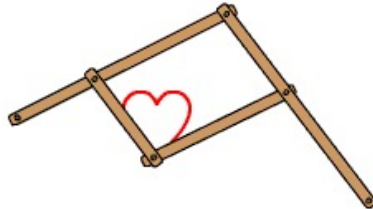


Lecture 7: Forward Kinematics I

Prof. DC

Feb. 17, 2026

Modern Robotics Ch. 4

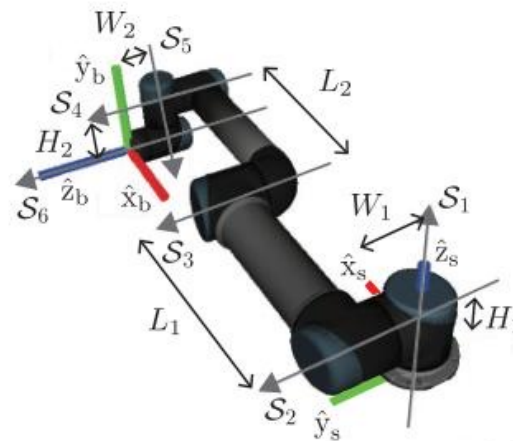


Exam 1 Stats

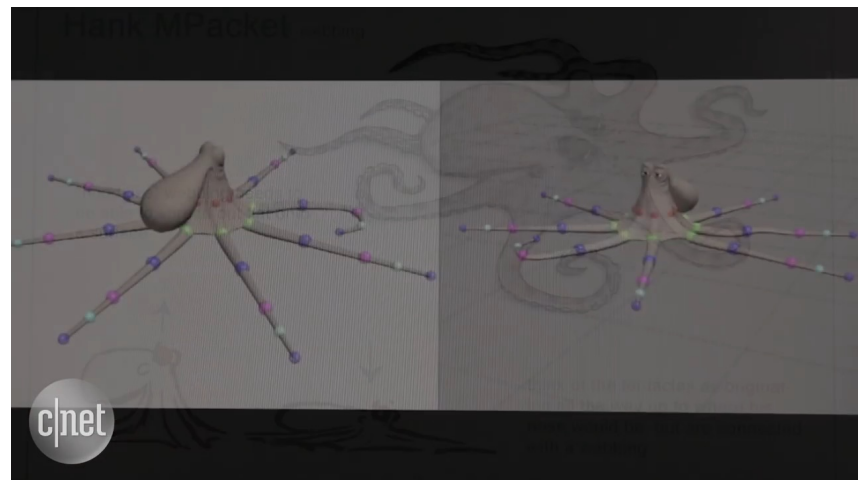
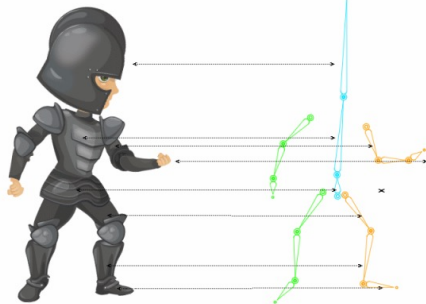
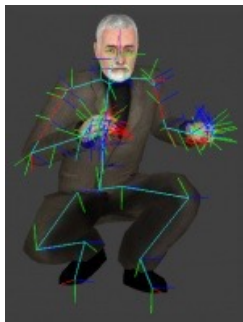
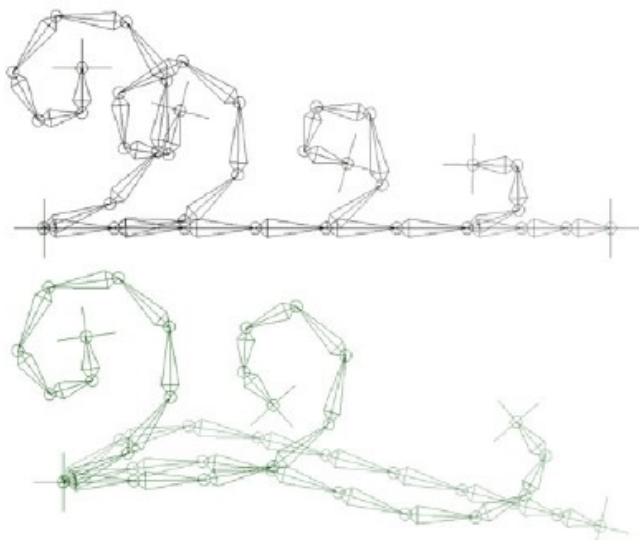
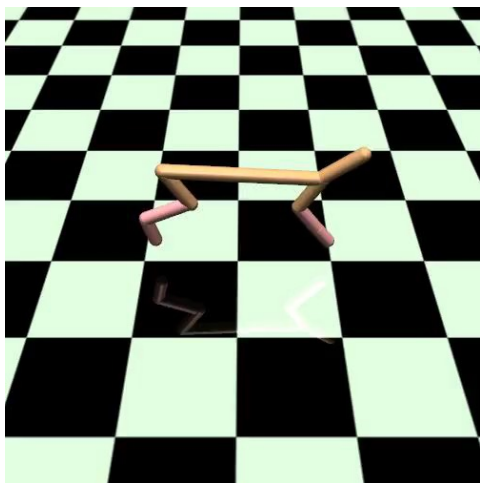
- Mean Score: 85%
 - Maximum Score: 100%
 - Mean Time: 38 minutes
-
- We'll be adding one point to everyone's grade (not reflected above)

What is Forward Kinematics?

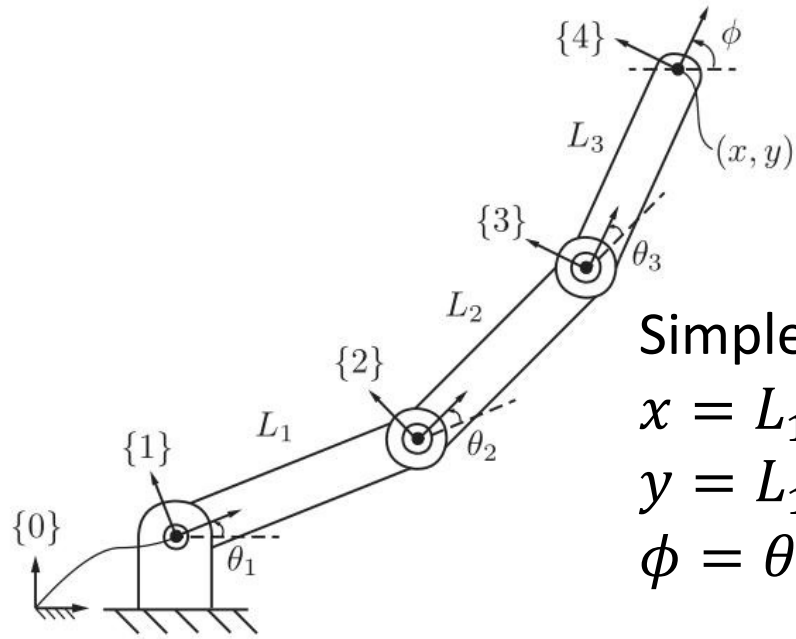
- **Kinematics:** a branch of classical mechanics that describes motion of bodies without considering forces. AKA “the geometry of motion”
- **Forward kinematics:** a core problem in robotics. Given the individual state of each joint of the robot (in a local frame), what is the position of a given point on the robot in the global frame?



Where is forward kinematics used?



Forward Kinematics of a Simple Chain



Simple Geometry!

$$x = L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$

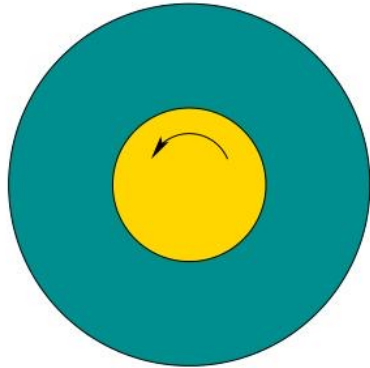
Can we use transformations instead?

$$T_{04} = T_{01} T_{12} T_{23} T_{34}$$

$$T_{sb} = T_{s1} T_{12} \cdots T_{n-1,n} M$$

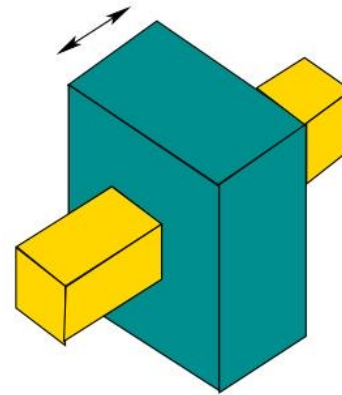
Assumptions

- Our robot is a kinematic chain, made of rigid links and movable joints
- No branches or loops
- All joints have one degree of freedom and are revolute or prismatic



Revolute

1 Degree of Freedom



Prismatic

1 Degree of Freedom

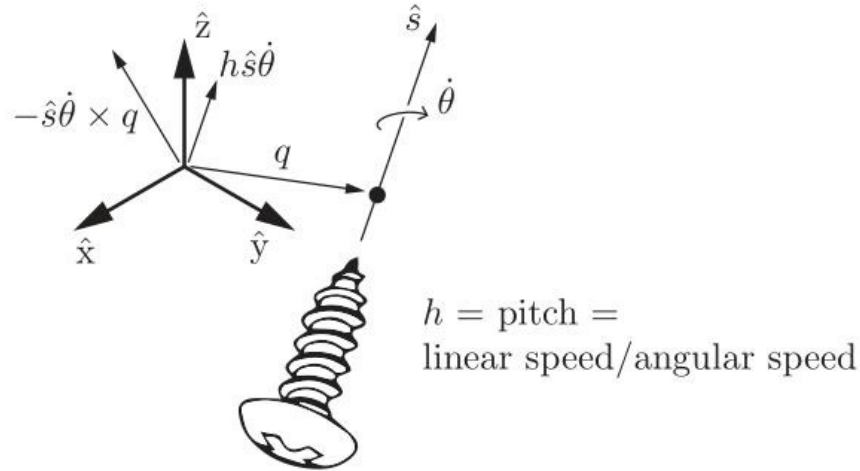
Review of Screw Motions

We derived a representation for the screw axis:

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

where either

- $\|\omega\| = 1$
 - where $v = -\omega \times q + h\omega$, where q is the point on the axis of the screw and h is the pitch of the screw
- $\|\omega\| = 0$ and $\|v\| = 1$



Screw Motions as Matrix Exponential

The screw axis $\mathcal{S}_i$ can be expressed in matrix form as:

$$[\mathcal{S}_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

To express a **screw motion** given a screw axis, we use the matrix exponential:

$$e^{[\mathcal{S}]\theta} \in SE(3)$$

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1 - \cos \theta)[\omega] + (\theta - \sin \theta)[\omega]^2) v \\ 0 & 1 \end{bmatrix}. \quad (3.88)$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Modeling Robot Joints as Screw Motions

Case 1: Revolute Joint

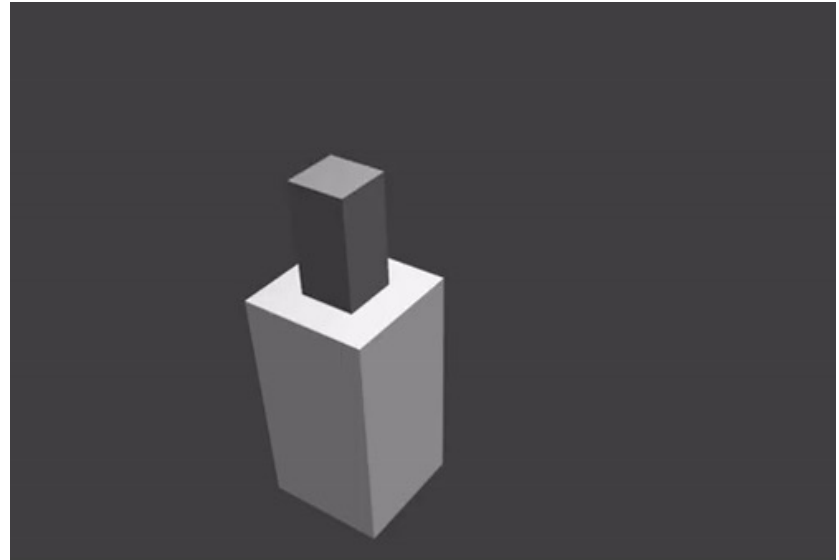
- $\|\omega\| = 1$
- $v = -\omega \times q + h\omega$
- $v = -\omega \times q$



Modeling Robot Joints as Screw Motions

Case 2: Prismatic Joint

- $\|\omega\| = 0$
- $\|v\| = 1$
- Axis of movement defines v



Product of Exponentials Approach

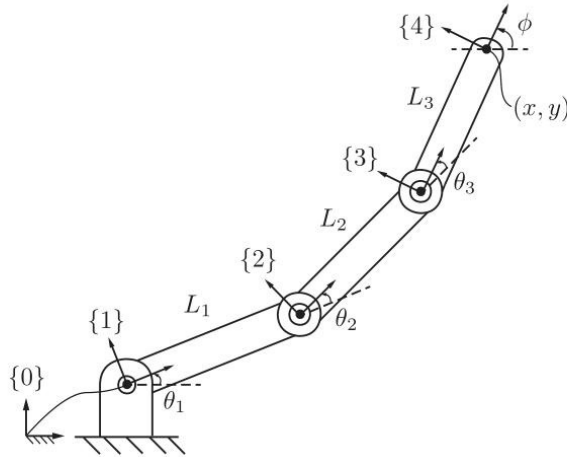


Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$ - and $\hat{y}$ -axis is shown; the $\hat{z}$ -axes are parallel and out of the page.

Let each joint i have a configuration defined by θ_i

Initialization steps:

- Choose a fixed frame $\{s\}$
- Choose an end-effector (tool) frame attached to the robot $\{b\}$
- Put all joints in *zero position*
- Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when the robot is in the zero position

Product of Exponentials Approach

- Given zero position M
- For each joint i , define the screw axis $\mathcal{S}$
- For each screw axis and joint movement θ_i , define the screw motion $e^{[\mathcal{S}_i]\theta_i}$
- How does the motion of each joint affect the tool?
→ these computed screw motions!
- These operations compose nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

Visualizing the Product of Exponentials

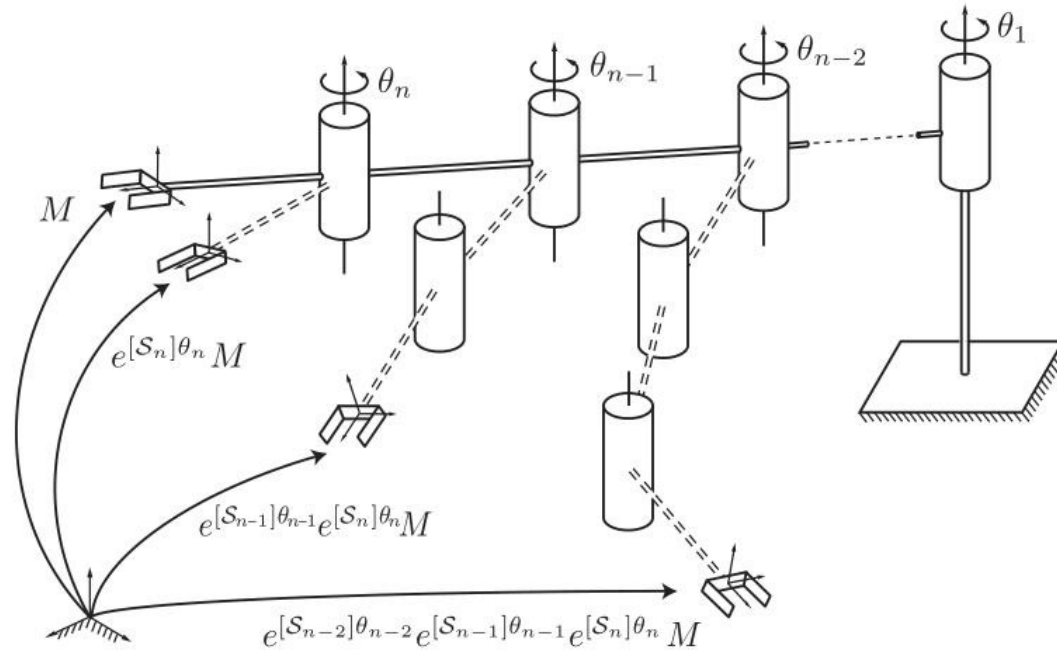


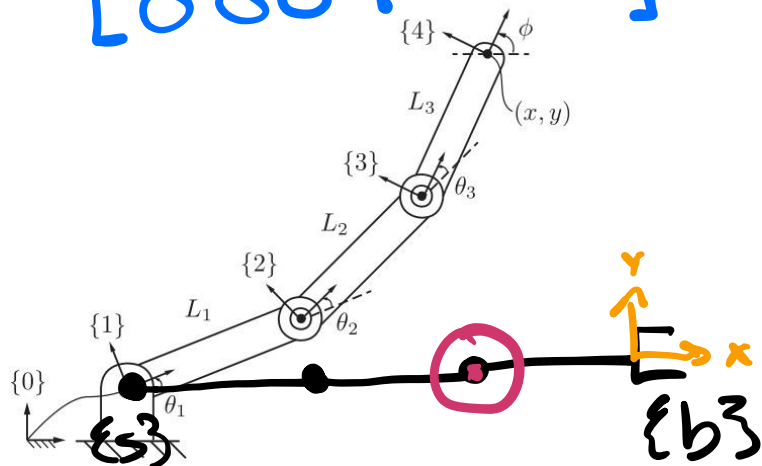
Figure 4.2: Illustration of the PoE formula for an n -link spatial open chain.

Example 1

① Find M

$$M = \begin{bmatrix} R_0 & P_0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} & p \\ 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



② find w_i

$$w_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \forall i$$

③ compute v_i

$$v_3 = -w_3 \times q_3, q_3 = \begin{bmatrix} L_1 + L_2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -(L_1 + L_2) \end{bmatrix}$$

$$g_3 = \begin{bmatrix} w_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -(L_1 + L_2) \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$ - and $\hat{y}$ -axis is shown; the $\hat{z}$ -axes are parallel and out of the page.

Example 1: PoE

Compute $e^{[s_i]\theta_i}$ for each joint:

$$e^{[s_i]\theta_i} = \begin{bmatrix} e^{[\omega_i]\theta_i} & (I\theta_i + (1 - \cos \theta_i)[\omega_i] + (\theta_i - \sin \theta_i)[\omega_i]^2)v_i \\ 0 & 1 \end{bmatrix}$$

and compose with M :

$$T(\theta) = e^{[s_1]\theta_1} e^{[s_2]\theta_2} e^{[s_3]\theta_3} M$$

Example 2

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

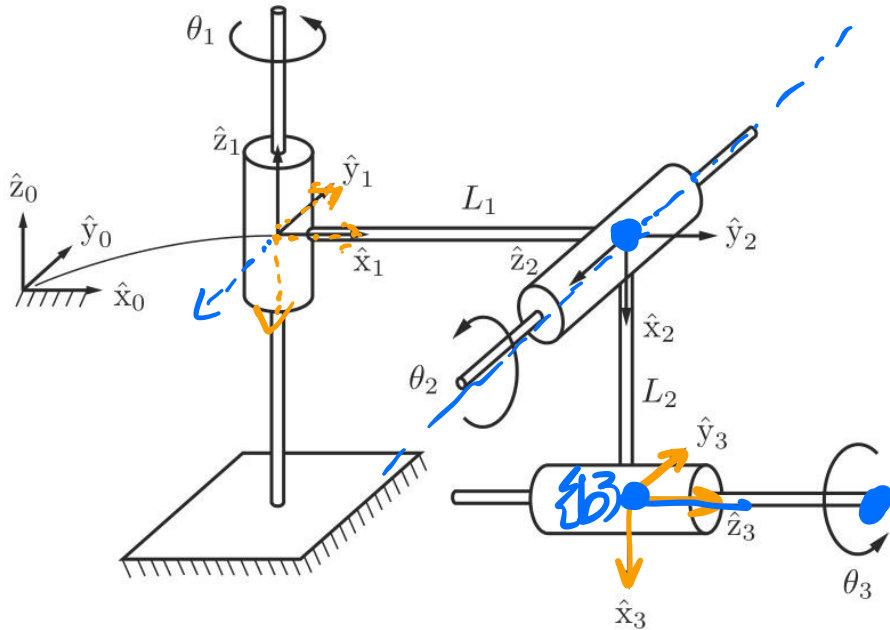


Figure 4.3: A 3R spatial open chain.

$$\textcircled{1} \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\textcircled{2} \omega_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = -\omega_2 \times q_2$$

$$q_2 = [L_1, 0, 0]^T$$

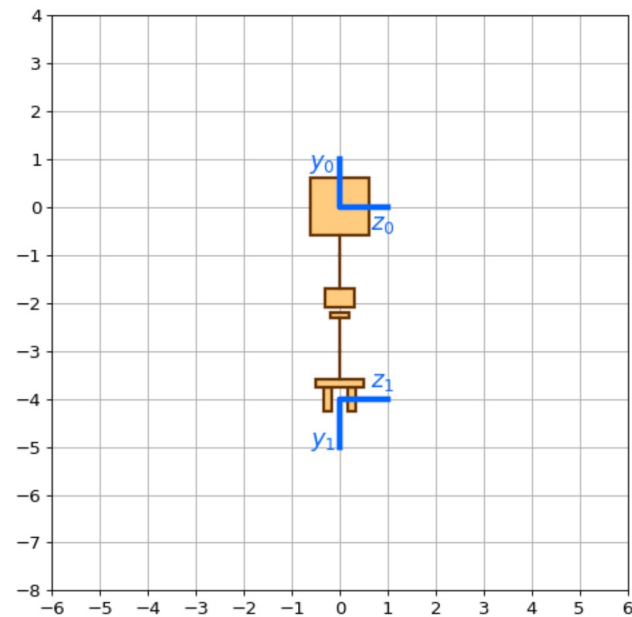
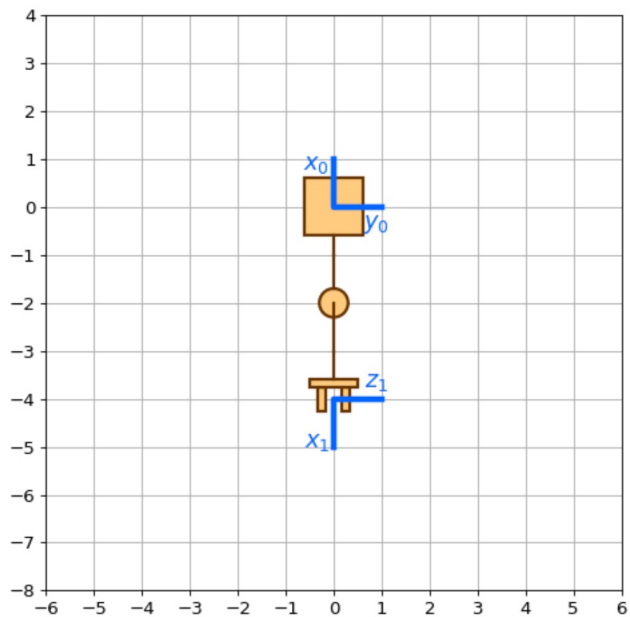
$$v_2 = [0, 0, -L_1]^T$$

$$\textcircled{3} \omega_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = -\omega_3 \times q_3$$

$$q_3 = [L_1, 0, -L_2]^T$$

$$v_3 = \begin{bmatrix} 0 \\ -L_2 \\ 0 \end{bmatrix}$$

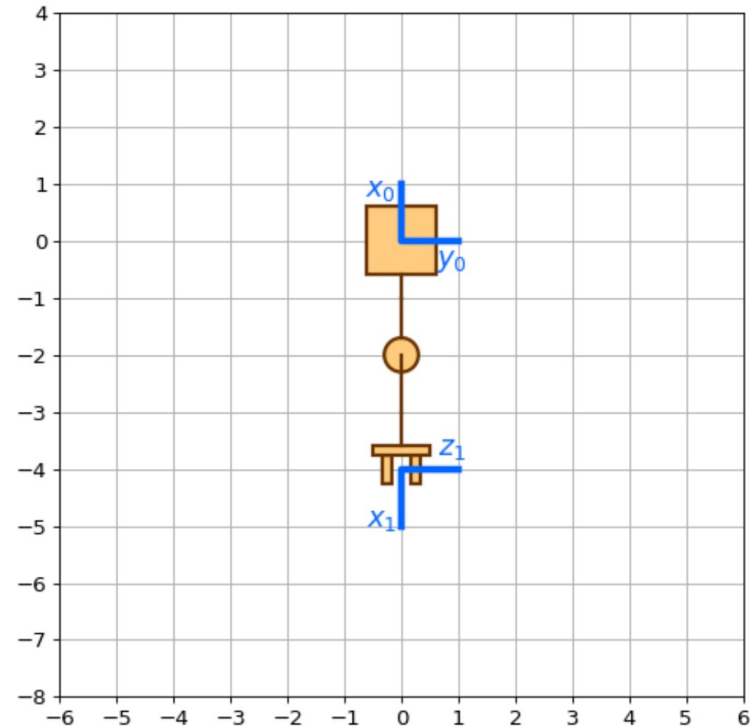
How to read this diagram (1)



Modeling Robot Joints as Screw Motions

Case 1: Revolute Joint

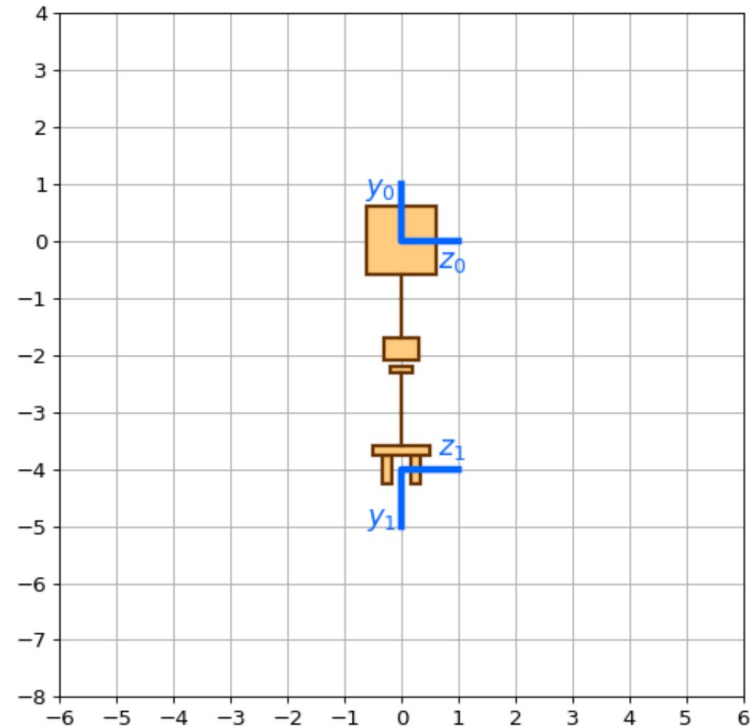
- $\|\omega\| = 1$
- $v = -\omega \times q + h\omega$
- $v = -\omega \times q$



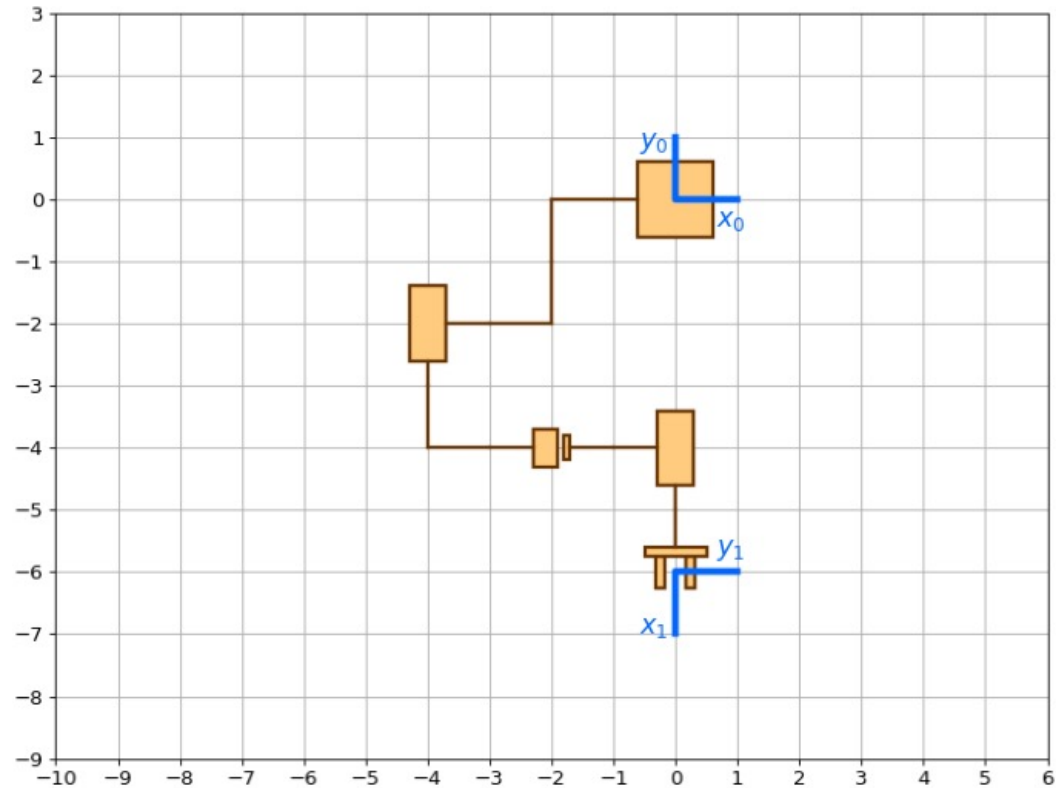
Modeling Robot Joints as Screw Motions

Case 2: Prismatic Joint

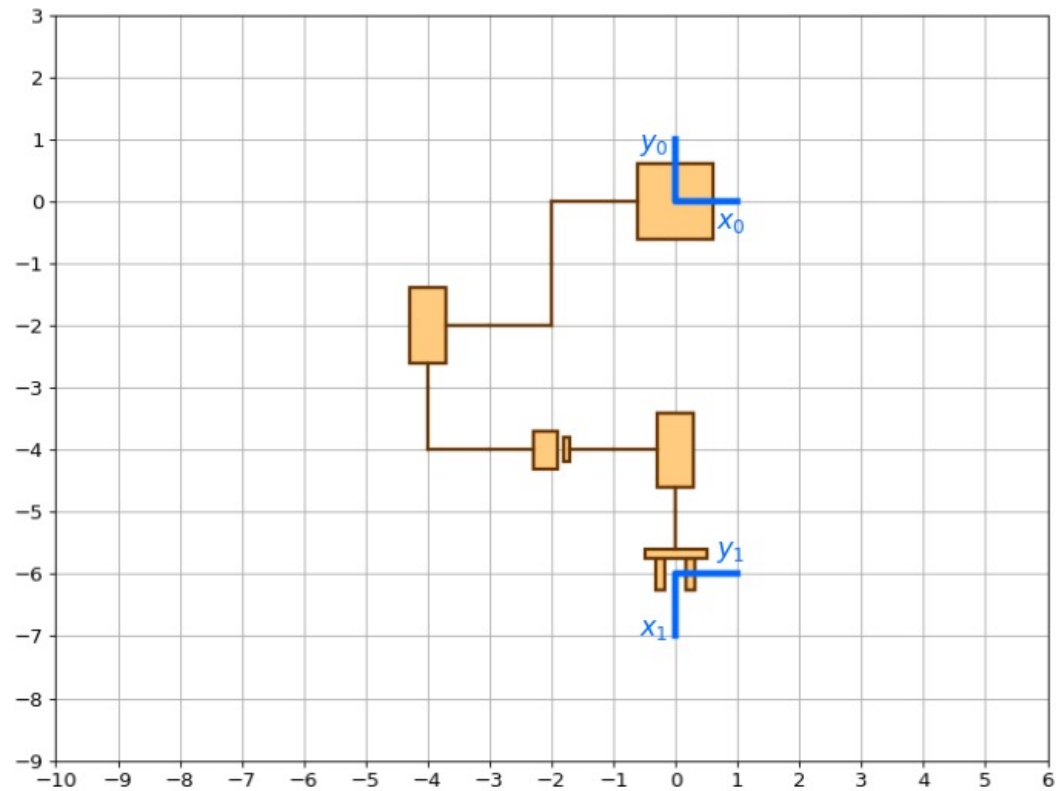
- $\|\omega\| = 0$
- $\|v\| = 1$
- Axis of movement defines v



How to read this diagram (2)



Example 3



PoE in the End-Effector (Body) Frame

- Recall some identities:

$$e^{M^{-1}PM} = M^{-1}e^PM \rightarrow e^PM = Me^{M^{-1}PM}$$

PoE in the End-Effector (Body) Frame

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- Applying this to PoE:

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} \dots e^{[S_{n-1}]\theta_{n-1}} e^{[S_n]\theta_n} M$$

$$T(\theta) = e^{[S_1]\theta_1} \dots e^{[S_{n-1}]\theta_{n-1}} M e^{M^{-1}[S_n]M\theta_n}$$

PoE in the End-Effector (Body) Frame

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$$T(\theta) = M e^{M^{-1}[S_1]M\theta_1} \dots e^{M^{-1}[S_{n-1}]M\theta_{n-1}} e^{M^{-1}[S_n]M\theta_n}$$

- What is $M^{-1}[S_i]M$ (or $[Ad_{M^{-1}}]S_i$)?
 - Screw axis in the body frame: $[B_i]$

PoE in the End-Effector (Body) Frame

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$$e^{M^{-1}PM} = M^{-1}e^PM \rightarrow e^PM = Me^{M^{-1}PM}$$

- Applying this to PoE:

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} \dots e^{[S_{n-1}]\theta_{n-1}} e^{[S_n]\theta_n} M$$

$$T(\theta) = e^{[S_1]\theta_1} \dots e^{[S_{n-1}]\theta_{n-1}} M e^{M^{-1}[S_n]M\theta_n}$$

$$T(\theta) = e^{[S_1]\theta_1} \dots M e^{M^{-1}[S_{n-1}]M\theta_{n-1}} e^{M^{-1}[S_n]M\theta_n}$$

$$T(\theta) = M e^{M^{-1}[S_1]M\theta_1} \dots e^{M^{-1}[S_{n-1}]M\theta_{n-1}} e^{M^{-1}[S_n]M\theta_n}$$

- What is $M^{-1}[S_i]M$ (or $[Ad_{M^{-1}}]S_i$)?

- Screw axis in the body frame: $[B_i]$

- Body form of PoE:

$$T(\theta) = M e^{[B_1]\theta_1} \dots e^{[B_{n-1}]\theta_{n-1}} e^{[B_n]\theta_n}$$

Summary

- The **screw axis $\mathcal{S}$** was defined and used to define the **exponential coordinates of homogeneous transformations**
- These transformations capture **screw motions** that correspond to the movements induced by joints
- Learned the basics of **forward kinematics**, which gives us a model for computing position and orientation of the end-effector
- Uses the **product of exponentials formula** to define this transformation as composed matrix multiplications