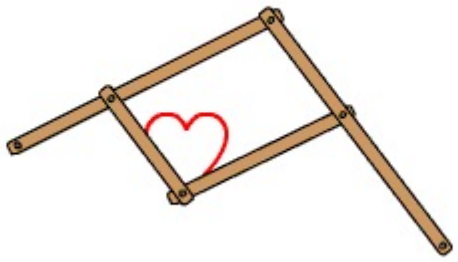
Lecture 7:
Forward Kinematics I

Prof. DC

Feb. 17, 2026

Modern Robotics Ch. 4



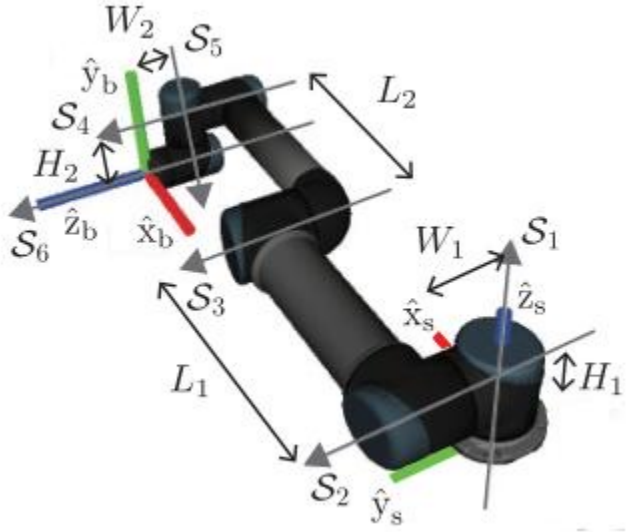
Exam 1 Stats

- Mean Score: 85%
- Maximum Score: 100%
- Mean Time: 38 minutes

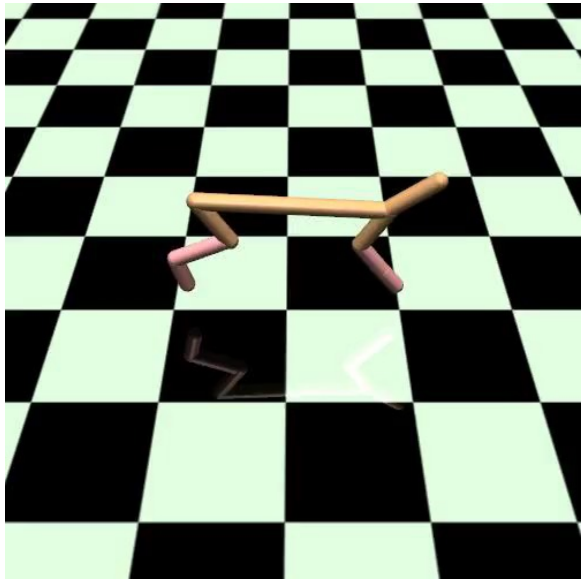
- We'll be adding one point to everyone's grade (not reflected above)

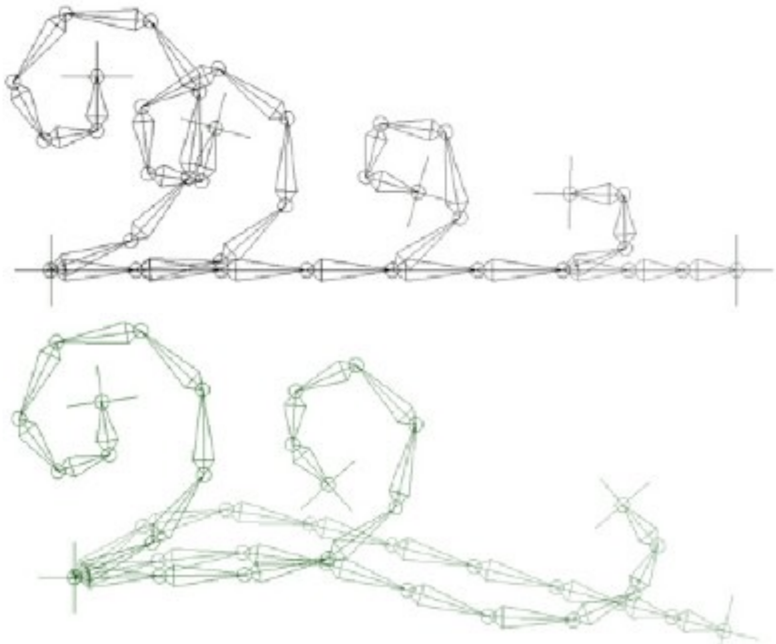
What is Forward Kinematics?

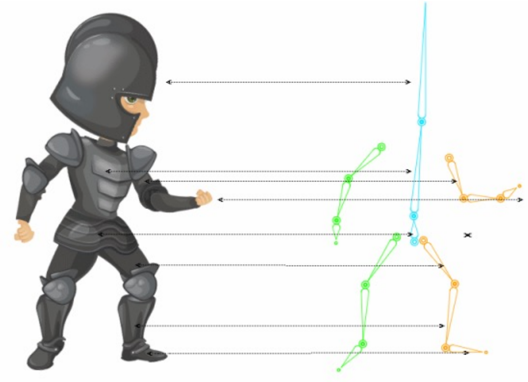
- **Kinematics:** a branch of classical mechanics that describes motion of bodies without considering forces. AKA “the geometry of motion”
- **Forward kinematics:** a core problem in robotics. Given the individual state of each joint of the robot (in a local frame), what is the position of a given point on the robot in the global frame?

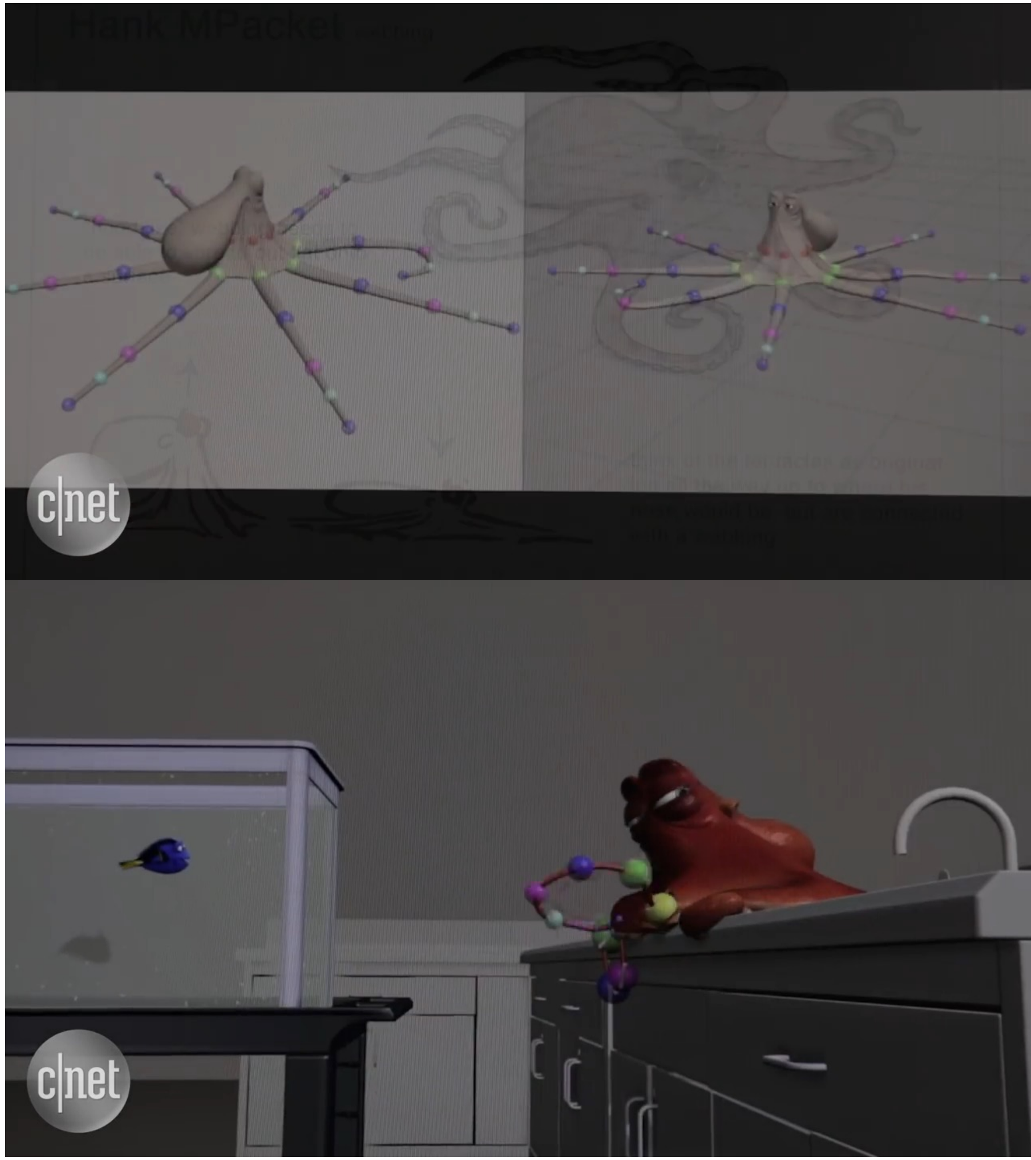


Where is forward kinematics used?

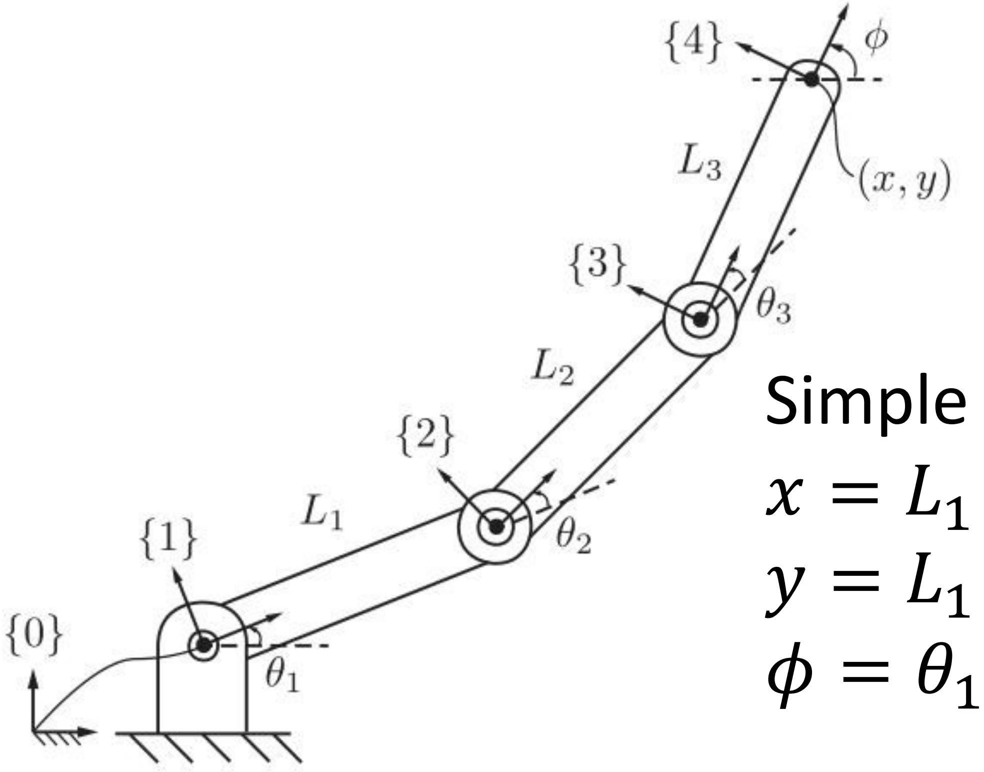








Forward Kinematics of a Simple Chain



Simple Geometry!

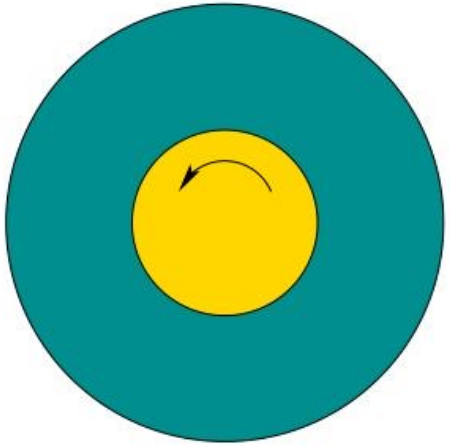
$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

Can we use transformations instead?

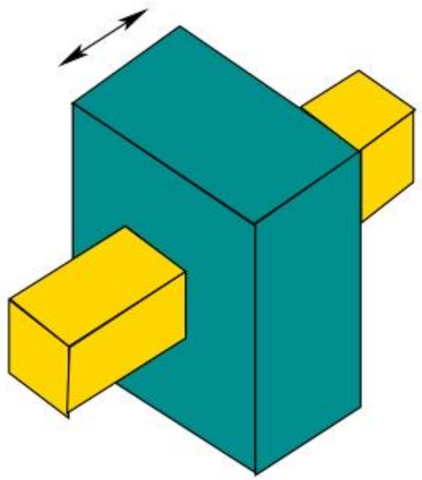
$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$
$$T_{sb} = T_{s1}T_{12} \cdots T_{n-1,n}M$$

Assumptions

- Our robot is a kinematic chain, made of rigid links and movable joints
- No branches or loops
- All joints have one degree of freedom and are revolute or prismatic

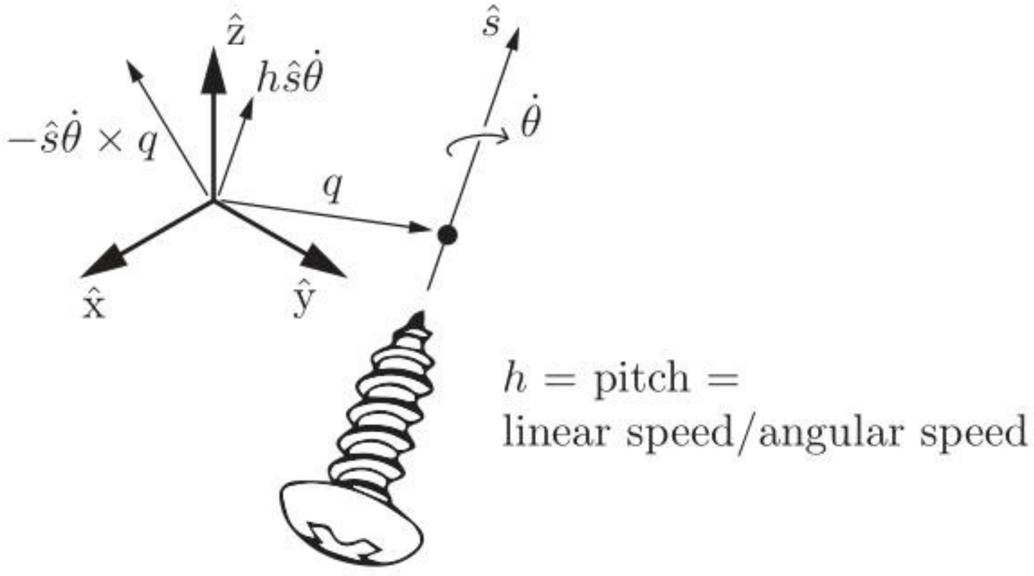


Revolute
1 Degree of Freedom



Prismatic
1 Degree of Freedom

Review of Screw Motions



We derived a representation for the screw axis:

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

where either

- $\|\omega\| = 1$
  - where $v = -\omega \times q + h\omega$, where q is the point on the axis of the screw and h is the pitch of the screw
- $\|\omega\| = 0$ and $\|v\| = 1$

Screw Motions as Matrix Exponential

The screw axis $\mathcal{S}_i$ can be expressed in matrix form as:

$$[\mathcal{S}_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

To express a **screw motion** given a screw axis, we use the matrix exponential:

$$e^{[\mathcal{S}]\theta} \in SE(3)$$

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

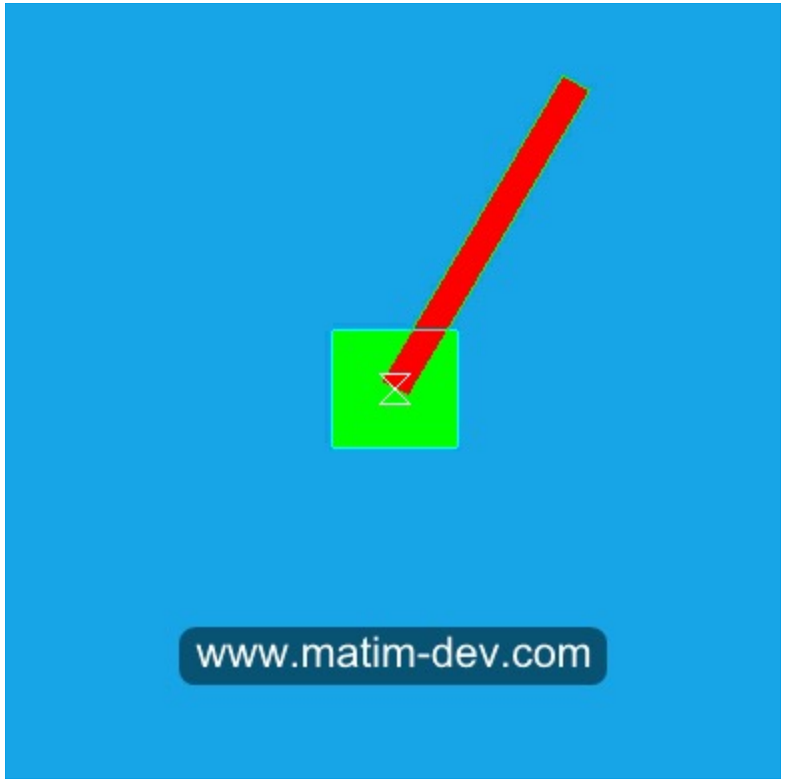
If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Modeling Robot Joints as Screw Motions

Case 1: Revolute Joint

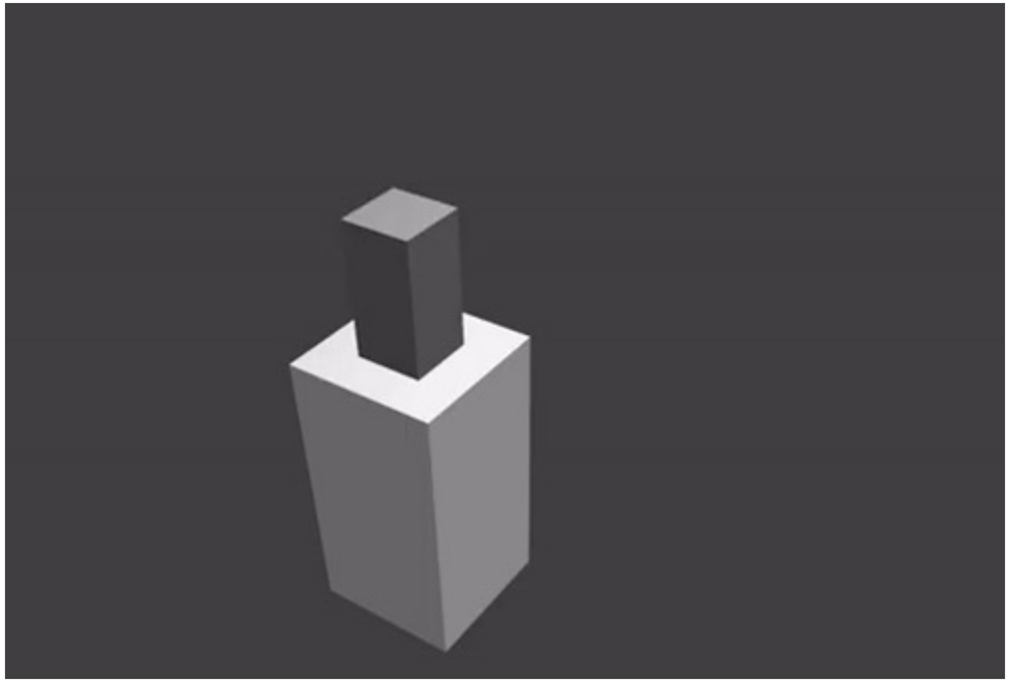
- $\|\omega\| = 1$
- $v = -\omega \times q + h\omega$
- $v = -\omega \times q$



Modeling Robot Joints as Screw Motions

Case 2: Prismatic Joint

- $\|\omega\| = 0$
- $\|v\| = 1$
- Axis of movement defines v



Product of Exponentials Approach

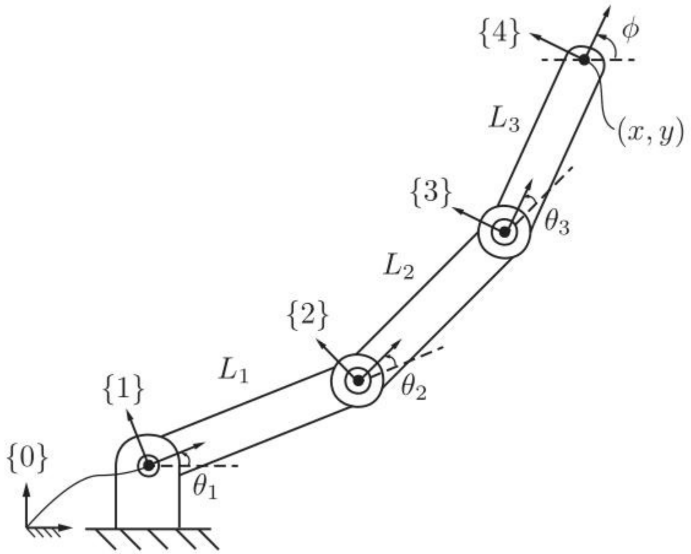


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the x̂- and ŷ-axis is shown; the ẑ-axes are parallel and out of the page.

Let each joint i have a configuration defined by θ_i

Initialization steps:

- Choose a fixed frame $\{s\}$
- Choose an end-effector (tool) frame attached to the robot $\{b\}$
- Put all joints in *zero position*
- Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position

Product of Exponentials Approach

- Given zero position M
- For each joint i, define the screw axis $\mathcal{S}$
- For each screw axis and joint movement θ_i, define the screw motion $e^{[\mathcal{S}_i]\theta_i}$
- How does the motion of each joint affect the tool?

  → these computed screw motions!

- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Visualizing the Product of Exponentials

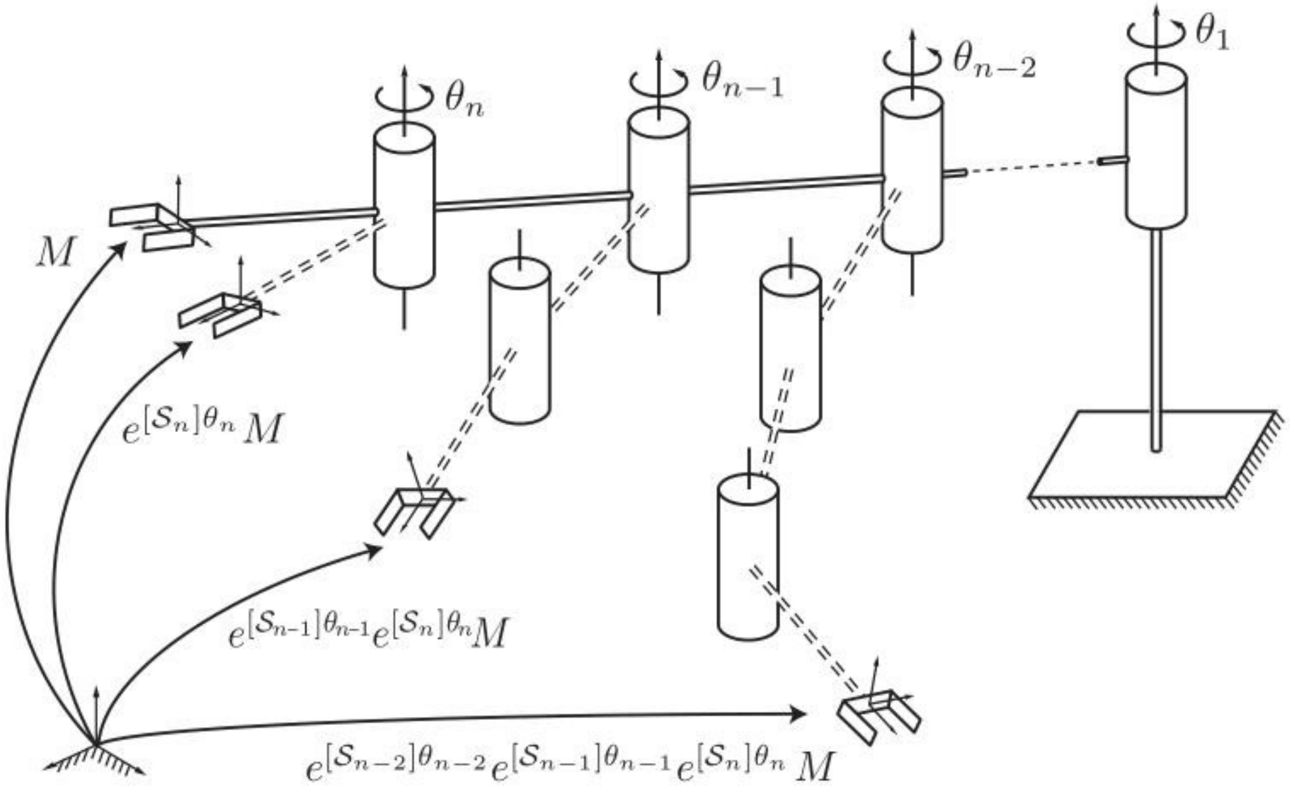
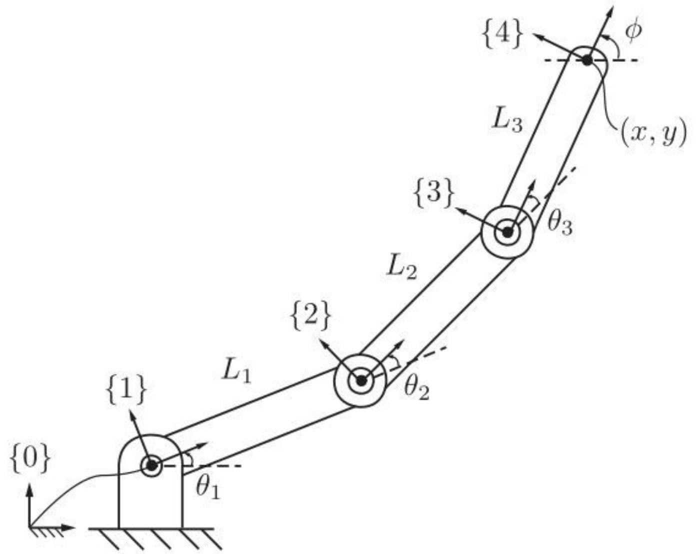


Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

Example 1



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Example 1: PoE

Compute $e^{[\mathcal{S}_i]\theta_i}$ for each joint:

$$e^{[\mathcal{S}_i]\theta_i} = \begin{bmatrix} e^{[\omega_i]\theta_i} & (I\theta_i + (1 - \cos\theta_i\,)[\omega_i] + (\theta_i - \sin\theta_i)[\omega_i]^2)v_i \\ 0 & 1 \end{bmatrix}$$

and compose with M:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Example 2

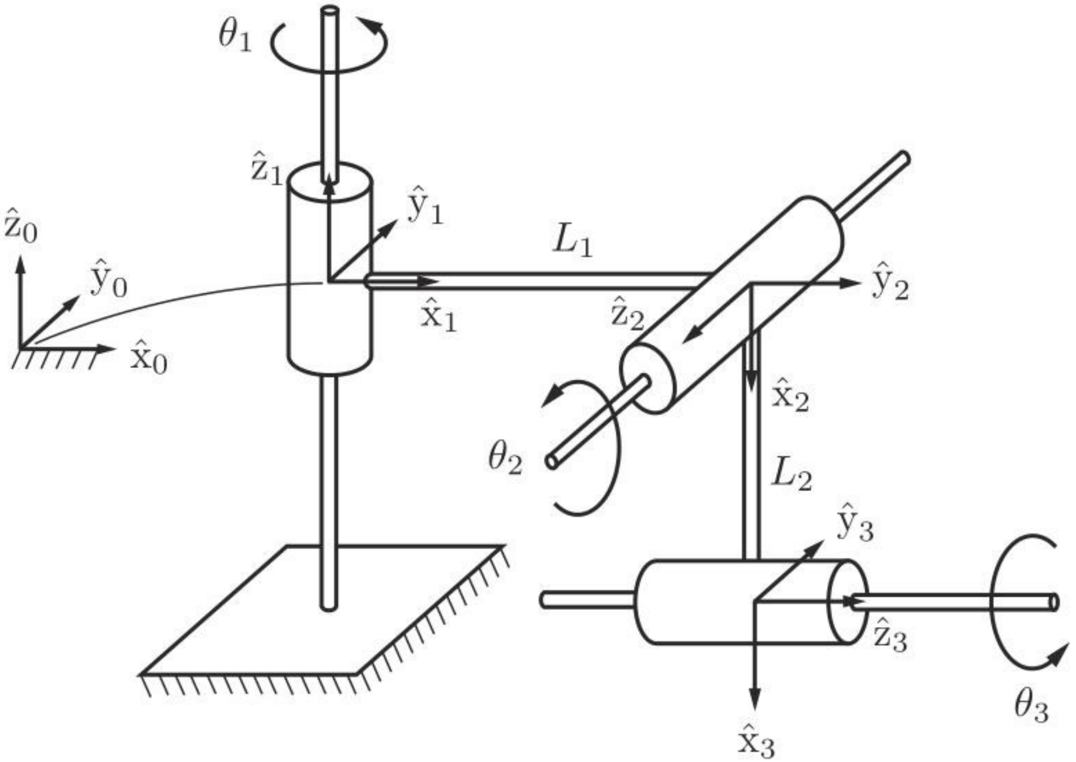
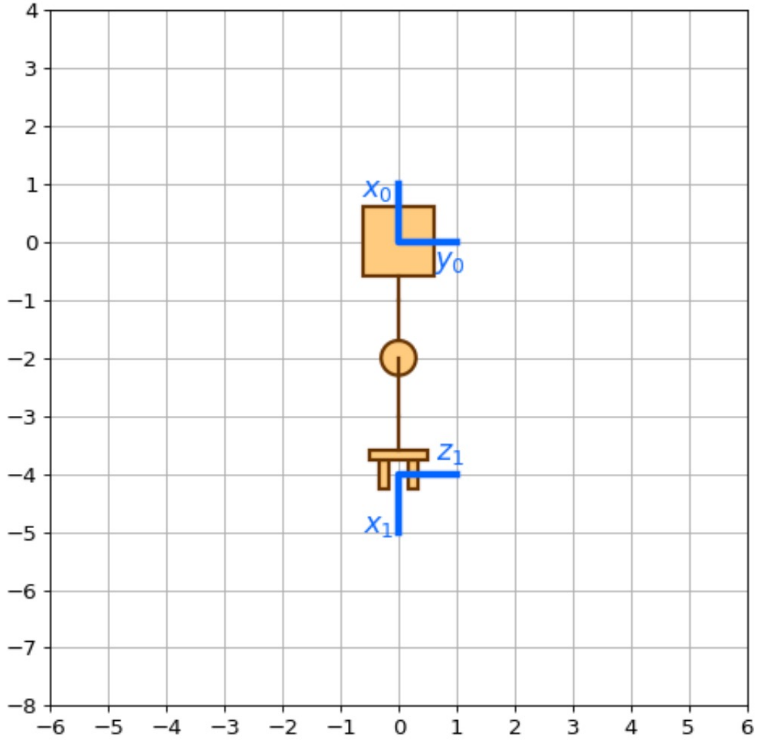
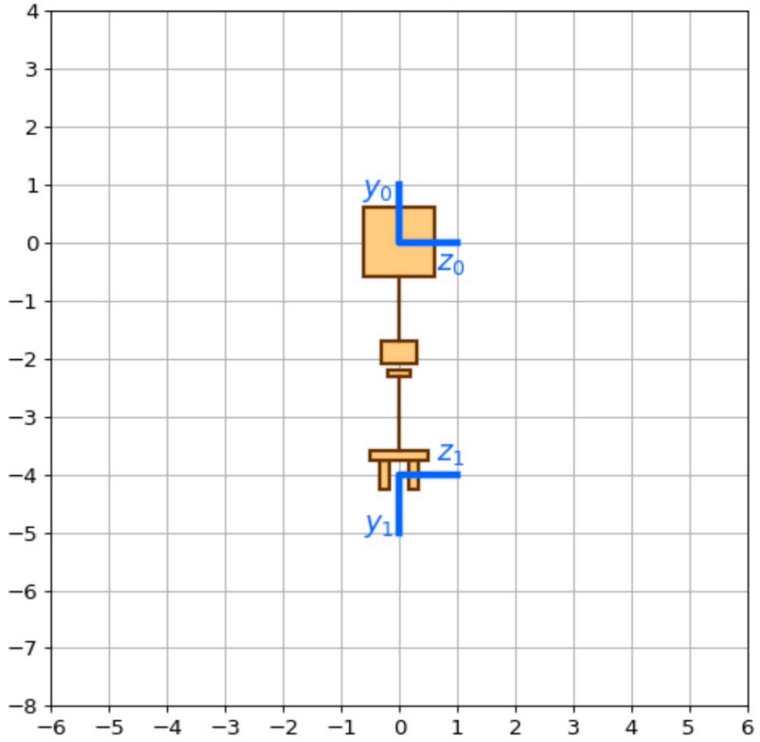


Figure 4.3: A 3R spatial open chain.

How to read this diagram (1)





Modeling Robot Joints as Screw Motions

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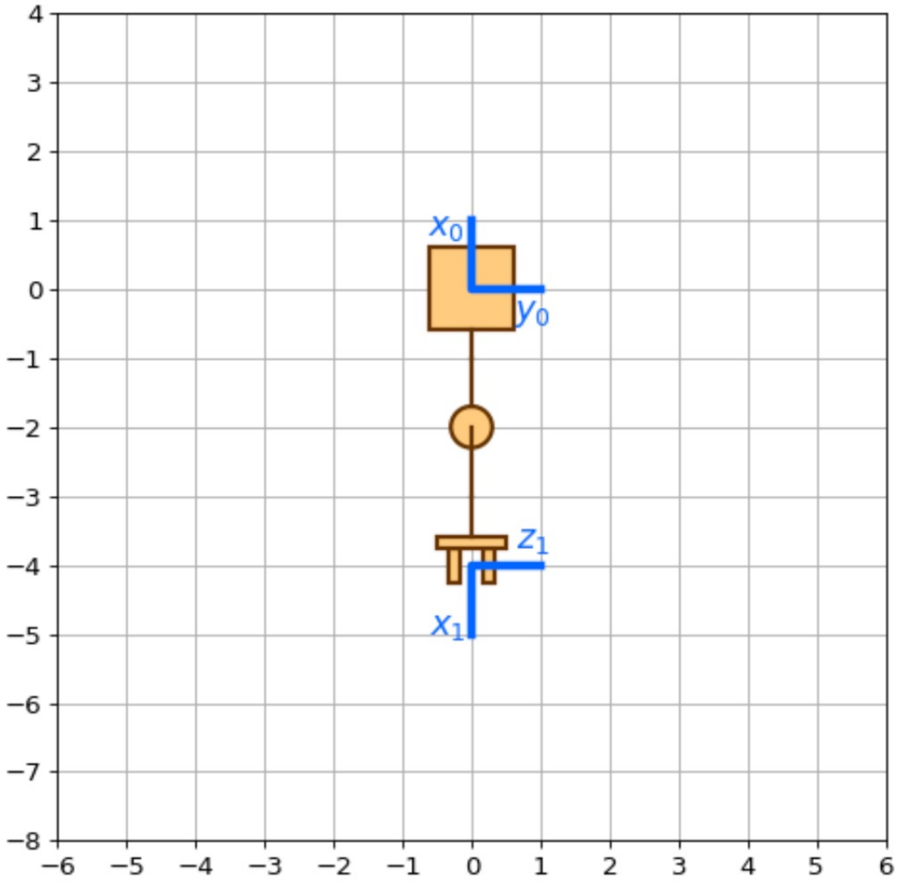
- $v = -\omega \times q$



Modeling Robot Joints as Screw Motions

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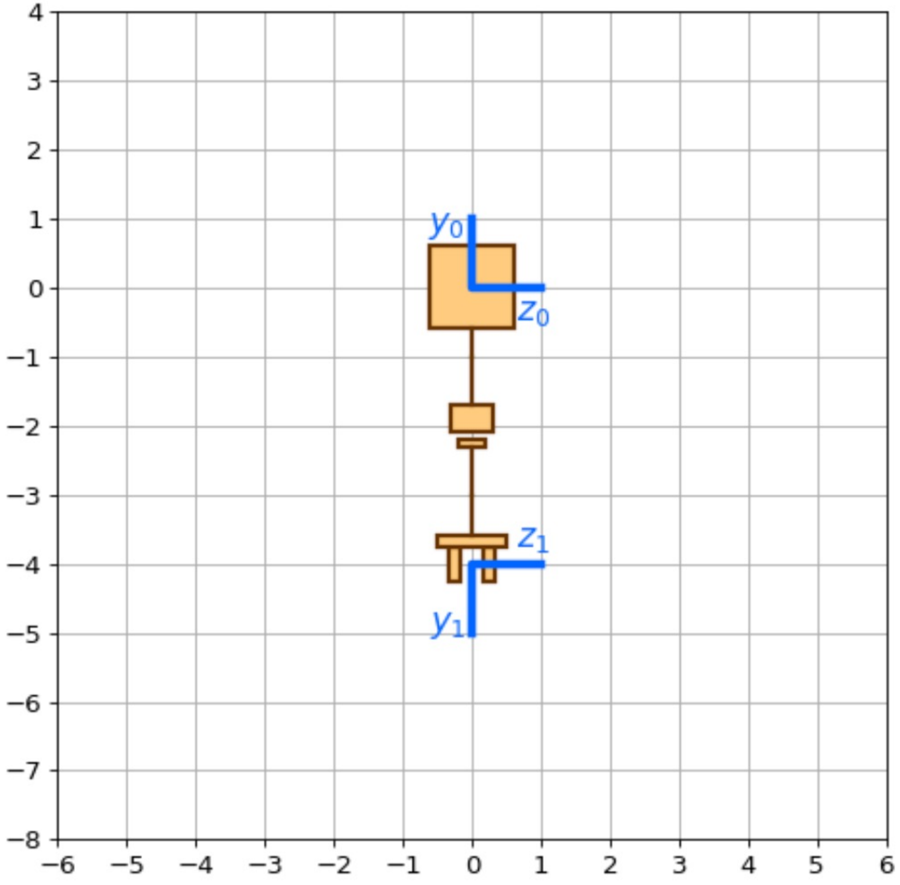
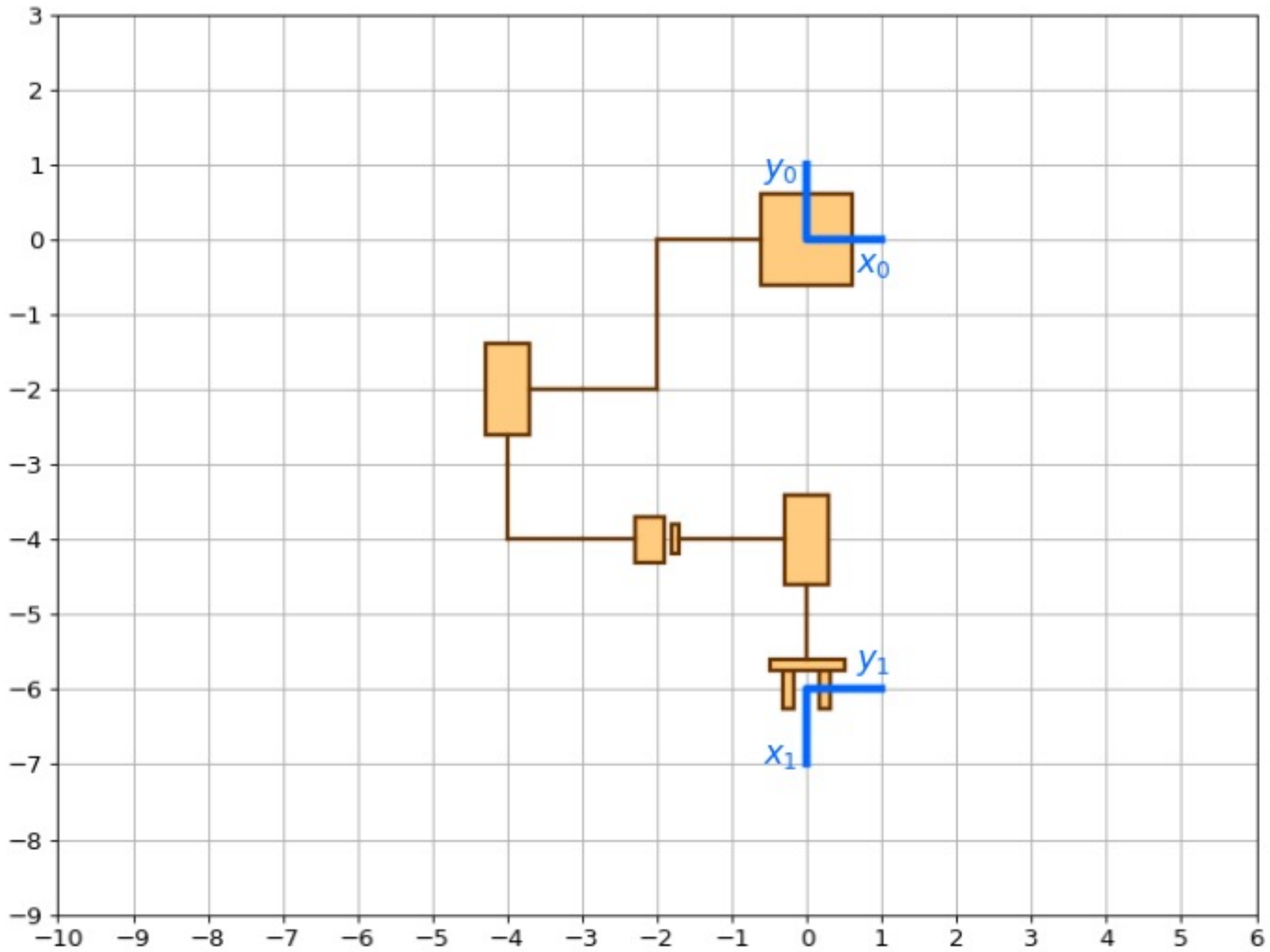
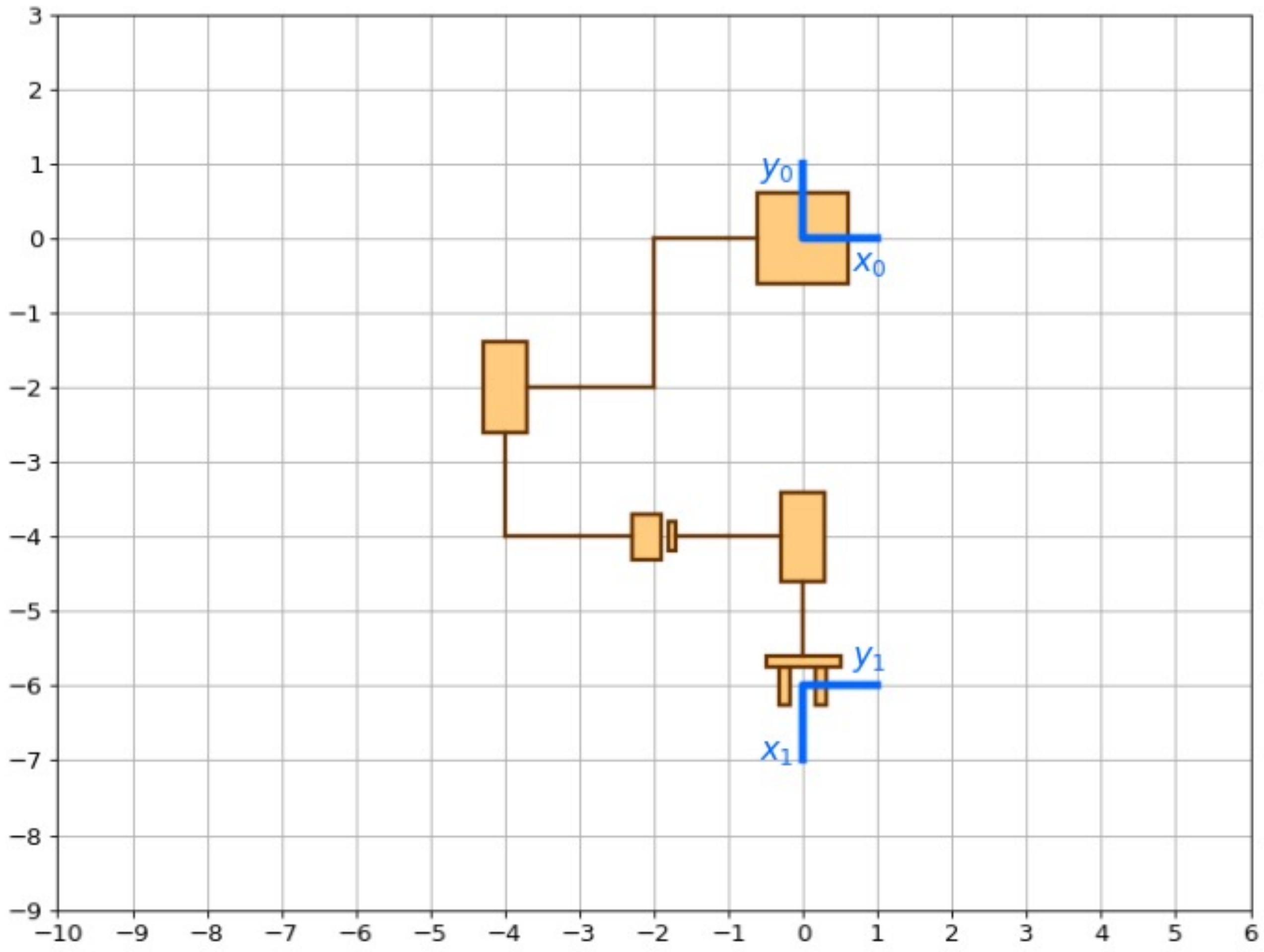


How to read this diagram (2)



Example 3



PoE in the End-Effector (Body) Frame

- Recall some identities:

$$e^{M^{-1}PM} = M^{-1}e^{P}M \rightarrow e^{P}M = Me^{M^{-1}PM}$$

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- Applying this to PoE:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}}e^{[\mathcal{S}_n]\theta_n}M$$

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- What is $M^{-1}[\mathcal{S}_i]M$ (or $[Ad_{M^{-1}}]\mathcal{S}_i$)?
  - Screw axis in the body frame: $[\mathcal{B}_i]$

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- Applying this to PoE:

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- What is $M^{-1}[\mathcal{S}_i]M$ (or $[Ad_{M^{-1}}]\mathcal{S}_i$)?
  - Screw axis in the body frame: $[\mathcal{B}_i]$
- Body form of PoE:

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}e^{[\mathcal{B}_n]\theta_n}$$

Summary

- The **screw axis $\mathcal{S}$** was defined and used to define the **exponential coordinates of homogeneous transformations**
- These transformations capture **screw motions** that correspond to the movements induced by joints
- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications