Lecture 8:
Forward Kinematics II

Prof. DC

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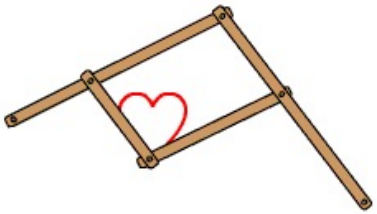
Modern Robotics Ch. 4



Denavit-Hartenberg Parameters

- In the 1950s, when Dick Hartenberg, a professor, and Jacques Denavit, a PhD student, developed a way to represent mathematically how mechanisms move
- They showed that the position of one link connected to another could be represented **minimally** using only four parameters
  - Known as the Denavit-Hartenberg (DH) parameters
- In 1981, Richard Paul demonstrated its value for the kinematic analysis of robotic systems

https://robotics.northwestern.edu/history.html

Screw Motions as Matrix Exponential

The screw axis $\mathcal{S}_i$ can be expressed in matrix form as:

$$[\mathcal{S}_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

To express a **screw motion** given a screw axis, we use the matrix exponential:

$$T = \underline{e}^{[\mathcal{S}]\theta} \in SE(3)$$

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)\, v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Product of Exponentials Approach

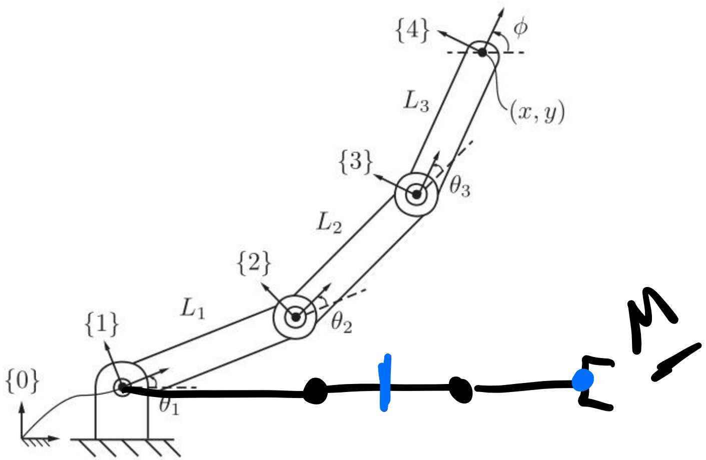


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

Let each joint i have a configuration defined by θ_i

Initialization steps:

- Choose a fixed frame $\{s\}$
- Choose an end-effector (tool) frame attached to the robot $\{b\}$
- Put all joints in *zero position*
- Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position

Product of Exponentials Approach

(Find)

- Given zero position M
- For each joint i, define the screw axis
- For each motion of a joint, define the screw motion $\rightarrow e^{[S]\theta}$
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = \underbrace{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M}$$

Finding $M = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

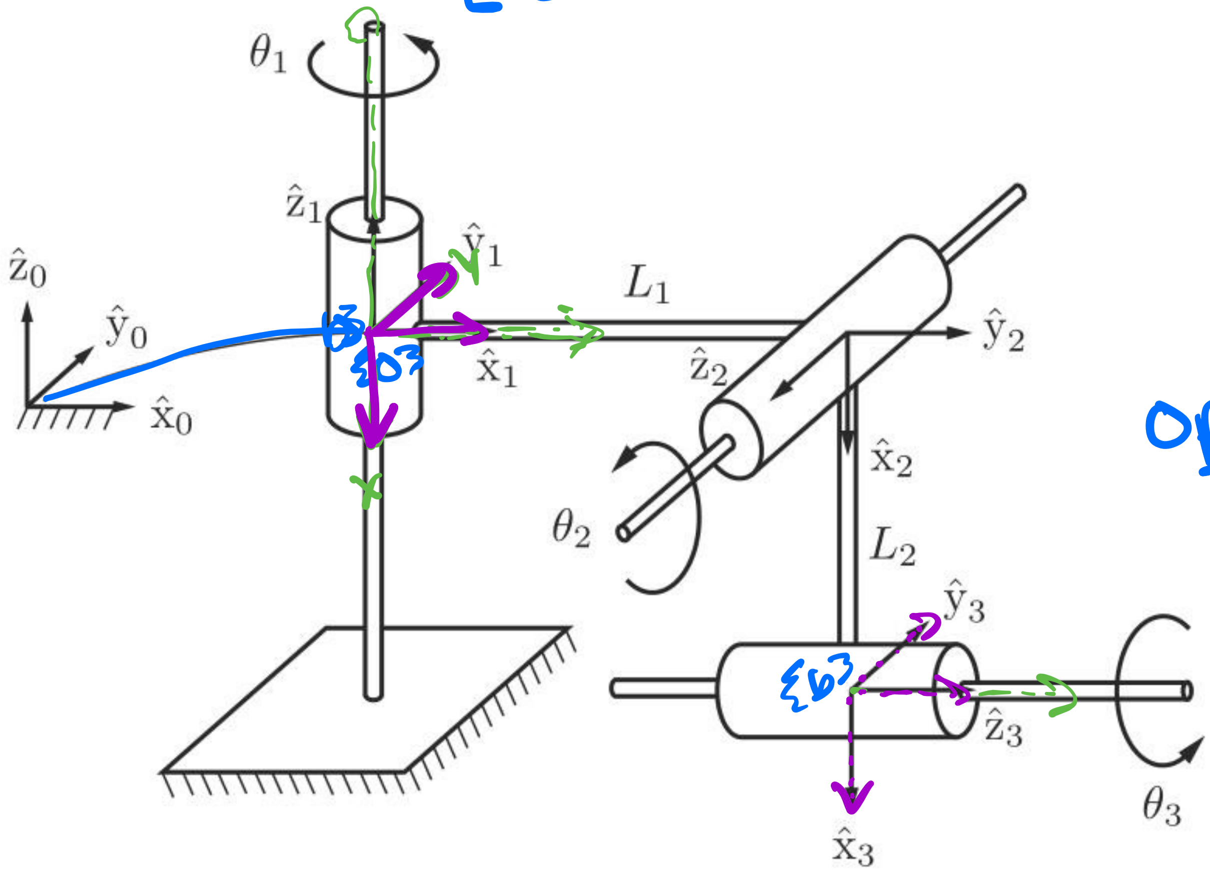


Figure 4.3: A 3R spatial open chain.

opt 1) identify axis of rotation(s)

$$R_y = \begin{bmatrix} c_\theta = 0 & 0 & +s_\theta = 1 \\ 0 & 1 & 0 \\ -s_\theta = -1 & 0 & c_\theta = 0 \end{bmatrix}$$

opt 2) by inspection

$$R = \begin{array}{c c} & \begin{array}{ccc} \hat{x}_3 & \hat{y}_3 & \hat{z}_3 \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{array}$$

$$p = [L_1, 0, -L_2]^T$$

How to read this diagram (1)

▨ denotes {s} {0}
π denotes our tool {b}

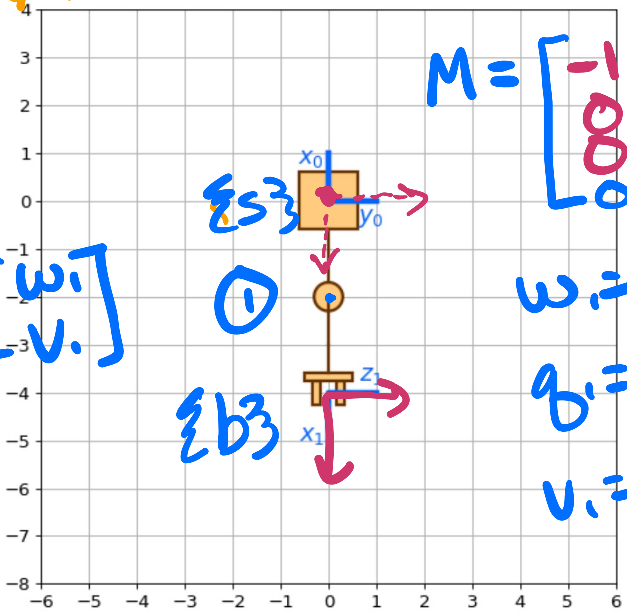
Revolute joint

top side

prismatic joint

side top

top: axis always of of pg



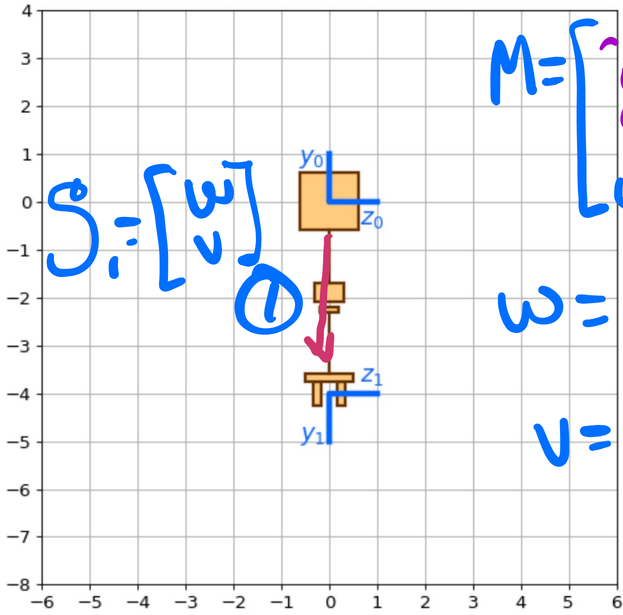
$$M=\begin{bmatrix}-1&0&0&-4\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$$S_1=\begin{bmatrix}\omega_1\\v_1\end{bmatrix}$$

$\omega_1=[0,0,-1]^T$

$q_1=[-2\ 0\ 0]^T$

$v_1=[0\ -2\ 0]^T$



$$S_1=\begin{bmatrix}\omega\\v\end{bmatrix}$$

$$M=\begin{bmatrix}-1&0&0&0\\0&-1&0&4\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

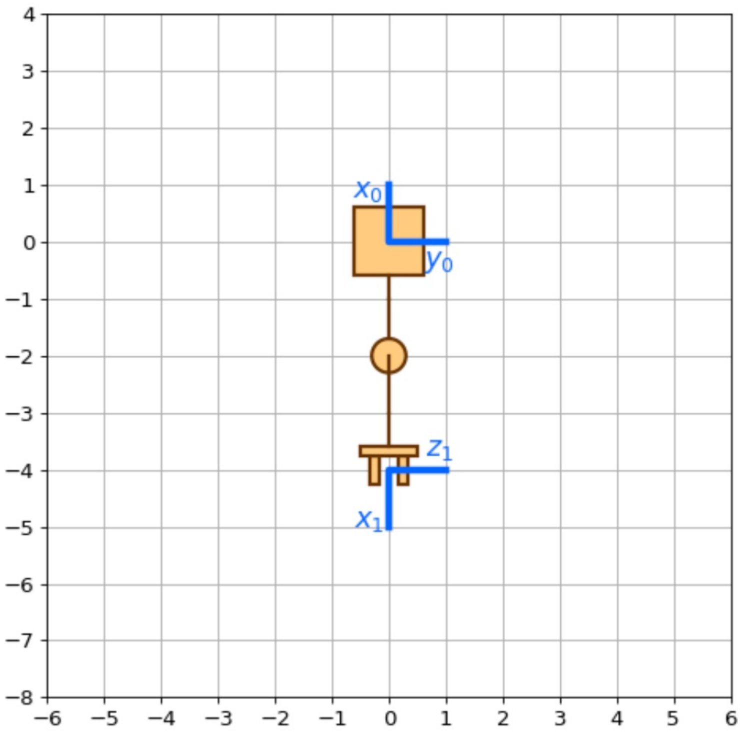
$\omega=0$

$$v=\begin{bmatrix}0\\-1\\0\end{bmatrix}$$

Modeling Robot Joints as Screw Motions

Case 1: Revolute Joint

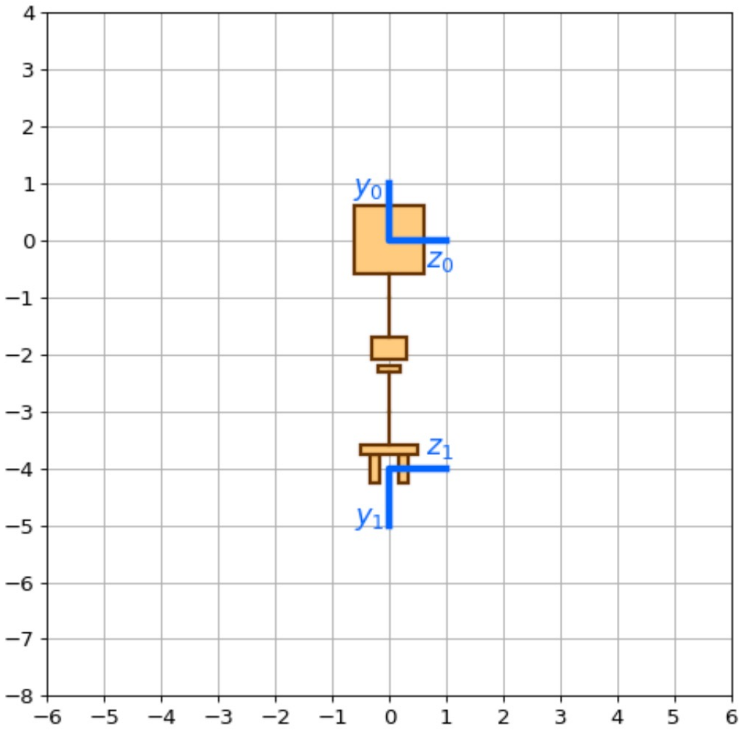
- $\|\omega\| = 1$
- $v = -\omega \times q + h\omega$
- $v = -\omega \times q$



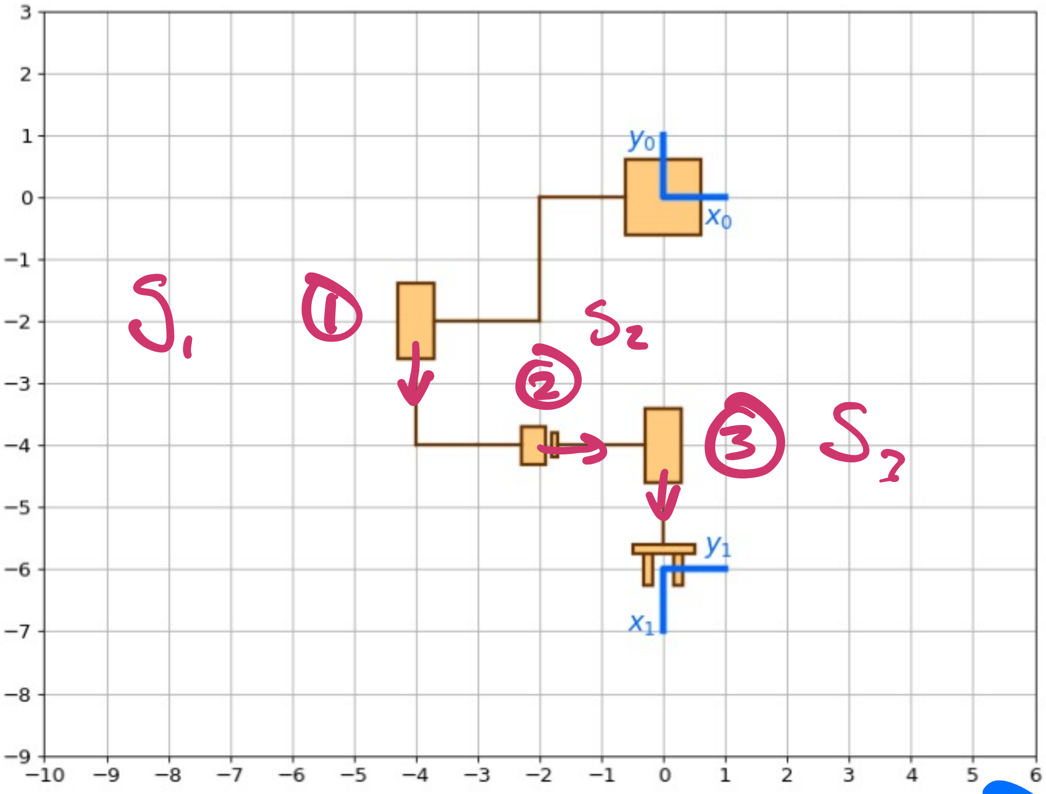
Modeling Robot Joints as Screw Motions

Case 2: Prismatic Joint

- $\|\omega\| = 0$
- $\|v\| = 1$
- Axis of movement defines v

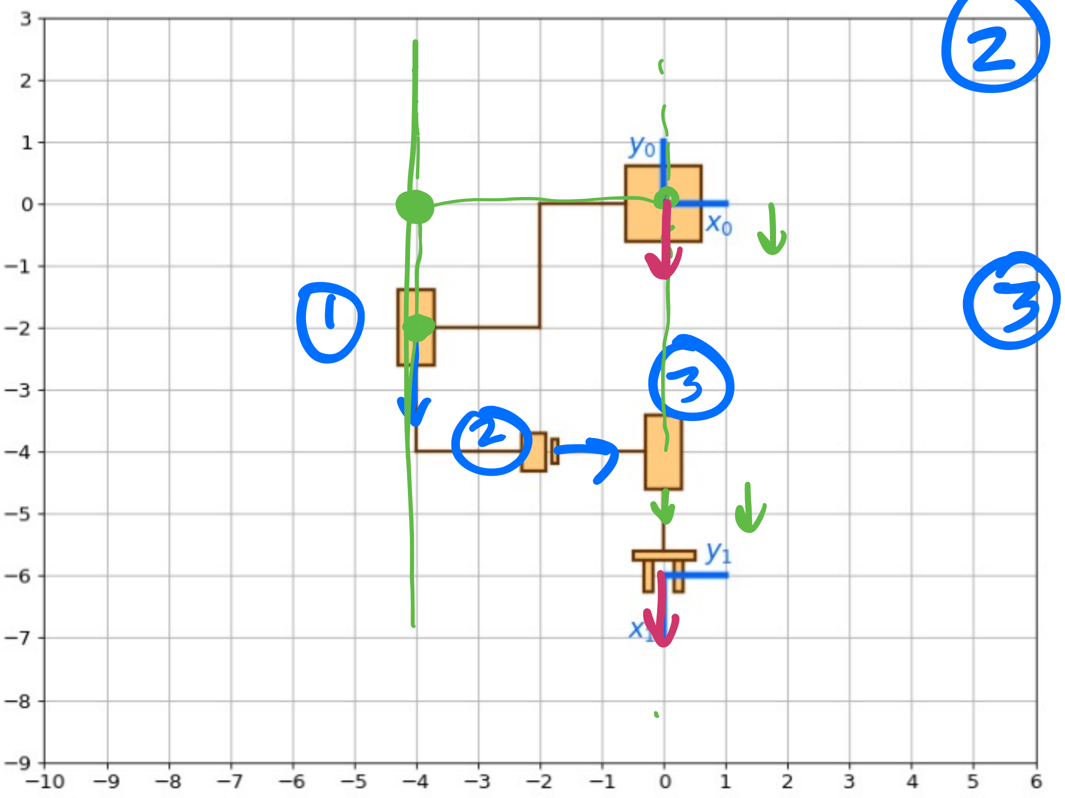


How to read this diagram (2)



Example 1

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



1. $w_1 = [0 \ -1 \ 0]^T$

   $q_1 = [-4 \ -2 \ 0]^T$

   $v = -w \times q$

2. $w_2 = 0$

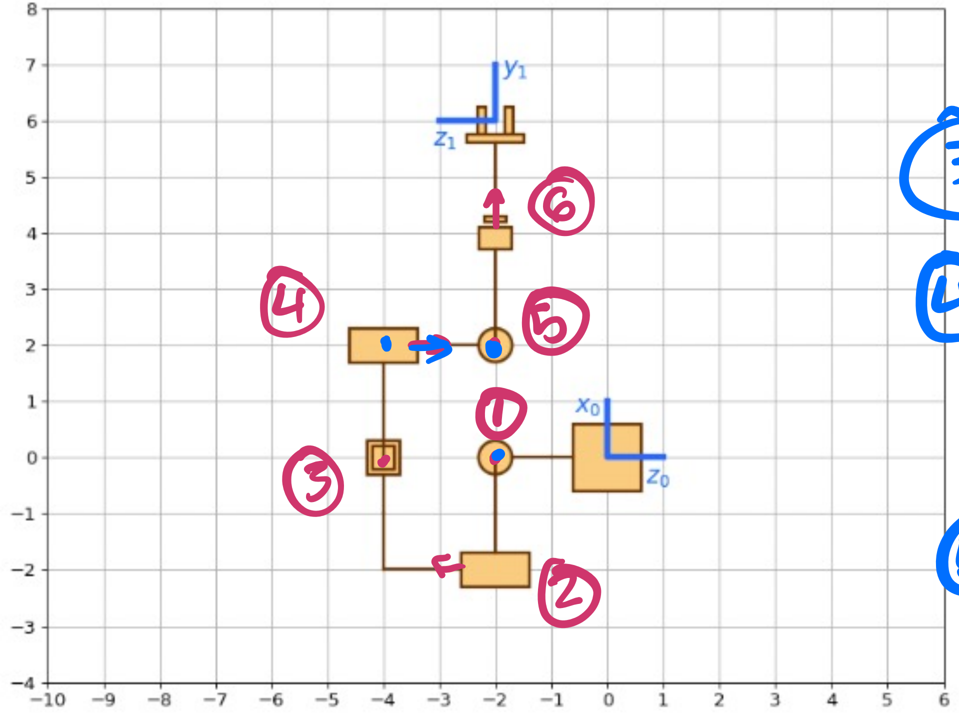
   $v_2 = [1 \ 0 \ 0]^T$

3. $w_3 = [0 \ -1 \ 0]^T$

   $q_3 = [0 \ -4 \ 0]^T$

Example 2

$$M = \begin{bmatrix} 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



1. $\omega_1 = [0 \;\; 1 \;\; 0]^T$

   $q_1 = [0 \;\; 0 \;\; -2]^T$, $v = [2 \;\; 0 \;\; 0]^T$

2. $\omega_2 = [0 \;\; 0 \;\; -1]^T$

   $q_2 = [-2 \;\; 0 \;\; -2]^T$

   $v_2 = [0 \;\; -2 \;\; 0]^T$

3. $\omega_3 = 0$, $v_3 = [0 \;\; 1 \;\; 0]^T$

4. $\omega_4 = [0 \;\; 0 \;\; 1]^T$

   $q_4 = [2 \;\; 0 \;\; -4]^T$

   $v_4 = [0 \;\; -2 \;\; 0]^T$

5. $\omega_5 = [0 \;\; 1 \;\; 0]^T$

   $q_5 = [2 \;\; 0 \;\; -2]^T$ $v_5 = [2 \;\; 0 \;\; 2]^T$

6. $\omega_6 = 0$

   $v = [1 \;\; 0 \;\; 0]^T$

So Far: Spatial Forward Kinematics

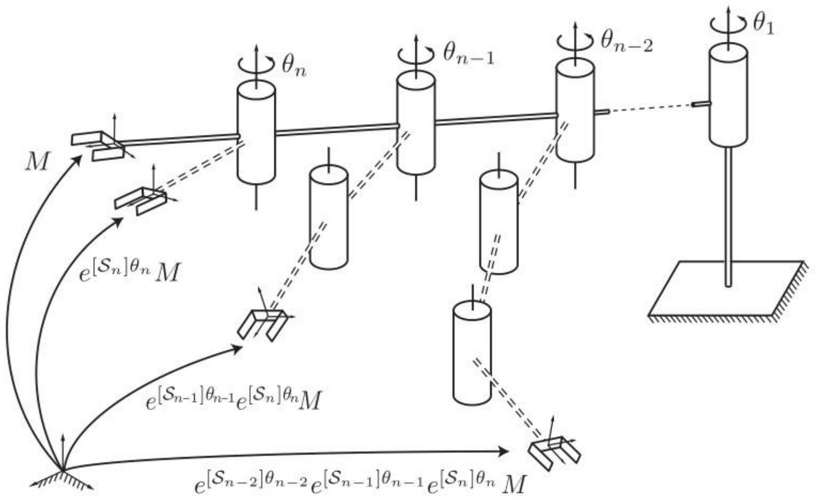


Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

- Initialization steps:
  - Choose a fixed frame $\{s\}$
  - Choose an end-effector (tool) frame $\{b\}$
  - Put all joints in *zero position*
  - Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position
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PoE in the End-Effector (Body) Frame

- Recall some identities:

$$e^{M^{-1}PM} = M^{-1}e^{P}M \rightarrow e^{P}M = Me^{M^{-1}PM}$$

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- Applying this to PoE:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}\dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}}e^{[\mathcal{S}_n]\theta_n}M$$

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1}\dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}}Me^{M^{-1}[\mathcal{S}_n]M\theta_n}$$

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- What is $M^{-1}[\mathcal{S}_i]M$ (or $[Ad_{M^{-1}}]\mathcal{S}_i$)?
  - Screw axis in the body frame: $[\mathcal{B}_i]$

PoE in the End-Effector (Body) Frame

- Recall some identities:

$$e^{M^{-1}PM} = M^{-1}e^{P}M \rightarrow e^{P}M = Me^{M^{-1}PM}$$

- Applying this to PoE:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}\dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}}e^{[\mathcal{S}_n]\theta_n}M$$
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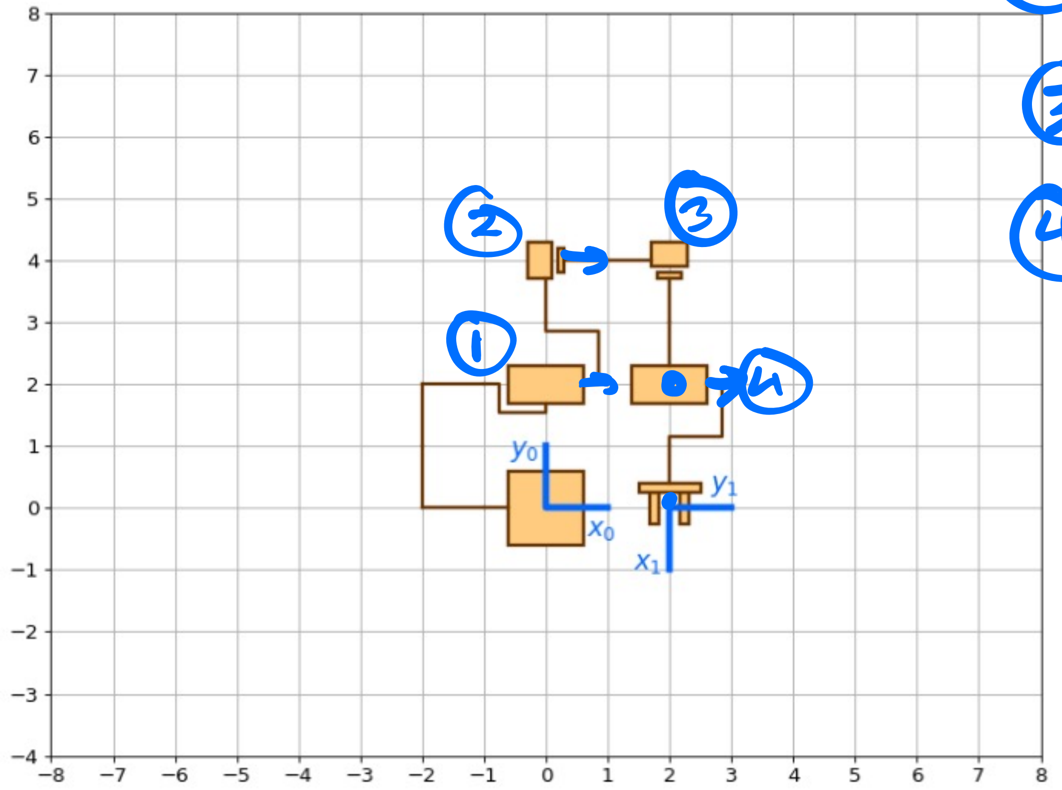
- What is $M^{-1}[\mathcal{S}_i]M$ (or $[Ad_{M^{-1}}]\mathcal{S}_i$)?
  - Screw axis in the body frame: $[\mathcal{B}_i]$
- Body form of PoE:

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1}\dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}e^{[\mathcal{B}_n]\theta_n}$$

Example 3: Body Form

$$M = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



① $\omega_1 = [0 \; 1 \; 0]^T$

$q_2 = [-2 \; -2 \; 0]^T$

② $\omega_2 = 0, \; v_2 = [0 \; 1 \; 0]^T$

③ $\omega_3 = 0, \; v_3 = [1 \; 0 \; 0]^T$

④ $\omega_4 = [0 \; 1 \; 0]^T$

$q_4 = [-2 \; 0 \; 0]^T$

$v_4 = [0 \; 0 \; 2]^T$

$$B_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix}$$

Summary

- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications
- Our **fancy screw motion matrix exponential** is just another way to write down **homogenous transformations!**

Summary

- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications
- Our **fancy screw motion matrix exponential** is just another way to write down **homogenous transformations!**
- Our usual homogenous transformation matrices can also be used for forward kinematics (the Denavit-Hartenberg (DH) representation)
  - We define a frame for each link in the frame of the previous link. So to compute the position of the end effector, a frame for each link must be defined in terms of the previous link
  - **With screw motions, we have only two reference frames** (base and tool), and then each joint screw motion is defined in the base frame