Lecture 8:
Forward Kinematics II

Prof. DC

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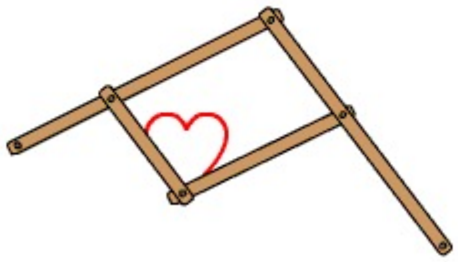
Modern Robotics Ch. 4



Denavit-Hartenberg Parameters

- In the 1950s, when Dick Hartenberg, a professor, and Jacques Denavit, a PhD student, developed a way to represent mathematically how mechanisms move
- They showed that the position of one link connected to another could be represented **minimally** using only four parameters
  - Known as the Denavit-Hartenberg (DH) parameters
- In 1981, Richard Paul demonstrated its value for the kinematic analysis of robotic systems

https://robotics.northwestern.edu/history.html

Screw Motions as Matrix Exponential

The screw axis $\mathcal{S}_i$ can be expressed in matrix form as:

$$[\mathcal{S}_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

To express a **screw motion** given a screw axis, we use the matrix exponential:

$$e^{[\mathcal{S}]\theta} \in SE(3)$$

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \qquad (3.88)$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Product of Exponentials Approach

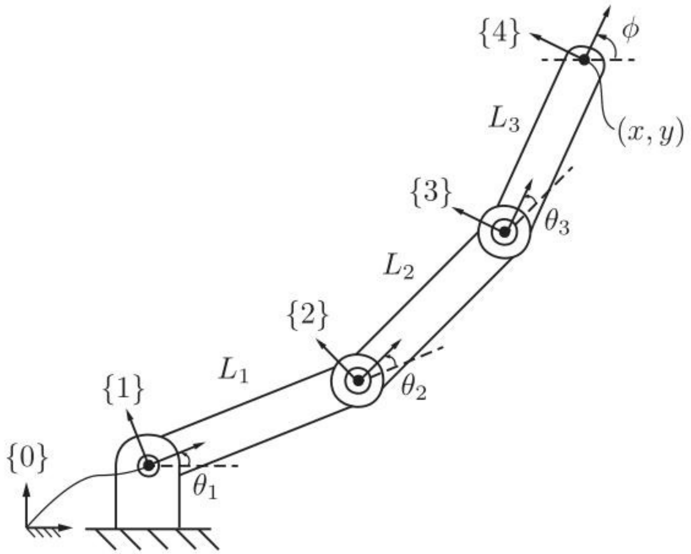


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the x̂- and ŷ-axis is shown; the ẑ-axes are parallel and out of the page.

Let each joint i have a configuration defined by θ_i

Initialization steps:

- Choose a fixed frame $\{s\}$
- Choose an end-effector (tool) frame attached to the robot $\{b\}$
- Put all joints in *zero position*
- Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position

Product of Exponentials Approach

- Given zero position M
- For each joint i, define the screw axis
- For each motion of a joint, define the screw motion
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Finding M

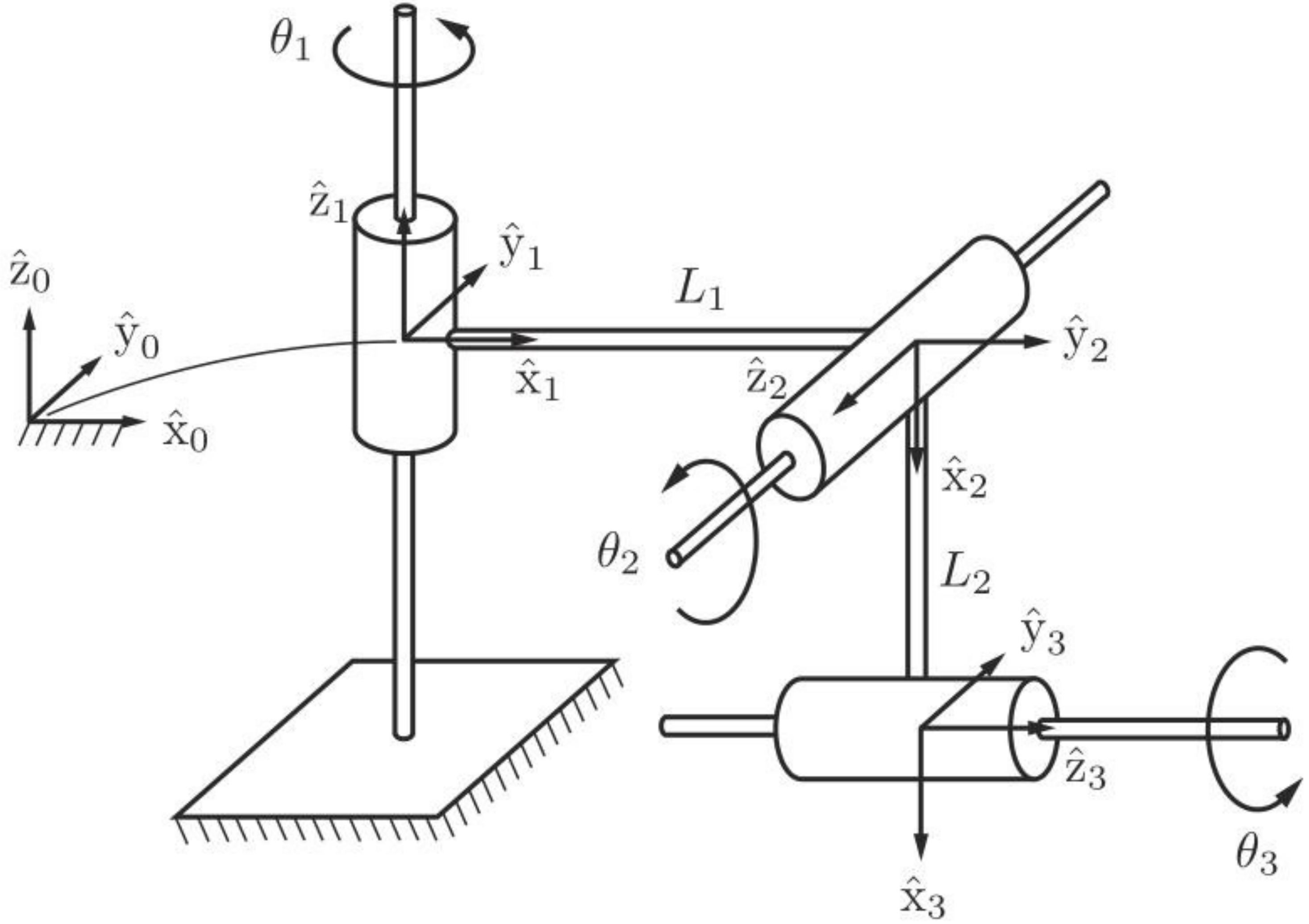
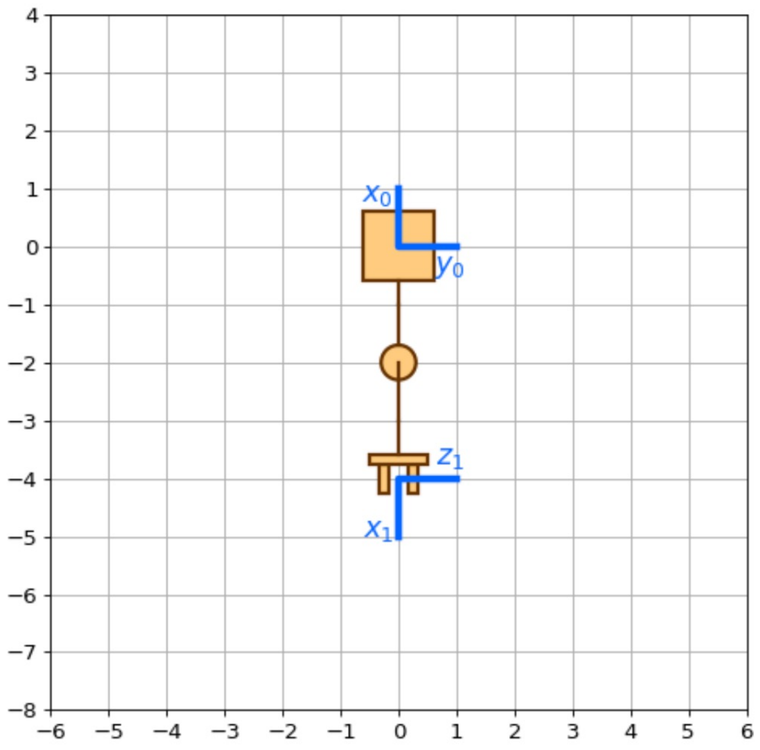
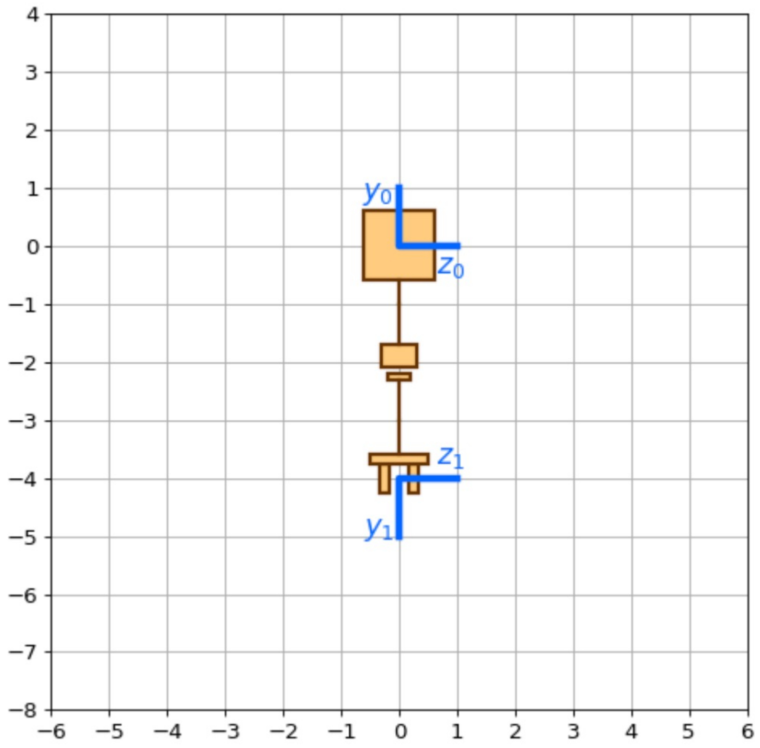


Figure 4.3: A 3R spatial open chain.

How to read this diagram (1)

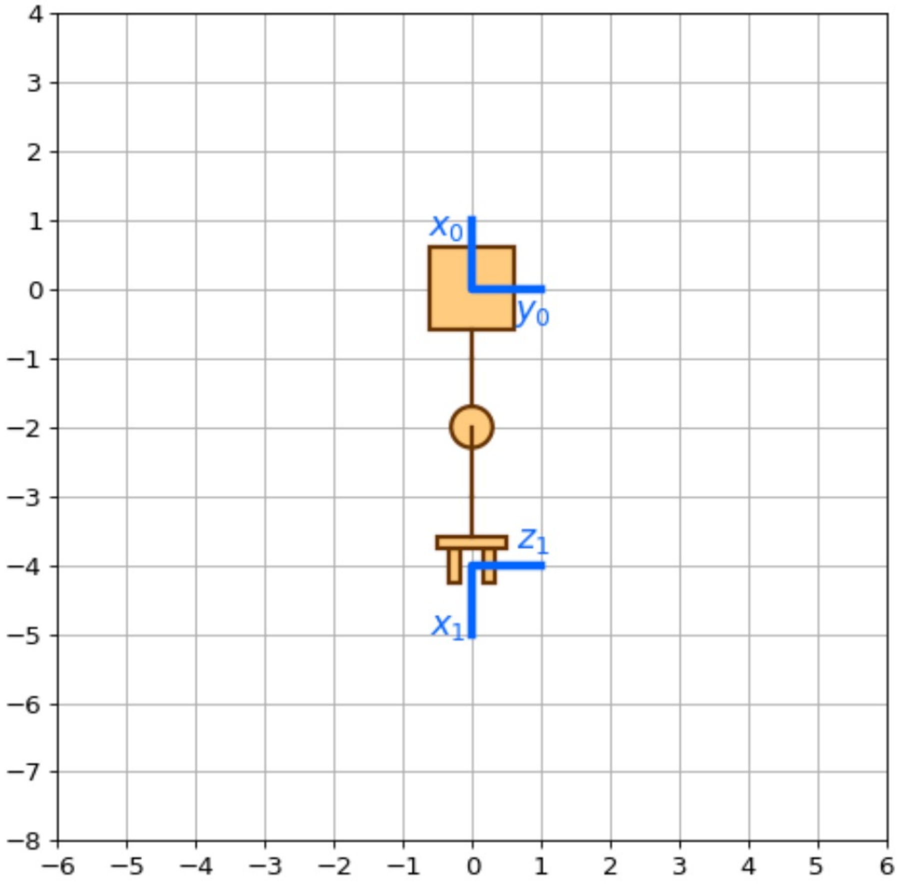




Modeling Robot Joints as Screw Motions

Case 1: Revolute Joint

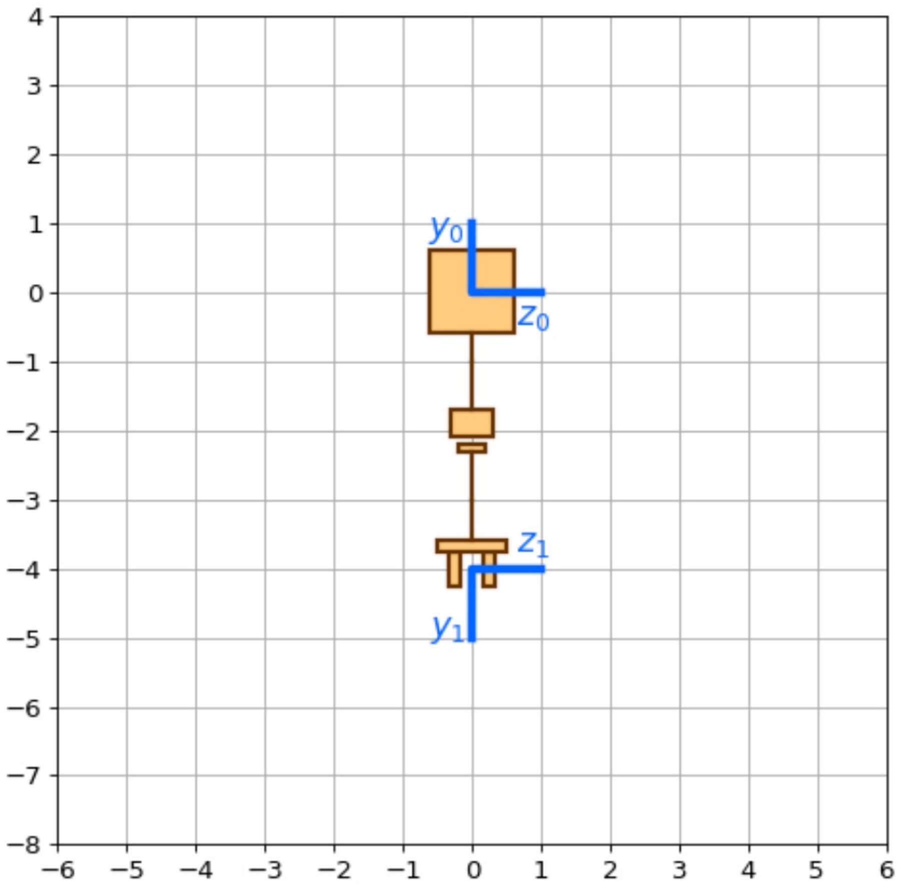
- $\|\omega\| = 1$
- $v = -\omega \times q + h\omega$
- $v = -\omega \times q$



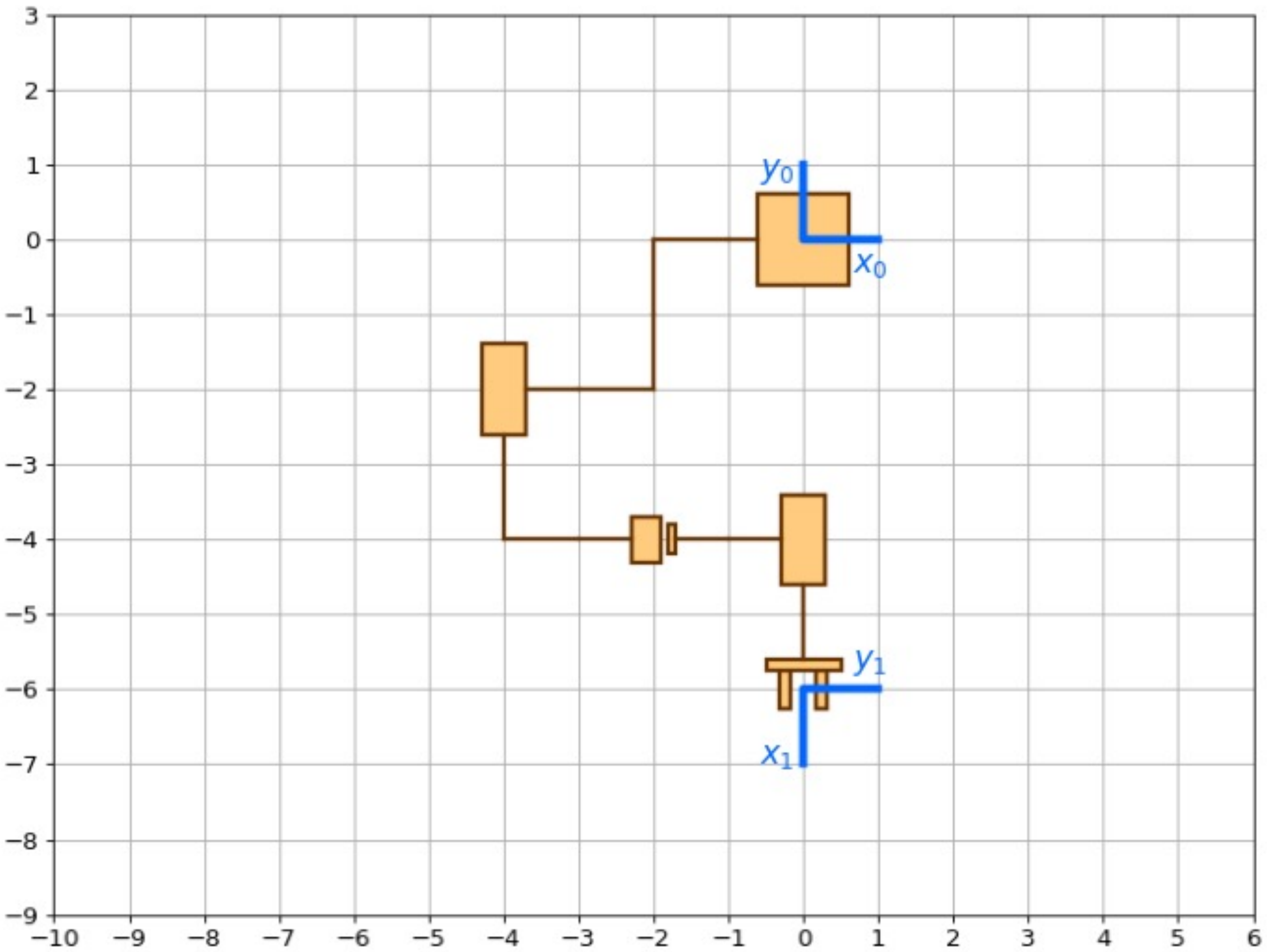
Modeling Robot Joints as Screw Motions

Case 2: Prismatic Joint

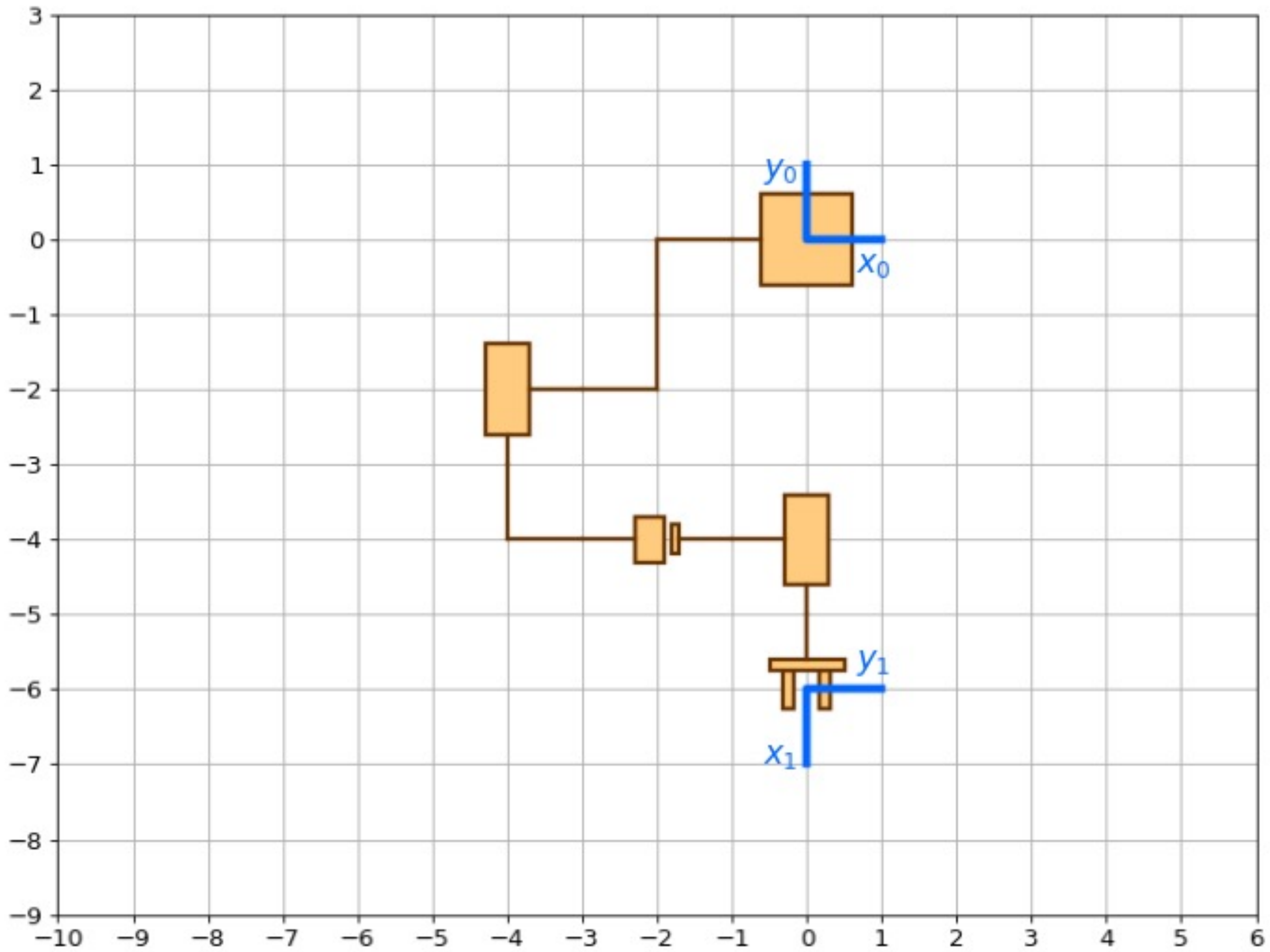
- $\|\omega\| = 0$
- $\|v\| = 1$
- Axis of movement defines v



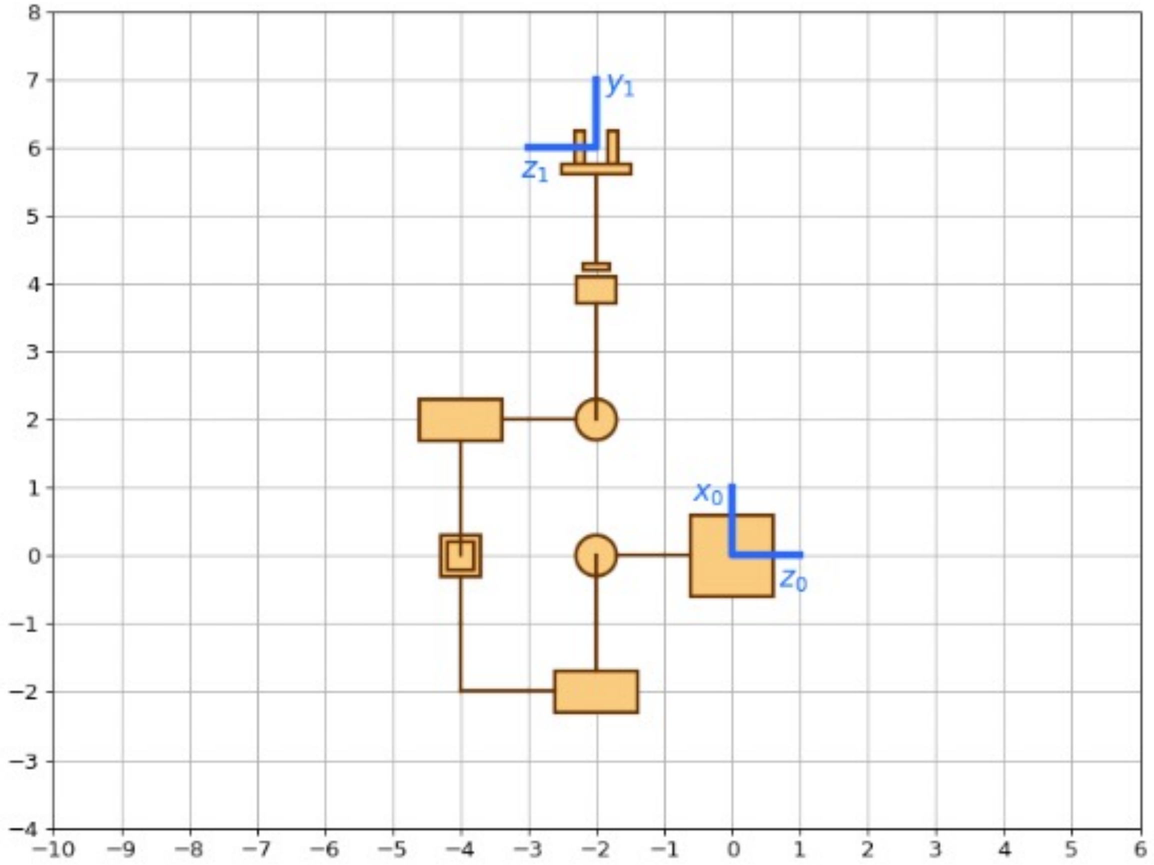
How to read this diagram (2)



Example 1



Example 2



So Far: Spatial Forward Kinematics

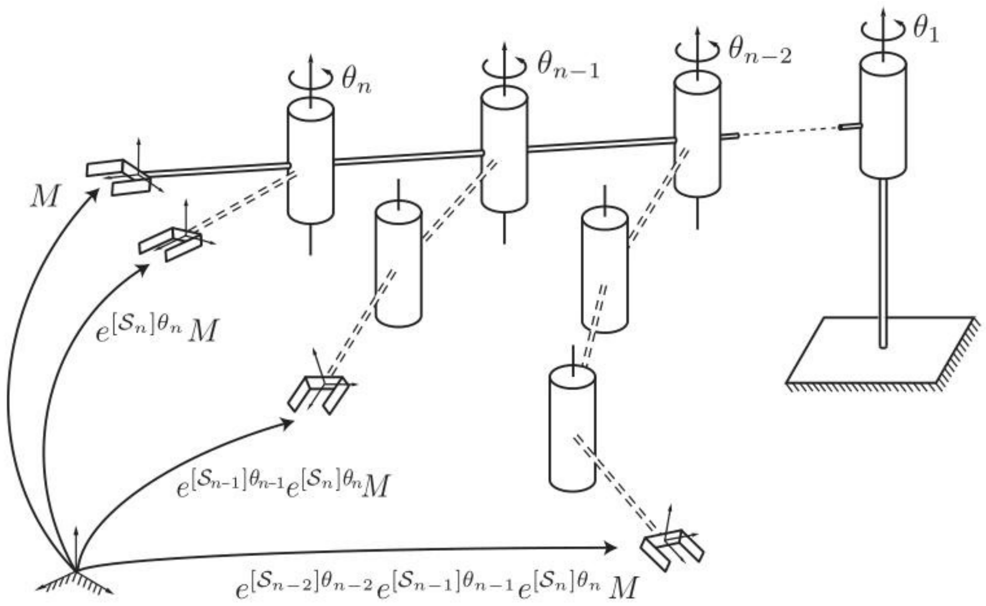


Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

- Initialization steps:
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  - Choose an end-effector (tool) frame $\{b\}$
  - Put all joints in *zero position*
  - Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position
- For each joint i, define the screw axis
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PoE in the End-Effector (Body) Frame

- Recall some identities:

$$e^{M^{-1}PM} = M^{-1}e^{P}M \rightarrow e^{P}M = Me^{M^{-1}PM}$$

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    - Screw axis in the body frame: $[\mathcal{B}_i]$

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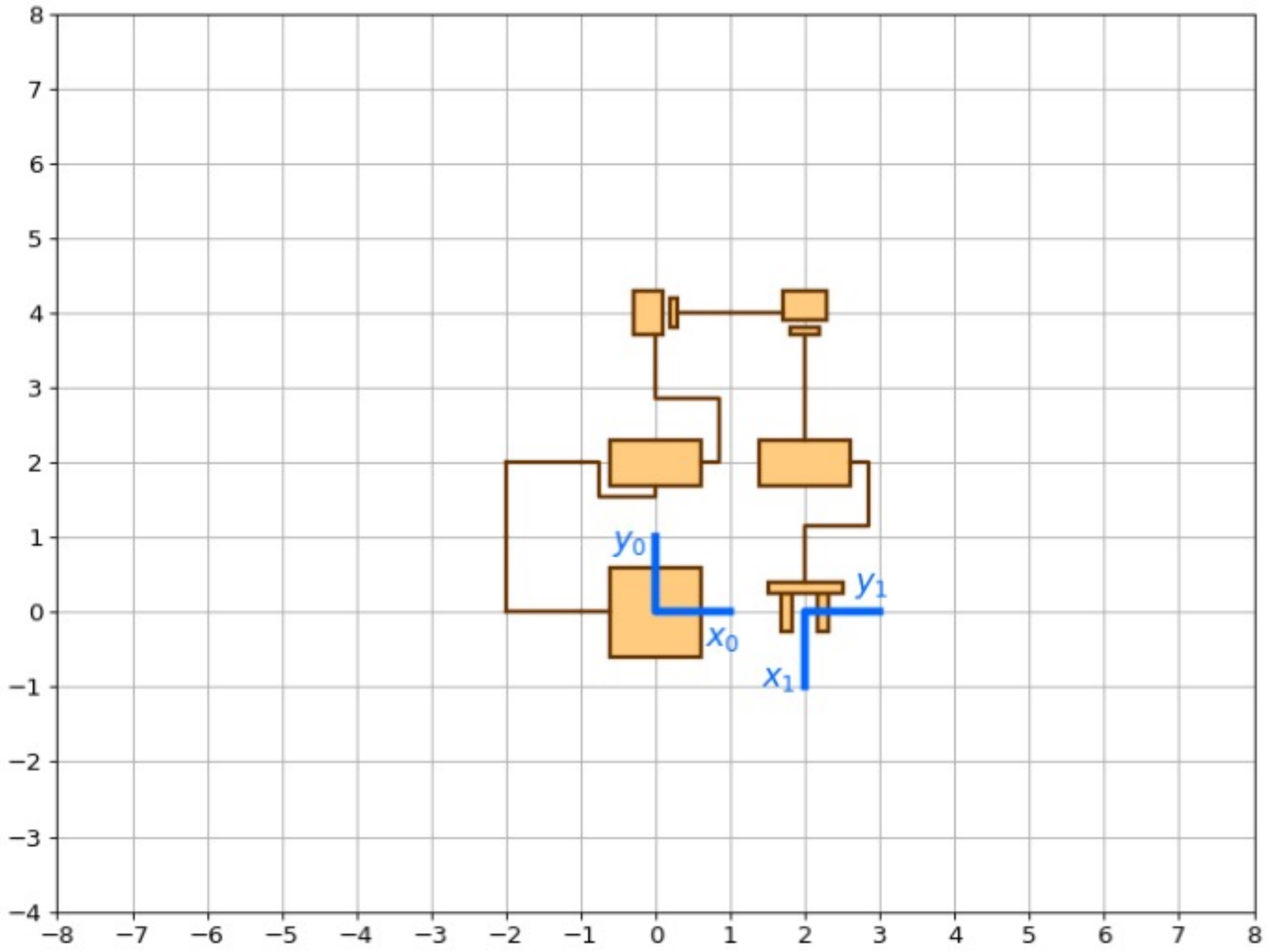
- What is $M^{-1}[\mathcal{S}_i]M$ (or $[Ad_{M^{-1}}]\mathcal{S}_i$)?
  - Screw axis in the body frame: $[\mathcal{B}_i]$
- Body form of PoE:

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1}\dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}e^{[\mathcal{B}_n]\theta_n}$$

Example 3: Body Form



Summary

- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications
- Our **fancy screw motion matrix exponential** is just another way to write down **homogenous transformations!**

Summary

- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications
- Our **fancy screw motion matrix exponential** is just another way to write down **homogenous transformations!**
- Our usual homogenous transformation matrices can also be used for forward kinematics (the Denavit-Hartenberg (DH) representation)
    - We define a frame for each link in the frame of the previous link. So to compute the position of the end effector, a frame for each link must be defined in terms of the previous link
    - **With screw motions, we have only two reference frames** (base and tool), and then each joint screw motion is defined in the base frame