Lecture 9
Velocity Kinematics I

Prof. DC

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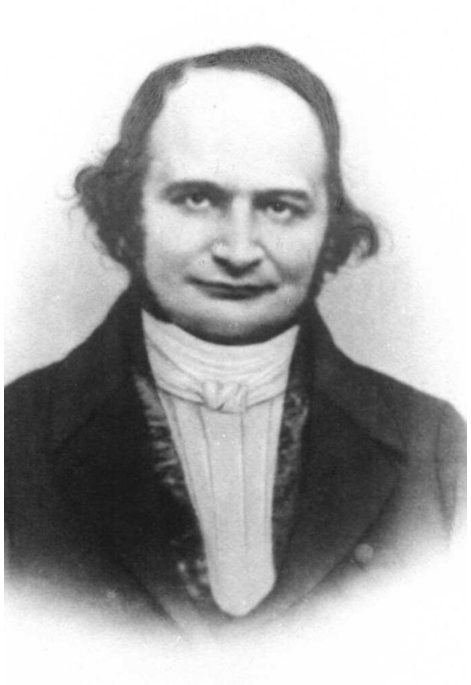
Modern Robotics Ch. 5.1-5.3



Who is Carl Gustav Jacob Jacobi?

- A German mathematician who contributed to elliptic functions, dynamics, differential equations, determinants, and number theory
- In Germany during the Revolution of 1848, Jacobi was politically involved with the Liberal club, which after the revolution ended led to his royal grant being temporarily cut off
- His PhD student, Otto Hesse, is known for the Hessian Matrix (second-order partial derivatives of a scalar-valued function, or scalar field)

Credit: Wikipedia

Velocity Kinematics

Forward Kinematics

Calculate the position of the end-effector of an open chain given joint angles

$$\underbrace{x(t)}_{\text{pose of tool}} = f(\underbrace{\theta(t)}_{\text{joint angles}})$$

$f \rightarrow$ FK map / $T(\theta)$ / PoE

Velocity Kinematics

Calculate the velocity (twist!) of the end-effector frame

$$\dot{x} = \frac{d}{dt}x(t) = \underbrace{\frac{\partial f}{\partial \theta}(\theta)}_{J(\theta) \rightarrow \text{Jacobian}} \cdot \dot{\theta}$$

Refresher on Twists

$\dot{T} = ?\,T$

$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

- What is a **twist?**
  - Spatial velocity: $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}$, $[\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1}$
- What is the **adjoint map?**
  - Given $T = (R, p)$, $[\mathrm{Ad}_\mathrm{T}] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times6}$
  - $\mathcal{V}' = [Ad_T]\mathcal{V} = Ad_T(\mathcal{V})$
  - $[\mathcal{V}'] = T[\mathcal{V}]T^{-1}$

Example

FK: $x_1 = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$

$x_2 = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$

Differentiate:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \dot{\theta}_1 \sin\theta_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2) \\ L_1 \dot{\theta}_1 \cos\theta_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2) \end{bmatrix}$$

$J_1(\theta)$ J_1 $J_2(\theta)$

rearrange: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix}}_{J(\theta)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\dot{x} = J_1(\theta) \cdot \dot{\theta}_1 + J_2(\theta) \cdot \dot{\theta}_2$$

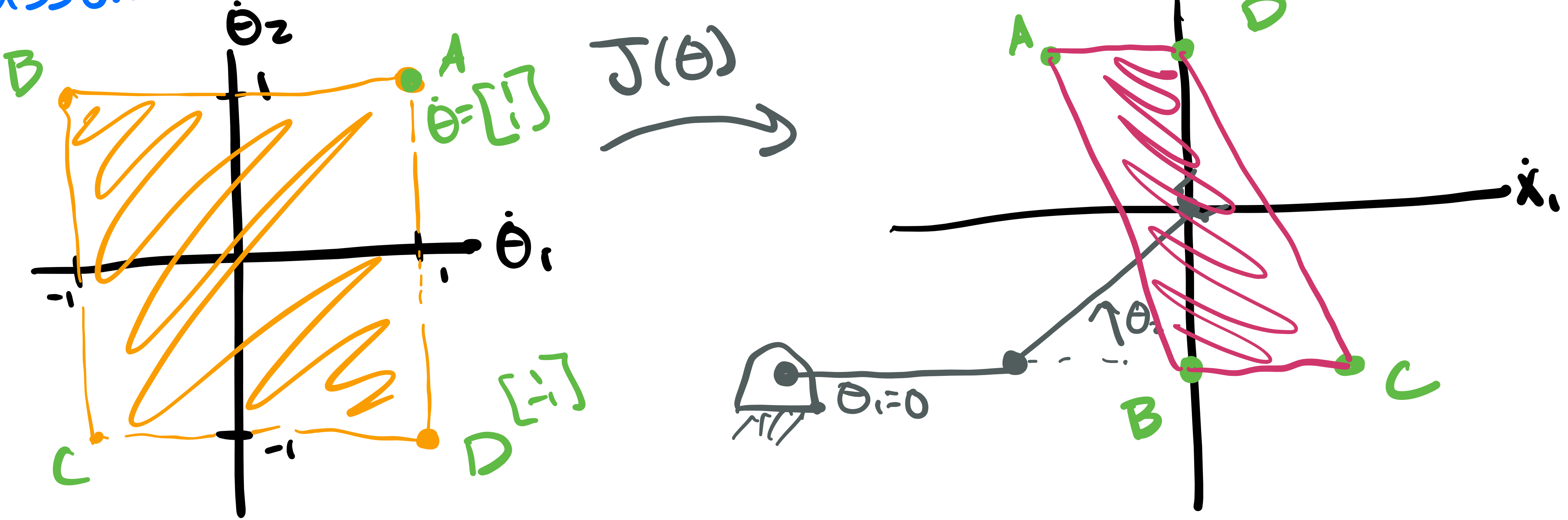
Extra

Jacobian Mapping

if $L_1 = L_2 = 1$, $\theta_1 = 0$, $\theta_2 = \pi/4$, then

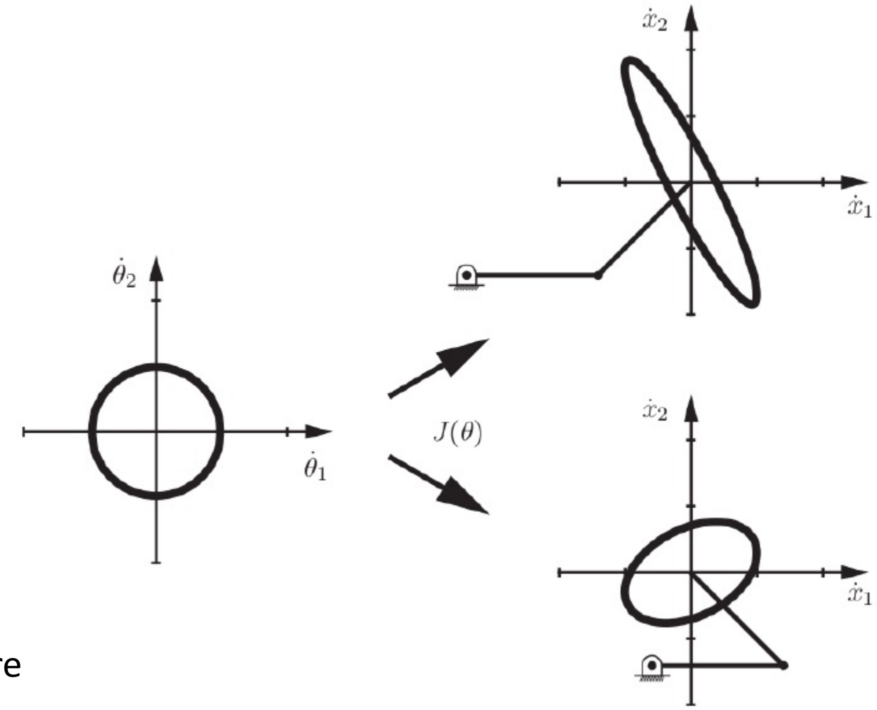
$$J\left(\begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}\right) = \begin{bmatrix} -.71 & -.71 \\ 1.71 & .71 \end{bmatrix}$$

assume actuator limits: $\dot{\theta}_i \in [-1, 1]$



Jacobian and Manipulability

- Suppose joint limits are $\dot{\theta}_1^2 + \dot{\theta}_2^2 \leq 1$
- The ellipsoid obtained by mapping through the Jacobian is called **the manipulability ellipsoid**



Configurations for which the end-effector cannot move in one or more directions instantaneously is a kinematic singularity

$J(\theta)$ is not full rank, <6 linearly independent columns

Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

- Take the derivative:

$$\begin{aligned}\dot{T}(\theta) &= \frac{d}{dt}\left(e^{[\mathcal{S}_1]\theta_1}\right) e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots \frac{d}{dt}\left(e^{[\mathcal{S}_n]\theta_n}\right) M \\ &= [\mathcal{S}_1]\, \dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} \dots [\mathcal{S}_n]\, \dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M\end{aligned}$$

Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

- Take the derivative:

$$\dot{T}(\theta) = \frac{d}{dt}\left(e^{[\mathcal{S}_1]\theta_1}\right) e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots \frac{d}{dt}\left(e^{[\mathcal{S}_n]\theta_n}\right) M$$

$$= [\mathcal{S}_1]\, \dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} \dots [\mathcal{S}_n]\, \dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M$$

- Take the inverse:

$$T^{-1}(\theta) = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \dots e^{-[\mathcal{S}_1]\theta_1}$$

$$T T^{-1} = e^{[S_1]\theta_1} e^{[S_2]\theta_2} M M^{-1} e^{-[S_2]\theta_2} e^{-[S_1]\theta_1}$$

Computing the Jacobian (1)

- Recall forward kinematics:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$$

- Take the derivative:

$$\dot{T}(\theta) = \frac{d}{dt}\left(e^{[\mathcal{S}_1]\theta_1}\right) e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots \frac{d}{dt}\left(e^{[\mathcal{S}_n]\theta_n}\right) M$$

$$= [\mathcal{S}_1]\,\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M + \dots + e^{[\mathcal{S}_1]\theta_1} \dots [\mathcal{S}_n]\,\dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M$$

- Take the inverse:

$$T^{-1}(\theta) = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \dots e^{-[\mathcal{S}_1]\theta_1}$$

- Recall: $[\mathcal{V}_s] = \dot{T}\, T^{-1}$

$$\dot{T}T^{-1} = [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_3 + \cdots$$

T_1 T_1^{-1}

$Ad_{e^{[S_1]\theta_1} e^{[S_2]\theta_2}}(S_3) \cdot \dot{\theta}_3$

Computing the Jacobian (2)

- Expressed via Adjoint:

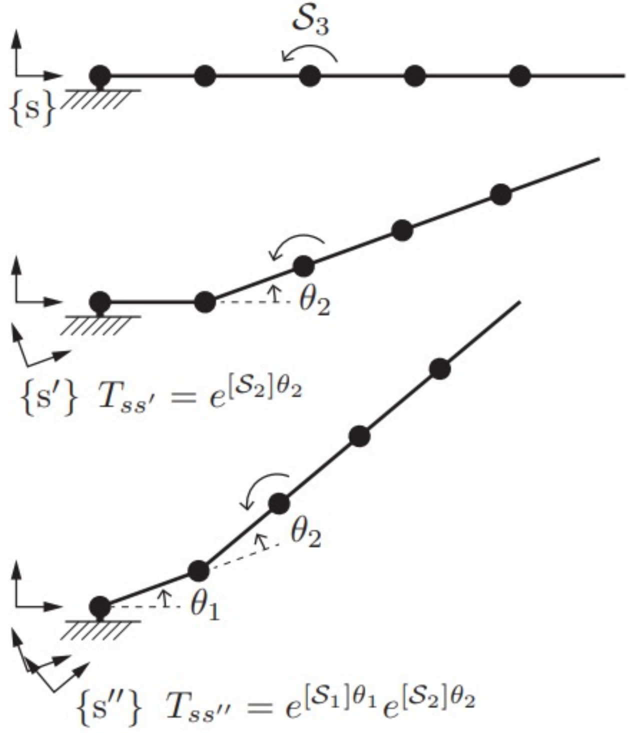
$$\mathcal{V}_s = \underbrace{\mathcal{S}_1}_{J_{s1}}\dot{\theta}_1 + \underbrace{Ad_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}}\dot{\theta}_2 + \underbrace{Ad_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}}\dot{\theta}_3 + \cdots$$

- Gives our Space Jacobian: $\mathcal{V}_s = J_s(\theta)\dot{\theta}$ $\quad J_s(\theta) = [J_{s1} \; J_{s2} \; \cdots \; J_{sn}]$

$$J_{s1} = \mathcal{S}_1, \quad J_{si} = Ad_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i)$$

- **<u>Intuition:</u>** If $T_i M$ is the configuration when you set joints $\theta_1, \ldots, \theta_i$ and leave remaining joints at 0, then J_{si} is the screw vector at joint i, for an arbitrary θ

Visualizing the Jacobian

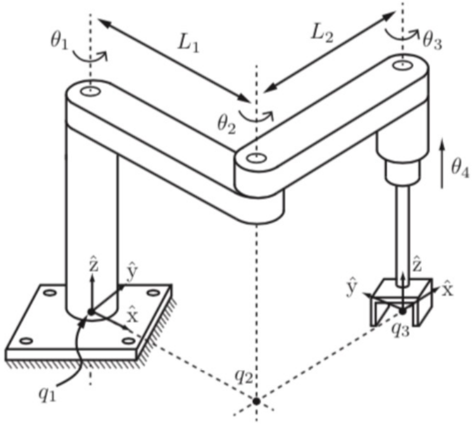


- By inspection, let's find J_{s3}
- Ignore joints $\theta_3, \theta_4, \theta_5$, as they do not displace axis 3 relative to {s}
- If $\theta_1 = 0$ and θ_2 is arbitrary, then $T_{ss'} = e^{[\mathcal{S}_2]\theta_2}$
- If θ_1 is arbitrary, then $T_{ss''} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}$
- $J_{s3} = [Ad_{T_{ss''}}]\mathcal{S}_3 = [Ad_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}]\mathcal{S}_3$

Methods for Computing the Jacobian

By Inspection

$J_{si} = (\omega_{si}, v_{si})$ where $Ad_{T_{i-1}}$ is implicit



By Definition

The **space Jacobian** $J_s(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$ via

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}. \tag{5.10}$$

The ith column of $J_s(\theta)$ is

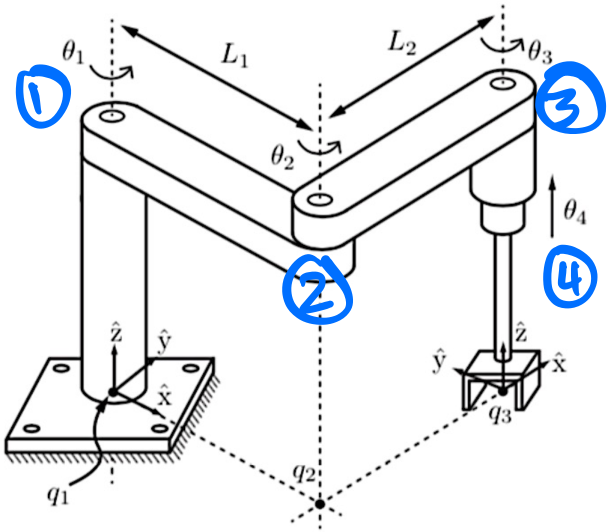
$$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i), \tag{5.11}$$

for $i = 2, \ldots, n$, with the first column $J_{s1} = \mathcal{S}_1$.

Example 1-1

① $J_{s1} = S_1$, $\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

② $\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $q_2 = \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \\ 0 \end{bmatrix}$



$$v_2 = -\omega_2 \times q_2 = \begin{bmatrix} L_1 \sin\theta_1 \\ -L_1 \cos\theta_1 \\ 0 \end{bmatrix}$$

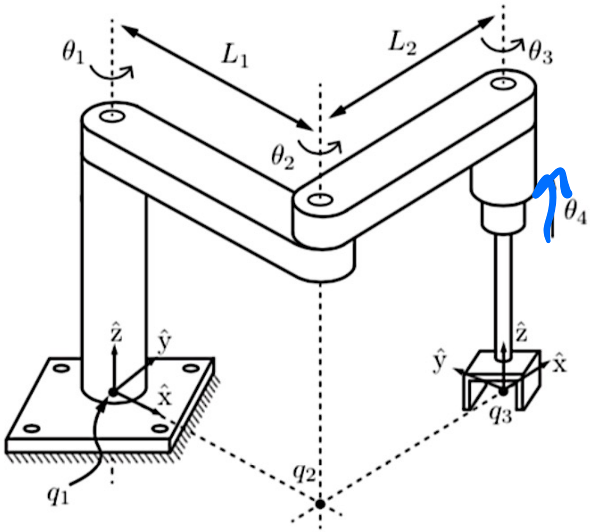
$$J_{s2} = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix}$$

Example 1-2

③ $\omega_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $q_3 = \begin{bmatrix} L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \\ 0 \end{bmatrix}$

④ $\omega_4 = 0$, $v_4 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & L_1 S_1 & L_1 S_1 + L_2 S_{12} & 0 \\ 0 & -L_1 C_1 & -L_1 C_1 - L_2 C_{12} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

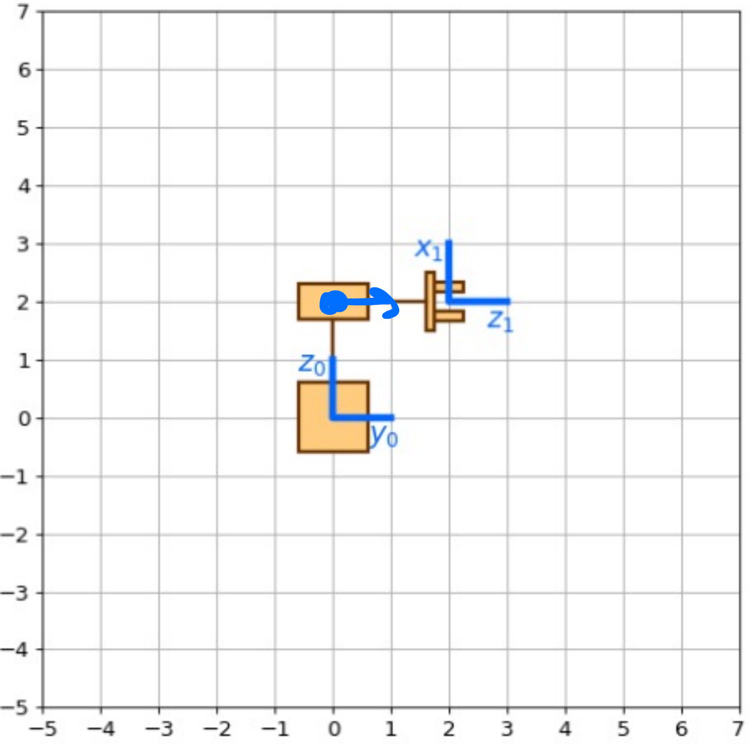


Example 2

Find S_1

$$\omega_{s1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad q_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$v_{s1} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

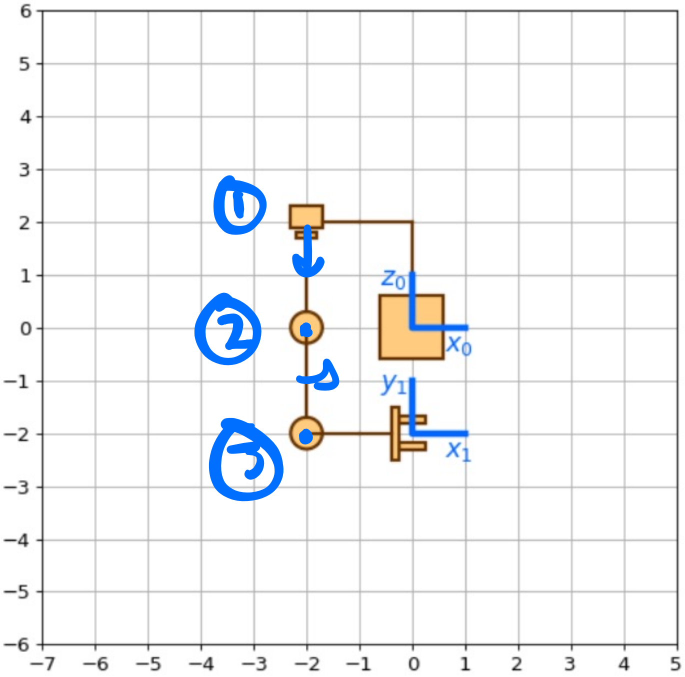


$$J_{s1} = S_1$$

$$\mathcal{V} = J_{s1} \cdot \dot{\theta}_1$$

Example 3

① $J_{31} = S_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$



② $w_2 = [0 \;\; -1 \;\; 0]^T$

$q_2 = [-2 \;\; 0 \;\; -\theta_1]^T$

$v_2 = [-\theta_1 \;\; 0 \;\; 2]^T$

③ $w_3 = [0 \;\; -1 \;\; 0]^T$

$q_3 = \begin{bmatrix} -2 + 2\sin\theta_2 \\ 0 \\ -\theta_1 - 2\cos\theta_2 \end{bmatrix}$

$v_3 = \begin{bmatrix} -\theta_1 - 2\cos\theta_2 \\ 0 \\ 2 - 2\sin\theta_3 \end{bmatrix}$

Body Jacobians

Our space Jacobian is $[\mathcal{V}_s] = \dot{T}\,T^{-1}$, and our body Jacobian is $[\mathcal{V}_b] = T^{-1}\dot{T}$

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M = M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} \dots e^{[\mathcal{B}_n]\theta_n}$$

$$\dot{T}(\theta) = M[\mathcal{B}_1]\,\dot{\theta}_1 e^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_n]\theta_n} + \dots + M e^{[\mathcal{B}_1]\theta_1} \dots [\mathcal{B}_n]\,\dot{\theta}_n e^{[\mathcal{B}_n]\theta_n}$$

$$T^{-1}(\theta) = e^{-[\mathcal{B}_n]\theta_n} \dots e^{-[\mathcal{B}_1]\theta_1} M^{-1}$$

$$\begin{aligned} &T^{-1}\dot{T} \\ &= [\mathcal{B}_n]\dot{\theta}_n + e^{-[\mathcal{B}_n]\theta_n}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} \\ &+ e^{-[\mathcal{B}_n]\theta_n} e^{-[\mathcal{B}_{n-1}]\theta_{n-1}}[\mathcal{B}_{n-2}]e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} \\ &+ e^{-[\mathcal{B}_n]\theta_n} \dots e^{-[\mathcal{B}_2]\theta_2}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2} \dots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1 \end{aligned}$$

The relationship between the Space and Body Jacobian

Recall that:

$$[\mathcal{V}_s] = \dot{T}_{sb}T_{sb}^{-1} \qquad [\mathcal{V}_b] = T_{sb}^{-1}\dot{T}_{sb}$$
$$\mathcal{V}_s = Ad_{T_{sb}}(\mathcal{V}_b) \qquad \mathcal{V}_b = Ad_{T_{bs}}(\mathcal{V}_s)$$
$$\mathcal{V}_s = J_s(\theta)\dot{\theta} \qquad \mathcal{V}_b = J_b(\theta)\dot{\theta}$$

$$Ad_{T_{sb}}(\mathcal{V}_b) = J_s(\theta)\dot{\theta}$$

- Apply $[Ad_{T_{bs}}]$ to both sides and recall that $\left[Ad_{T_X}\right][Ad_Y] = [Ad_{XY}]$:

$$Ad_{T_{bs}}(Ad_{T_{sb}}(\mathcal{V}_b)) = \mathcal{V}_b = Ad_{T_{bs}}(J_s(\theta)\dot{\theta})$$
$$J_b(\theta)\dot{\theta} = Ad_{T_{bs}}(J_s(\theta)\dot{\theta})$$

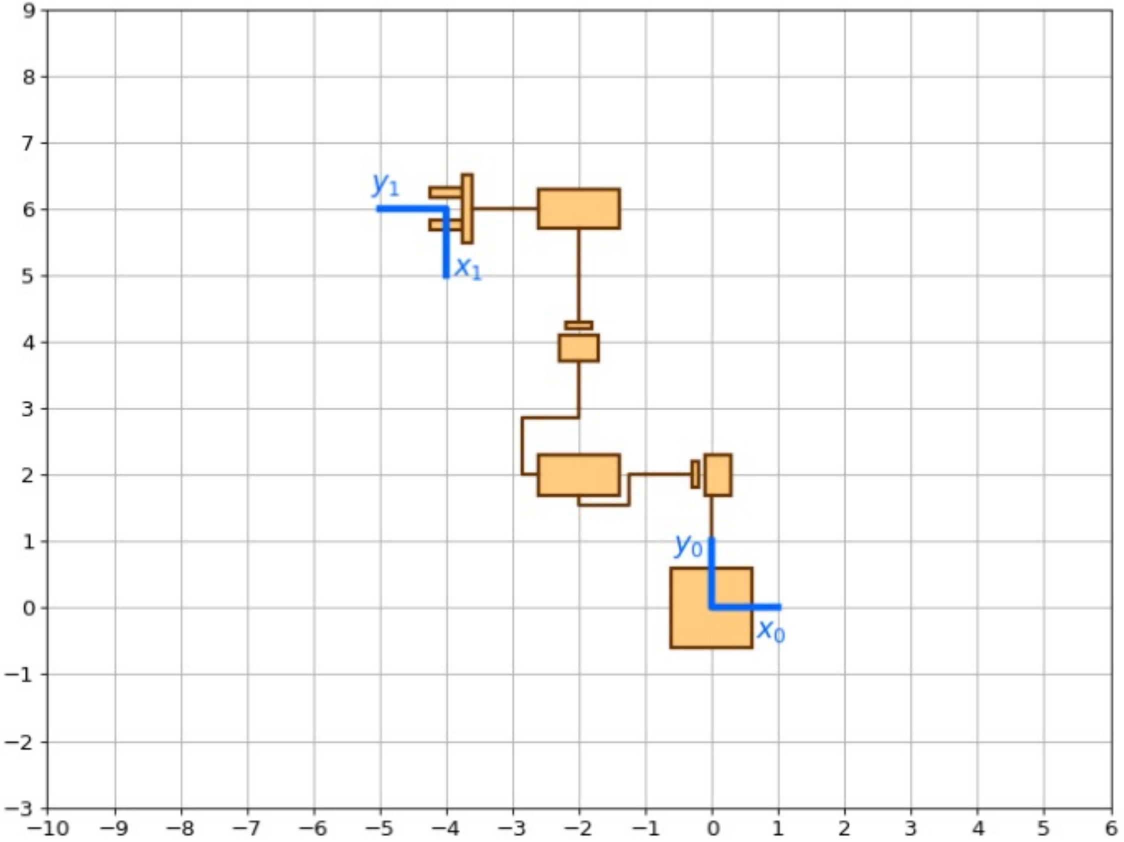
- Since this holds for all $\dot{\theta}$:

$$J_b(\theta) = Ad_{T_{bs}}\left(J_s(\theta)\right) = \left[Ad_{T_{bs}}\right]J_s(\theta)$$
$$J_s(\theta) = Ad_{T_{sb}}\left(J_b(\theta)\right) = \left[Ad_{T_{sb}}\right]J_b(\theta)$$

Summary

- Defined **velocity kinematics** as means to compute the twist at the end-effector frame
- Interpreted the end-effector twist as a way to map actuator limits to **possible end-effector velocities** (directions of motion) and insight as to how close a pose is to a **singularity**
- Learned **two methods for computing the Jacobian**: (1) by inspection and (2) by using equations for defining the spatial and body twists

Example 4



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