e^{-[B_{n-1}]\theta_{n-1}} [B_{n-2}] e^{[B_{n-1}]\theta_{n-1}} e^{[B_n]\theta_n} \dot{\theta}_{n-1} \\ &+ e^{-[B_n]\theta_n} \dots e^{-[B_2]\theta_2} [B_1] e^{[B_2]\theta_2} \dots e^{[B_n]\theta_n} \dot{\theta}_1 \end{aligned}$$

The relationship between the Space and Body Jacobian

Recall that:

$$\left. \begin{array}{ll} [\mathcal{V}_s] = \dot{T}_{sb} T_{sb}^{-1} & [\mathcal{V}_b] = T_{sb}^{-1} \dot{T}_{sb} \\ \mathcal{V}_s = Ad_{T_{sb}}(\mathcal{V}_b) & \mathcal{V}_b = Ad_{T_{bs}}(\mathcal{V}_s) \\ \mathcal{V}_s = J_s(\theta) \dot{\theta} & \mathcal{V}_b = J_b(\theta) \dot{\theta} \end{array} \right\} Ad_{T_{sb}}(\mathcal{V}_b) = J_s(\theta) \dot{\theta}$$

- Apply $[Ad_{T_{bs}}]$ to both sides and recall that $[Ad_{T_X}][Ad_Y] = [Ad_{XY}]$:

$$\begin{aligned} Ad_{T_{bs}}(Ad_{T_{sb}}(\mathcal{V}_b)) &= \mathcal{V}_b = Ad_{T_{bs}}(J_s(\theta) \dot{\theta}) \\ J_b(\theta) \dot{\theta} &= Ad_{T_{bs}}(J_s(\theta) \dot{\theta}) \end{aligned}$$

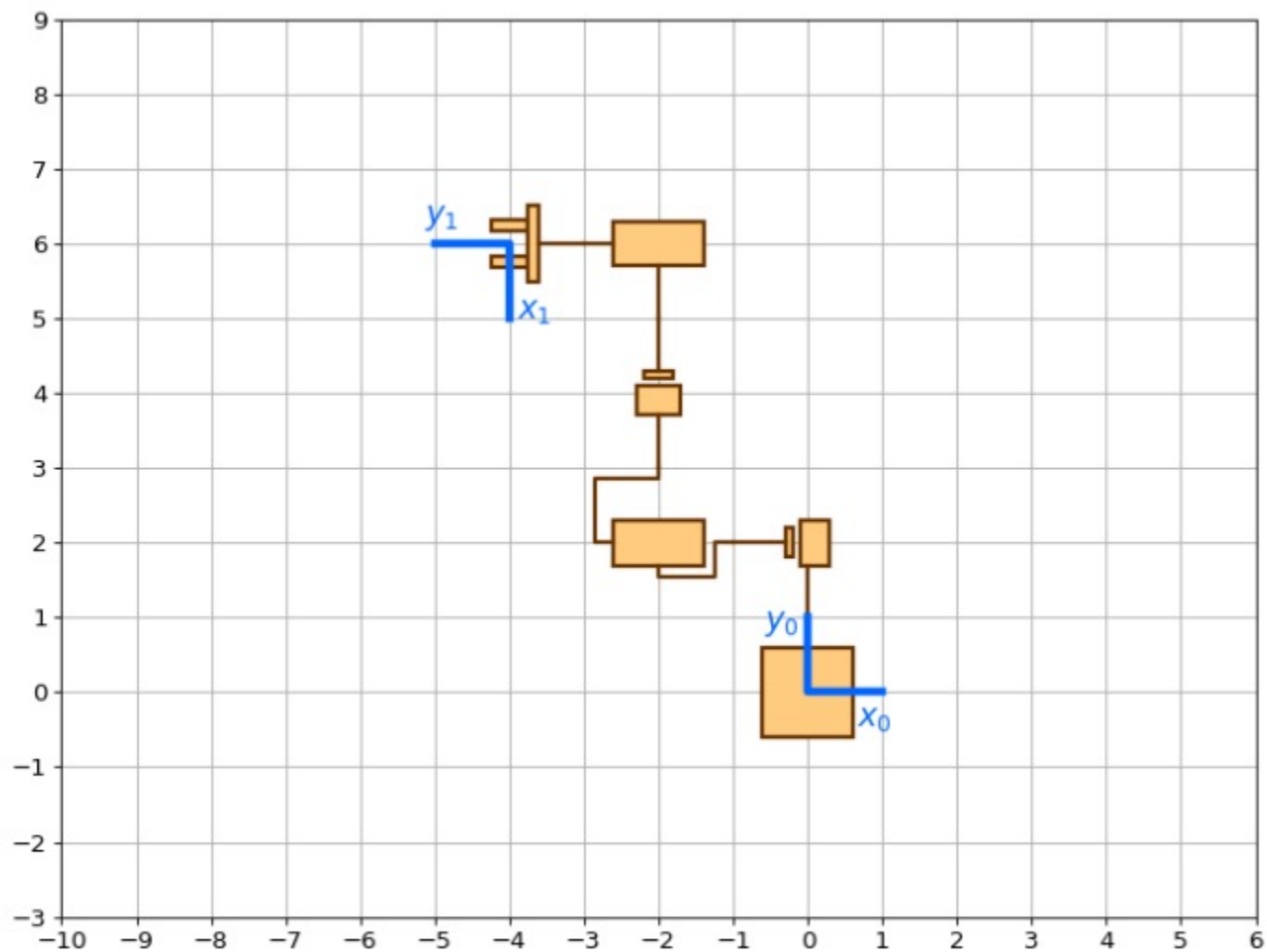
- Since this holds for all $\dot{\theta}$:

$$\begin{aligned} J_b(\theta) &= Ad_{T_{bs}}(J_s(\theta)) = [Ad_{T_{bs}}] J_s(\theta) \\ J_s(\theta) &= Ad_{T_{sb}}(J_b(\theta)) = [Ad_{T_{sb}}] J_b(\theta) \end{aligned}$$

Summary

- Defined **velocity kinematics** as means to compute the twist at the end-effector frame
- Interpreted the end-effector twist as a way to map actuator limits to **possible end-effector velocities** (directions of motion) and insight as to how close a pose is to a **singularity**
- Learned **two methods for computing the Jacobian**: (1) by inspection and (2) by using equations for defining the spatial and body twists

Example 4



The **space Jacobian** $J_s(\theta) \in \mathbb{R}^{6 \times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$ via

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The i th column of $J_s(\theta)$ is

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